Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 477--484

Persistent URL: http://dml.cz/dmlcz/106663

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29,3 (1988)

COMPACTIFICATIONS AND L-SEPARATION

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Abstract: In the paper, the notion of L-separation introduced by J.L. Blasco is applied to characterizing subsets of $C^*(X)$ which generate compactifications of a Tychonoff space X (i.e. sets $F \subset C^*(X)$ such that the diagonal mapping $\triangle_{f \in P} f$ is a homeomorphic embedding).

Key words: Compactifications, continuous functions, L-separation, homeomorphic embeddings, proximities, functional bases.

Classification: 43D35, 54D40, 54C20

Throughout this paper, X denotes a Tychonoff space. The algebra of all bounded real-valued continuous functions on X is denoted by $C^*(X)$.

Let K(X) be the family of all compactifications of X. If αX, $\gamma X \in K(X)$ and there is a continuous $\varphi : \alpha X \to \gamma X$ such that $\varphi \circ \alpha = \gamma$, then we write $\gamma X \leq \alpha X$. For $\alpha X \in K(X)$, let C_α denote the set of all functions $f \in C^*(X)$ continuously extendable to αX. For $f \in C_\alpha$, let f^α be the continuous extension of f to αX and, for $F \subset C_\alpha$, let $F^\alpha = \{f^\alpha : f \in F\}$.

If $F \subset C^*(X)$ and the family $\{\alpha X \in K(X) : F \subset C_\alpha\}$ has a minimal (with respect to the partial order $\leq$) element $\alpha_F X$, then $\alpha_F X$ is said to be determined by F. Denote by $\mathcal{D}(X)$ the family of subsets of $C^*(X)$ which determine compactifications of X.

Let $\mathcal{E}(X)$ be the family of all sets $F \subset C^*(X)$ such that the diagonal mapping $e_F = \triangle_{f \in F} f$ is a homeomorphic embedding. If $F \in \mathcal{E}(X)$, then the closure of $e_F(X)$ in $R^{|F|}$ is a compactification of X. This compactification is said to be generated by F and is denoted by $e_F X$. If $\alpha X \in K(X)$, $F \in \mathcal{E}(X)$ and $e_F X = \alpha X$, then we say that F generates αX.

Finally, let $\mathcal{S}(X)$ be the family of all sets $F \subset C^*(X)$ which separate points from closed sets. It is well known that $\mathcal{S}(X) \subset \mathcal{E}(X) \subset \mathcal{D}(X)$; however, in general, both inclusions are proper.

The families $\mathcal{G}(X)$ and $\mathcal{D}(X)$ were considered in [1] - [3] and [7]. J.L. Blasco introduced in [4] the notion of L-separation and used it to characterize those functions from $C^*(X)$ which are continuously extendable to $e_F X$ where $F \in \mathcal{S}(X)$. In this paper we apply the notion of L-separation to investigate the family $\mathcal{E}(X)$.

For notation and terminology not defined here, see [5] and [6].

Before proceeding to the body of the article, let us recall two more definitions and establish some useful facts.

Definition 1 (cf. [4]). A set $G \subset C^*(X)$ L-separates a set $A \subset X$ from a set $B \subset X$ if there exist real numbers $a_{j,k} < b_{j,k} \leq c_{j,k} < d_{j,k}$ and functions $g_{j,k} \in G$ $(j=1,\ldots,m;\ k=1,\ldots,n)$ such that $A \subset \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} g_{j,k}^{-1}([b_{j,k};\ c_{j,k}])$ and $B \subset \bigcap_{j=1}^{m} \bigcup_{k=1}^{n} g_{j,k}^{-1}((-\infty;\ a_{j,k}] \cup [d_{j,k};\ +\infty))$.

Proposition 1. Suppose that $G \subset C^*(X)$ and let A_i, B_i be subsets of X for $i=1,2$.

(1) If G L-separates A_i from B_i, then G L-separates B_i from A_i.

(2) If G L-separates A_i form B_i for $i=1,2$, then G L-separates $A_1 \cup A_2$ from $B_1 \cap B_2$.

(3) Subsets A and B of X are completely separated if and only if $C^*(X)$ L-separates A from B.

We omit simple proofs of (1) and (2). To show (3), it suffices to observe that if $C^*(X)$ L-separates A from B, then $(cl_{\beta X} A) \cap (cl_{\beta X} B) = \emptyset$.

Definition 2 (cf. [4]). A set $G \subset C^*(X)$ L-separates a function $f \in C^*(X)$ if, for any real numbers $a < b$, the sets $f^{-1}((-\infty; a])$ and $f^{-1}([b;\ +\infty))$ are L-separated by G. A set $F \subset C^*(X)$ is L-separated by G if G L-separates any function $f \in F$.

Proposition 2. A set $G \subset C^*(X)$ L-separates a function $f \in C^*(X)$ if and only if, for any real numbers $a < b \leq c < d$, the sets $f^{-1}([b;\ c])$ and $f^{-1}((-\infty;\ a] \cup [d;\ +\infty))$ are L-separated by G.

Proof. Let $a < b \leq c < d$. If G L-separates $f^{-1}((-\infty;\ a])$ from $f^{-1}([b;\ +\infty))$ and $f^{-1}([d;\ +\infty))$ from $f^{-1}((-\infty;\ c])$, then, by Proposition 1 (2), the sets $f^{-1}((-\infty;\ a] \cup [d;\ +\infty))$ and $f^{-1}([b;\ c])$ are L-separated by G. On the other hand, since f is bounded, there is a real number $r > 0$ such that $f(X) \subset [-r;r]$ and $a,b \in (-r;r)$. Then $f^{-1}((-\infty;a])=f^{-1}([-r;a])$ and $f^{-1}([b;+\infty))=f^{-1}((-\infty;-2r] \cup [b;+\infty))$, which completes the proof.

Now, we are in a position to prove the main theorems of this paper.

Theorem 1. If $F \in \mathcal{D}(X)$, $G \subset C^*(X)$ and F is L-separated by G, then $G \in \mathcal{D}(X)$ and $\alpha_F X \leq \alpha_G X$.

Proof. Let us consider any $\alpha X \in K(X)$ for which $G \subset C_\alpha$. Since C_α L-separates F, it follows from [4; Theorem 4] that $F \subset C_\alpha$. Hence the set $C_F = \bigcap\{C_\alpha : \alpha X \in K(X)$ and $F \subset C_\alpha\}$ is contained in $C_G = \bigcap\{C_\alpha : \alpha X \in K(X)$ and $G \subset C_\alpha\}$. This, together with [1; Theorem 3.1] or [5; Theorem 2.18], implies that $C_G \in \mathcal{S}(X)$ because $C_F \in \mathcal{S}(X)$. Using [1; Theorem 3.1] again, we conclude that $G \in \mathcal{D}(X)$ and $\alpha_F X \leq \alpha_G X$.

The next theorem can be regarded as a generalization of Theorem 6 of [4].

Theorem 2. For sets $F \in \mathcal{E}(X)$ and $G \subset C^*(X)$, the following conditions are equivalent:

(1) $G \in \mathcal{E}(X)$ and $e_F X \leq e_G X$;

(2) F is L-separated by G.

Proof. That (1) implies (2) follows from [4; proofs of Proposition 2 and Theorem 6].

Assume (2). Let A be a closed subset of X and let $x \in X \setminus A$, By virtue of the theorem given in [6; Exercise 2.3.D], there exist $f_1,\dots,f_n \in F$ such that $\underset{i=1}{\overset{n}{\triangle}} f_i(x) \notin \mathrm{cl}_{R^n} \underset{i=1}{\overset{n}{\triangle}} f_i(A)$. We can find $\eta > 0$ such that

$(\prod_{i=1}^{n}[f_i(x)-\eta;f_i(x)+\eta]) \cap (\underset{i=1}{\overset{n}{\triangle}} f_i(A))=\emptyset$. By Proposition 2, for each $i \in \{1,\dots,n\}$, there exist functions $g_{i,j,k} \in G$ and real numbers $a_{i,j,k} < b_{i,j,k} \leq c_{i,j,k} < d_{i,j,k}$ $(j=1,\dots,n_i;\ k=1,\dots,m_i)$ such that

$$\{y \in X : |f_i(y)-f_i(x)| \leq \tfrac{\eta}{2}\} \subset \bigcup_{j=1}^{n_i} \bigcap_{k=1}^{m_i} g_{i,j,k}^{-1}([b_{i,j,k};c_{i,j,k}]) \text{ and}$$

$$\{y \in X : |f_i(y)-f_i(x)| > \eta\} \subset \bigcap_{j=1}^{n_i} \bigcup_{k=1}^{m_i} g_{i,j,k}^{-1}((-\infty;a_{i,j,k}] \cup [d_{i,j,k};+\infty)).$$

To each $i \in \{1,\dots,n\}$ assign some $j_i \in \{1,\dots,n_i\}$ such that

$x \in \bigcap_{k=1}^{m_i} g^{-1}_{i,j_i,k}([b_{i,j_i,k};c_{i,j_i,k}])$. Denote

$g=\Delta\{g_{i,j_i,k}$: $i=1,\dots,n$ and $k=1,\dots,m_i\}$ and

$V=\prod\{(a_{i,j_i,k};d_{i,j_i,k})$: $i=1,\dots,n$ and $k=1,\dots,m_i\}$.

Then V is an open subset of R^m where $m=\sum_{i=1}^{n} m_i$, and $g(x)\in V$. It is easily seen that $g(A)\cap V=\emptyset$, so $g(x)\notin \mathrm{cl}_{R^m}\, g(A)$. Using the theorem of [6; Exercise 2.3.D], we obtain that $G\in\mathcal{C}(X)$. Theorem 1 yields that $e_FX\leq e_GX$.

For a nonempty set $F\subset C^*(X)$, let M_F denote the family of all functions of the form $\varphi\circ\Delta_{f\in F} f$ where $\varphi\in C^*(R^{|F|})$ (cf. [2],[3] and [7]). It follows from [7; Remark 1.5 and Corollary 1.12] that M_F is the smallest subalgebra of $C^*(X)$ closed under uniform convergence, containing F and all constant functions.

Corollary 1. For sets $F\in\mathcal{C}(X)$ and $G\subset C^*(X)$, the following conditions are equivalent:

(1) $G\in\mathcal{C}(X)$ and $e_FX\leq e_GX$;

(2) M_F is L-separated by G;

(3) F is L-separated by M_G.

Proof. By virtue of [7; Corollary 2.6] (or [2; Theorem 2.3]), M_F generates e_FX, so the implication (1) $\Rightarrow$ (2) follows from Theorem 2. The implication (2) $\Rightarrow$ (3) is obvious. If we assume (3), then Theorem 2 yields that $M_G\in\mathcal{C}(X)$ and, moreover, the compactification generated by M_G is not less than e_FX. From [7; Corollary 2.6] (or [2; Theorem 2.3]) we deduce that (3) $\Rightarrow$ (1).

Corollary 2. For sets $F\in\mathcal{C}(X)$ and $G\subset C^*(X)$, the following conditions are equivalent:

(1) $G\in\mathcal{C}(X)$ and $e_FX=e_GX$;

(2) M_F is L-separated by G and M_G is L-separated by F;

(3) F is L-separated by M_G and G is L-separated by M_F.

Since $M_F=C_\alpha$ for any $F\in\mathcal{C}(X)$ such that $e_FX=\alpha X$ (cf. [2; Theorem 2.3], [3; Theorem 3.1] or [7; Theorem 2.12]), our next corollary is an immediate consequence of Corollary 2.

Corollary 3. For any $F\subset C^*(X)$ and $\alpha X\in K(X)$, the following conditions are equivalent:

(1) $F \in \mathcal{C}(X)$ and $e_F X = \alpha X$;

(2) $F \subset C_\alpha$ and C_α is L-separated by F;

(3) $F \subset C_\alpha$ and C_α is L-separated by M_F.

Let $\overset{L}{=}$ be the equivalence relation on $\mathcal{C}(X)$ defined by the condition: $F \overset{L}{=} G$ if and only if F L-separates G and G L-separates F. The equivalence class of $\overset{L}{=}$ containing $F \in \mathcal{C}(X)$ will be denoted by $[F]_L$. For $F,G \in \mathcal{C}(X)$, putting $[F]_L \overset{L}{\leq} [G]_L$ if and only if G L-separates F, we define a partial order on the set $\mathcal{C}(X)/L$ of all equivalence classes of $\overset{L}{=}$. The corollaries from Theorem 2 imply the following

Theorem 3. By assigning to any $[F]_L \in \mathcal{C}(X)/L$ the compactification $e_F X$ of X, one establishes an isomorphism of the partially ordered set $(\mathcal{C}(X)/L, \overset{L}{\leq})$ onto the partially ordered set $(K(X), \leq)$.

Now, we are going to study interrelations between elements of $\mathcal{C}(X)$ and proximities on X.

For $\alpha X \in K(X)$, denote by $\delta(\alpha)$ the proximity on X induced by αX; i.e. $\delta(\alpha)$ is defined by letting: $A \delta(\alpha) B$ if and only $(cl_{\alpha X} A) \cap (cl_{\alpha X} B) \neq \emptyset$ (cf. [6; p. 561]).

Let $F \subset C^*(X)$. We shall say that two sets $A,B \subset X$ are close with respect to $\delta(F)$ if F does not L-separate A from B.

Theorem 4. For any $F \subset C^*(X)$, the following conditions are equivalent:

(1) $F \in \mathcal{C}(X)$, and $\delta(F)$ is a proximity on X such that $\delta(F) = \delta(e_F)$;

(2) $F \in \mathcal{C}(X)$;

(3) $\delta(F)$ is a proximity on X.

Proof. According to the proof of Proposition 2 in [4], we deduce that (2) $\Rightarrow$ (3).

Assume (3) and let $\alpha X \in K(X)$ be such that $\delta(F) = \delta(\alpha)$. By virtue of [4; Corollary 3], C_α is L-separated by F. On the other hand, if $f \in F$ and $a < b$ $(a,b \in R)$, then the sets $f^{-1}((-\infty;a])$ and $f^{-1}([b;+\infty))$ are L-separated by F, so their closures in αX are disjoint. Using [4; Corollary 3] again, we obtain that $F \subset C_\alpha$. By our Corollary 3, $F \in \mathcal{C}(X)$ and $e_F X = \alpha X$; hence (3) $\Rightarrow$ (1).

Theorem 5. By assigning to any $[F]_L \in \mathcal{C}(X)/L$ the proximity $\delta(F)$ on X, we establish a one-to-one correspondence between elements of $\mathcal{C}(X)/L$ and all proximities on the space X.

To give another necessary and sufficient condition for F to be in $\mathcal{E}(X)$, we need some notation.

Suppose that $F \subset C^*(X)$. Denote by $\mathcal{Z}_F$ the family of all sets of the form $\bigcup_{j=1}^{m} \bigcap_{k=1}^{n} f_{j,k}^{-1}([a_{j,k};b_{j,k}])$ where $f_{j,k} \in F$ and $a_{j,k} \leq b_{j,k}(a_{j,k},b_{j,k} \in R)$ for $j=1,\dots,m;\ k=1,\dots,n\ (m,n \in N)$. One can easily check that the family $\mathcal{Z}_F$ is closed under finite unions and intersections; moreover, $\mathcal{Z}_F$ consists of zero-sets of X.

Theorem 6. A set $F \subset C^*(X)$ is an element of $\mathcal{E}(X)$ if and only if the family $\mathcal{Z}_F$ is a closed base for X.

Proof. Let A be a closed subset of X and let $x \in X \setminus A$. If $F \in \mathcal{E}(X)$, then from [4; proof of Proposition 2] we deduce that F L-separates A from $\{x\}$; hence there exists $Z \in \mathcal{Z}_F$ such that $A \subset Z$ and $x \notin Z$, which means that $\mathcal{Z}_F$ is a closed base for X.

Conversely, if $\mathcal{Z}_F$ is a closed base for X, then there exist functions $f_{j,k} \in F$ and real numbers $a_{j,k} \leq b_{j,k}$ $(j=1,\dots,m;\ k=1,\dots,n)$ such that

$$A \subset \bigcup_{j=1}^{m} \bigcap_{k=1}^{n} f_{j,k}^{-1}([a_{j,k};b_{j,k}]) \text{ and } x \in \bigcap_{j=1}^{m} \bigcup_{k=1}^{n} f_{j,k}^{-1}((-\infty;a_{j,k}) \cup (b_{j,k};+\infty)).$$

To each $j \in \{1,\dots,m\}$ assign some $k_j \in \{1,\dots,n\}$ such that

$x \in f_{j,k_j}^{-1}((-\infty;a_{j,k_j}) \cup (b_{j,k_j};+\infty))$. Denote $f = \triangle_{j=1}^{m} f_{j,k_j}$ and

$V = \prod_{j=1}^{m} [(-\infty;a_{j,k_j}) \cup (b_{j,k_j};+\infty)]$. Then $V \cap f(A) = \emptyset$, so $f(x) \notin \mathrm{cl}_{R^m} f(A)$.

Applying the theorem given in [6; Exercise 2.3.D], we obtain that $F \in \mathcal{E}(X)$.

Let $\alpha X \in K(X)$. In [1; p.9] B.J. Ball and Shoji Yokura introduced the cardinal number $e(\alpha X) = \min\{|F| : F \in \mathcal{E}(X) \text{ and } e_F X = \alpha X\}$. We shall call this number the functional weight of αX. As shown in [1; Theorem 4.2], if the functional weight of αX is infinite, then it is equal to the weight of αX. It seems natural to call every set generating αX a functional base for αX. It is worth mentioning that $F \subset C^*(X)$ is a functional base for αX if and only if $M_F = C_\alpha$ (cf. [2; Definition 1.2 and Theorem 2.3]). Our final theorem points out that functional bases have some property similar to that of open bases for topological spaces.

Theorem 7. If $\alpha X \in K(X)$ is of infinite functional weight, then every functional base for αX contains a functional base for αX of cardinality $e(\alpha X)$.

Proof. Consider any functional base F for αX. There exists a functional base H for αX such that $|H|=e(\alpha X)$. Denote by Q the set of rational numbers and let $P=\{\langle a,b\rangle \in Q^2 : a<b\}$. By Corollary 2, H is L-separated by F. Therefore, to each $h\in H$ and $\langle a,b\rangle \in P$ we can assign a finite set $F(h;\langle a,b\rangle)\subset F$ which L-separates $h^{-1}((-\infty;a])$ from $h^{-1}([b;+\infty))$. Let $G=\bigcup\{F(h;\langle a,b\rangle):h\in H \text{ and } \langle a,b\rangle\in P\}$. First of all, observe that $|G|\leq|H|$ and H is L-separated by G. Since $G\subset F$ and, by Corollary 2, F is L-separated by H, we have that G is L-separated by H. Applying Corollary 2 again, we deduce that G is a functional base for αX and, consequently, $|G|=|H|$.

The assumption that $e(\alpha X)$ is infinite cannot be omitted in the above theorem.

Example 1. Let $X=(-1;1)$, $\alpha X=[-1;1]$ and $F=\{f_1,f_2\}$ where

$$f_1(x)=\begin{cases} 0 & \text{for } -1<x\leq 0, \\ x & \text{for } 0<x<1 \end{cases} \quad \text{and} \quad f_2(x)=\begin{cases} x & \text{for } -1<x\leq 0, \\ 0 & \text{for } 0<x<1. \end{cases}$$

Then F is a functional base for αX (cf. [3; Theorem 2.3]), $e(\alpha X)=1$, but none of the sets $\{f_1\}$, $\{f_2\}$ generates αX.

Observe that Theorem 4.3 of [1], our Theorem 2 and the proof of Theorem 7 imply that if αX, $\gamma X\in K(X)$ are of infinite functional weight, $\alpha X\leq\gamma X$ and F is a functional base for γX, then there exists a set $G\subset F$ such that $G\in\mathcal{C}(X)$, $|G|=e(\alpha X)$ and $\alpha X\leq e_G X\leq\gamma X$; however, F need not contain any functional base for αX.

Example 2. Consider the space $[0;\omega_1)$ of ordinal numbers $<\omega_1$ with the order topology. Let $X=[0;\omega_1)\times\{0,1\}$ and $F=\{f\in C^*(X):f^\beta(\langle\omega_1,0\rangle)\neq f^\beta(\langle\omega_1,1\rangle)\}$. Since F^β separates points of βX, it follows from [3; Theorem 2.3] that F is a functional base for βX. No function from F is continuously extendable to the one-point compactification of X; hence, no subset of F is a functional base for the one-point compactification of X.

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(Oblatum 3.8. 1987, revisum 8.2. 1988)