Commentationes Mathematicae Universitatis Carolinae

Vladimír Puš

Chromatic number of products of graphs

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 457--463

Persistent URL: http://dml.cz/dmlcz/106661

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,3 (1988)

CHROMATIC NUMBER OF PRODUCTS OF GRAPHS

Vladimír PUŠ

Abstract: We give a description of all products $G * H$ of simple graphs (excepting the direct product) having the following property: the chromatic number $\chi(G*H)$ is a function of numbers $\chi(G)$ and $\chi(H)$. We also determine these functions.

Key words: Product of graphs, chromatic number.

Classification: 05C15

---

0. Introduction. L. Lovász's well-known problem is the following one: Is it true that the chromatic number of the direct product of simple graphs is given by the formula $\chi(G\times H)=\min(\chi(G), \chi(H))$?
(In other words: Does the function f exist such that $\chi(G\times H)=f(\chi(G), \chi(H))$ for every pair G,H of simple graphs?)

In this paper we describe all products $G*H$ of simple graphs (excepting the direct product) for which there exists a function f such that the chromatic number of $G*H$ is given by the formula $\chi(G*H)=f(\chi(G), \chi(H)$. The explicit expressions of the functions f are also given.

1. Definitions. The graphs we consider are simple graphs, i.e. undirected graphs without loops and multiple edges. The set of vertices of a graph G is denoted by V(G), E(G) is the set of edges. We will consider only graphs with a non-empty set of vertices.

By $\chi(G)$ we denote the chromatic number of G.

K_n is the complete graph on n vertices, D_n is the discrete graph on n vertices and C_n is the circuit of the length n.

Let us recall the general definition of products of simple graphs (see [1]).

Let $p: \{1,-1,0\}\times\{1,-1,0\}\rightarrow\{1,-1,0\}$ be a fixed mapping such that $p(i,j)=0$ iff $i=j=0$.

For a simple graph $G=(V,E)$ and a pair of vertices $x,y\in V$ define

$$s(x,y)=\begin{cases} 1 & \text{iff } \{x,y\}\in E \\ -1 & \text{iff } \{x,y\}\notin E \text{ and } x\neq y \\ 0 & \text{iff } x=y \end{cases}$$

(i.e. $s:V\times V\to\{1,-1,0\}$).

Given a pair G,H of simple graphs, define the <u>product</u> $\underline{G\overset{p}{\times}H}$ as follows:

$$V(G\overset{p}{\times}H)=V(G)\times V(H)$$

and

$$E(G\overset{p}{\times}H)=\{\{(x,x'),(y,y')\};\ p(s(x,y),s(x',y'))=1\}.$$

This definition covers all products of graphs (there exists $2^8=256$ different products).

For example, let $p(i,j)=1$ iff $i=j=1$. Then $\overset{p}{\times}$ is the <u>direct product</u>; we denote $\times$ instead of $\overset{p}{\times}$ in this case.

Let $p(i,j)=1$ iff $i=1$ or $j=1$. The product we obtain is in [2], p. 52, called the <u>cartesian sum</u> and denoted by $\oplus$.

Let $p(i,j)=1$ iff $i=0$ and $j=1$, or $i=1$ and $j=0$. Then $\overset{p}{\times}$ is the well-known <u>cartesian product</u>; this product will be denoted by $\square$.

Let $p(i,j)=-1$ iff $i=-1$ or $j=-1$. Then $\overset{p}{\times}$ is the so-called <u>strong product</u>; we denote it by $\boxtimes$.

Let $p(i,j)=1$ iff either $i=1$, or $i=0$ and $j=1$. Then we obtain the so-called <u>lexicographic product</u> (or the <u>substitution</u> of the graph H into G). In this we denote $G\overset{p}{\times}H=G[H]$.

2. Auxiliary results. First we notice that

$$(G\times H)\cup(G\,\square\,H)=G\boxtimes H\ \begin{matrix} \subseteq G[H] \subseteq \\ \subseteq H[G] \subseteq \end{matrix}\ G\oplus H$$

and that

$$\chi(G\,\square\,H)=\max(\chi(G),\chi(H)).$$

In the following proposition we show that generally $\chi(G\oplus H)<\chi(G)\cdot\chi(H)$.

Proposition 1. $\chi(C_{2m+1}\oplus C_{2n+1})\leq 8$ for $m,n\geq 2$.

Proof: Let G, H be graphs. For $v \in V(G)$ and $w \in V(H)$ denote $S_v = \{v\} \times V(H)$ and $R_w = V(G) \times \{w\}$. The mapping $\varphi : V(G) \times V(H) \to \{1,2,\dots,k\}$ is a colouring of the graph $G \oplus H$ by k colours if and only if the following conditions hold:

$$\{v_1, v_2\} \in E(G) \Longrightarrow S_{v_1} \cap S_{v_2} = \emptyset$$

and

$$\{w_1, w_2\} \in E(H) \Longrightarrow R_{w_1} \cap R_{w_2} = \emptyset.$$

Hence, the following matrix (with 2n+1 rows and 2m+1 columns) represents a colouring of the graph $C_{2m+1} \oplus C_{2n+1}$ by 8 colours.

$$\begin{array}{cccccccccc}
1 & 2 & 1 & 2 & \dots & 1 & 2 & 1 & 2 & 3 \\
4 & 5 & 4 & 5 & \dots & 4 & 5 & 4 & 5 & 6 \\
1 & 2 & 1 & 2 & \dots & 1 & 2 & 1 & 2 & 3 \\
4 & 5 & 4 & 5 & \dots & 4 & 5 & 4 & 5 & 6 \\
\cdot & & & & & & & & \cdot & \cdot \\
\cdot & & & & & & & & \cdot & \cdot \\
\cdot & & & & & & & & \cdot & \cdot \\
1 & 2 & 1 & 2 & \dots & 1 & 2 & 1 & 2 & 3 \\
4 & 5 & 4 & 5 & \dots & 4 & 5 & 4 & 5 & 6 \\
1 & 2 & 1 & 2 & \dots & 1 & 2 & 1 & 2 & 8 \\
4 & 5 & 4 & 5 & \dots & 4 & 5 & 4 & 5 & 3 \\
7 & 8 & 7 & 8 & \dots & 7 & 8 & 6 & 7 & 8
\end{array}$$

Proposition 2. Suppose that there exists a function f such that $\chi(G \stackrel{p}{\times} H) \leq f(\chi(G), \chi(H))$. Then the following condition holds:

(T) $\quad p(i,j)=1 \Longrightarrow i=1$ or $j=1$.

Conversely, if the condition (T) is fulfilled, then $\chi(G \stackrel{p}{\times} H) \leq \chi(G) \cdot \chi(H)$.

Proof: Suppose that there exists a function f such that $\chi(G \stackrel{p}{\times} H) \leq f(\chi(G), \chi H))$ and that $p(i,j)=1$. Assume that for contradiction $i \neq 1$ and $j \neq 1$.

If $(i,j)=(-1,-1)$ then $K_n \subseteq D_n \stackrel{p}{\times} D_n$, hence $n \leq \chi(D_n \stackrel{p}{\times} D_n) \leq f(1,1)$ for every n, a contradiction.

If $(i,j)=(-1,0)$ then $D_n \stackrel{p}{\times} D_1 \cong K_n$, hence $n = \chi(D_n \stackrel{p}{\times} D_1) \leq f(1,1)$, a contradiction. Similarly, the case $(i,j)=(0,-1)$ leads to a contradiction.

Conversely, let the condition (T) be fulfilled. Then the product $A \times B$

of discrete sets $A \subseteq V(G)$ and $B \subseteq V(H)$ is a discrete set in $G \overset{p}{\times} H$, which implies that $\chi(G \overset{p}{\times} H) \leq \chi(G) \cdot \chi(H)$.

3. The main result

Theorem

(I) Suppose that p fulfils the following conditions:

(1) $p(i,j)=1 \Longrightarrow i=1$ and

(2) $p(1,0)=1$.

Then $\chi(G \overset{p}{\times} H) = \chi(G)$.

(II) Suppose that

(3) $p(i,j)=1 \Longrightarrow j=1$ and

(4) $p(0,1)=1$.

Then $\chi(G \overset{p}{\times} H) = \chi(H)$.

(III) If $\overset{p}{\times}$ is the cartesian product, then $\chi(G \overset{p}{\times} H)=\max(\chi(G), \chi(H))$.

(IV) If p is identically equal to -1, then $\chi(G \overset{p}{\times} H)$ is identically equal to 1.

(V) Assume that there exists a function f such that $\chi(G \overset{p}{\times} H)=f(\chi(G), \chi(H))$ for every pair G,H of (finite) graphs. Then either $\overset{p}{\times}$ is the direct product or some of the cases (I)-(IV) occurs.

(VI) Assume that there exists a function f such that $\chi(G \times H)=f(\chi(G), \chi(H))$. Then $\chi(G \times H)=\min(\chi(G), \chi(H))$.

Proof: Suppose that there exists a function f such that $\chi(G \overset{p}{\times} H)= f(\chi(G), \chi(H))$. Then the condition (T) from Proposition 2 is satisfied. Now we distinguish four cases (α), (β), (γ) and (δ).

(α) Let $p(1,0)=1$ and $p(0,1)= -1$.

Then $f(n,m)=\chi(K_n \overset{p}{\times} K_m)=n$. For this, let $V(K_n)=\{1,2,\dots,n\}$ and $V(K_m)= \{1,2,\dots,m\}$. Then the function φ defined by $\varphi(i,j)=i$ is a colouring of $K_n \overset{p}{\times} K_m$ by n colours and moreover $K_n \subseteq K_n \overset{p}{\times} K_m$.

It follows that $p(-1,1)= -1$. Indeed, $p(-1,1)=1$ implies $K_n \subseteq D_n \overset{p}{\times} K_n$ for every n, and so $n= \chi(D_n \overset{p}{\times} K_n)=f(1,n)=1$, a contradiction. Hence, according to

(T), the following condition holds:

(1) $p(i,j)=1 \Longrightarrow i=1$.

Since, moreover, by the assumption, $p(1,0)=1$, the conditions (1) and (2) in Part (I) of Theorem are fulfilled. Conversely we show that under these conditions $\chi(G \overset{p}{\times} H)= \chi(G)$.

Indeed, (1) follows from the fact that $A \times V(H)$ is a discrete set for every discrete set $A \subseteq V(G)$. Hence, $\chi(G \overset{p}{\times} H) \leqq \chi(H)$. Further, (2) follows from $G \subseteq G \overset{p}{\times} H$ and so $\chi(G \overset{p}{\times} H) \geqq \chi(G)$.

(β) Let $p(1,0)= -1$ and $p(0,1)=1$.

Then, similarly as in the case (α), the conditions (3) and (4) in Part II of Theorem follow. Conversely, these conditions imply that $\chi(G \overset{p}{\times} H)= \chi(H)$.

Now we suppose that

(P) $p(1,0)=p(0,1)$.

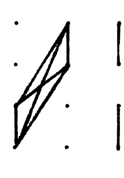
We divide this case into two partial cases (γ) and (δ).

(γ) In addition, let $p(1,1)= -1$.

By (P), either $p(1,0)=p(0,1)=1$ or $p(1,0)=p(0,1)= -1$.

(γ_1) In the first case we have $K_n \overset{p}{\times} K_m \cong K_n \square K_m$,

hence $\max(n,m)= \chi(K_n \overset{p}{\times} K_m)=f(\chi(K_n), \chi(K_m))=f(n,m)$. It follows that $p(1,-1)= =p(-1,1)= -1$. Indeed, if for example $p(1,-1)=1$ then $K_2 \overset{p}{\times} (K_2+K_2)$ contains K_4 (see the figure) and so $4 \leqq \chi(K_2 \overset{p}{\times} (K_2+K_2))=f(2,2)=2$, a contradiction. Thus, $\overset{p}{\times}$ is the cartesian product; hence, the case (III) in Theorem has occurred.



Figure

(γ_2) In the second case we have $K_n \overset{p}{\times} K_m \cong D_{n \cdot m}$, so $f(n,m)=1$. But this means that p is identically equal to -1 and $\chi(G \overset{p}{\times} H)$ is identically equal to

-1 and $\chi(G \overset{p}{\times} H)$ is identically equal to 1, which is the situation described in Theorem, Part (IV).

(δ) Let p(1,1)=1.

By (P) we again consider two cases.

(δ_1) Let p(1,0)=p(0,1)=1. Then $K_n \overset{p}{\times} K_m \cong K_{n\cdot m}$, Hence f(n,m)= $= \chi(K_n \overset{p}{\times} K_m)=n\cdot m$. Further, by (P), $G \overset{p}{\times} H \subseteq G \oplus H$ (more exactly, this case includes the strong product, the lexicographic product and the cartesian sum). Therefore, by Proposition 1, we have

$$9=3\cdot 3=f(3,3)= \chi(C_{2m+1} \overset{p}{\times} C_{2n+1}) \leq \chi(C_{2m+1} \oplus C_{2n+1})=8,$$

a contradiction.

(δ_2) Let p(1,0)=p(0,1)= -1. Then $\chi(K_n \overset{p}{\times} K_m) \leq \min(n,m)$. For this, if $V(K_n)=\{1,2,\dots,n\}$ and $V(K_m)=\{1,2,\dots,m\}$, then the function φ defined by $\varphi((i,j))=i$ (or $\varphi((i,j))=j$) is a colouring of the graph $K_n \overset{p}{\times} K_m$ by n (or m) colours. Conversely, since p(1,1)=1, we have $K_{\min(n,m)} \subseteq K_n \overset{p}{\times} K_m$. Therefore, $f(n,m)= \chi(K_n \overset{p}{\times} K_m)=\min(n,m)$.

The last formula implies that p(-1,1)=p(1,-1)= -1. To see this, let us suppose without loss of generality that p(-1,1)=1. Then $n= \chi(D_n \overset{p}{\times} K_n)=f(1,n)=$ $=\min(1,n)=1$ for every n, a contradiction. Thus, $\overset{p}{\times}$ is the direct product (the case (VI) in Theorem).

It is clear that the discussion just given includes proofs of all propositions (I)-(VI) in Theorem.

The previous theorem gives also the answer to the question of C. Thomassen whether there exists a product $\overset{p}{\times}$ such that $\chi(G \overset{p}{\times} H)= \chi(G)\cdot\chi(H)$.

Corollary. There is no product $\overset{p}{\times}$ of simple graphs with the following property: $\chi(G \overset{p}{\times} H)= \chi(G)\cdot\chi(H)$ for every pair G,H of (finite) graphs.

References

[1] J. NEŠETŘIL, V. RÖDL: Three remarks on dimensions of graphs, Annals of Discrete Math. 28(1985), 199-207.

[2] O. ORE: Theory of Graphs, 1962.

[3] C. THOMASSEN: personal communication.

Forestry and Game Management Research Institute, Jíloviště-Strnady, 15604
Praha 5 - Zbraslav, Czechoslovakia

(Oblatum 25.5. 1988)