Commentationes Mathematicae Universitatis Carolinae

Tadeusz Kuczumow; Adam Stachura

Extensions of nonexpansive mappings in the Hilbert ball with the hyperbolic metric. II.

Commentationes Mathematicae Universitatis Carolinae, Vol. 29 (1988), No. 3, 403--410

Persistent URL: `http://dml.cz/dmlcz/106656`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
29,3 (1988)

EXTENSIONS OF NONEXPANSIVE MAPPINGS IN THE HILBERT BALL WITH THE HYPERBOLIC METRIC. PART II.

Tadeusz KUCZUMOW and Adam STACHURA

Abstract: If in a real Hilbert space H_R we take an open unit ball B_R with the hyperbolic metric ρ_1, then every ρ_1-nonexpansive mapping T from a subset $X \subset B_R$ into B_R has a ρ_1-nonexpansive extension on the whole B_R.

Key words: Hyperbolic metric, nonexpansive mappings, fixed points.

Classification: 47H10, 32H15

Let H_R be a real Hilbert space and let B_R be an open unit ball in H_R. Then H_R (B_R) can be identified with the subset of a complex Hilbert space H (an open unit ball B in H). Thus the hyperbolic metric ρ_1 in B ([9]) may be restricted to B_R. There are three reasons, why we are interested in (B_R, ρ_1):

(i) there is an obvious connection of (B_R, ρ_1) with Klein's model of the hyperbolic geometry;

(ii) the distance ρ_1 is visibly a projective invariant ([7]);

(iii) (B_R, ρ_1) has metric properties different from properties of (B, ρ_1).

As a direct consequence of Theorem 1 in [5] we get that every mapping $U \circ M_a$, where U is a unitary operator in H_R and M_a is the Möbius transformation with $a \in B_R$ ([3]), is an isometry in B_R, ρ_1). Now we show something more.

Theorem 1. Every isometry from B_R onto B_R has the form $T = U \circ M_a$, where M_a is the Möbius transformation and U is a unitary linear mapping in H_R.

Proof: Let -a be equal to $T^{-1}(0)$. Then $U_1 = T \circ M_{-a}$ has the following properties:

(i) $U_1(0) = 0$,

(ii) $(U_1 x, U_1 y) = (x, y)$ for all $x, y \in B_R$ (it follows from the equality $\sigma(U_1 x, U_1 y) = \sigma(x, y)$),

(iii) $U_1(tx)=tU_1x$ for $x\in B_R\setminus\{0\}$ and $t\in(-1/\|x\|,1/\|x\|)$ because $\|U_1(tx)-tU_1x\|^2=\|U_1(tx)\|^2+\|tU_1x\|^2-2t(U_1(tx),U_1x)==0$.

Therefore the mapping

$$Ux=\begin{cases}0 & \text{if } x=0\\ 2\|x\|U_1\left(\frac{x}{2\|x\|}\right) & \text{if } x\neq 0\end{cases}$$

is well defined and unitary.

Corollary 1. If T is an isometry from B_R onto B_R and has no fixed point in B_R, then its fixed set in B_R closure consists of either one point or two points.

Corollary 2. If T is an isometry from B_R onto B_R which has two fixed points in $\bar{B}_R$ and no fixed points in B_R, then the iterates T^i of T converge to a fixed point of T. The convergence is uniform on the ball of radius $r<1$.

The above corollaries are consequences of Theorem 1, Theorem 4 from [5] and Theorem 3 from [12].

Now we consider a problem of extensions of nonexpansive mappings in B_R. The key role in our considerations will be played by the following

Theorem 2. If $x_1,\dots,x_m,x_1',\dots,x_m',x_m',p$ are points of B_R such that $\rho_1(x_i',x_j')\leq\rho_1(x_i,x_j)$ $(i,j=1,2,\dots,m)$, then in B_R there exists a point p' such that $\rho_1(x_i',p')\leq\rho_1(x_1,p)$ $(i=1,2,\dots,m)$.

Proof: For every $\mu\geq 0$ the set

$P_\mu=\{q\in B_R:\rho_1(x_i',q)\leq\mu\rho_1(x_i,p) \text{ for } i=1,2,\dots,m\}$

is bounded, closed and nonempty if μ is sufficiently large. Moreover, $\mu\leq\lambda$ implies $P_\mu\subset P_\lambda$. Hence there exists the smallest nonnegative number α for which the set P_α is nonempty ([3]). If $\alpha\leq 1$ the proof is finished.

Suppose that $\alpha>1$ and let p' be an element of P_α. Without loss of generality we may assume that $p=p'=0$,

$\rho_1(x_i',0)>\rho_1(x_i,0)$ for $i=1,2,\dots,k$

and

$\rho_1(x_i',0)\leq\rho_1(x_i,0)$ for $i=k+1,\dots,m$.

To our surprise this simple assumption allows us to apply the method due to Schoenberg ([11]).

The element 0 must lie in the ρ_1-convex hull (equal to the usual convex hull) of the set $\{x_1',\dots,x_k'\}$ ([3]). Hence we have $0=\sum_{i=1}^{k}\mu_i x_i'$, where $\mu_1,\dots,\mu_k\geq 0$ and $\sum_{i=1}^{k}\mu_i=1$. But then we have

$$\sigma(x_1',x_j')\geq \sigma(x_i,x_j)\ (i,j=1,2,\dots,k)$$

which imply

$$\frac{(1-(x_i',x_j'))^2}{(1-(x_i,x_j))^2}\leq\frac{(1-\|x_i'\|^2)(1-\|x_j'\|^2)}{(1-\|x_i\|^2)(1-\|x_j\|^2)}<1$$

and finally $(x_i',x_j')>(x_i,x_j)$ for $i,j=1,\dots,k$. Therefore we get

$$0=\|\sum_{i=1}^{k}\mu_i x_i'\|^2=\sum_{i,j=1}^{k}\mu_i\mu_j(x_i',x_j')>\sum_{i,j=1}^{k}\mu_i\mu_j(x_i x_j)=\|\sum_{i=1}^{k}\mu_i x_i\|^2.$$

This contradiction completes the proof.

As a simple consequence of the above theorem we obtain the following two equivalent theorems.

Theorem 3. Let $\{B(x_\mu,r_\mu)\}_{\mu\in I}$, $\{B(x_\mu',r_\mu)\}_{\mu\in I}$ be two families of closed balls in (B_R,ρ_1). If $\rho_1(x_\mu',x_\lambda')\leq\rho_1(x_\mu,x_\lambda)$ for all $\mu,\lambda\in I$ and the intersection $\bigcap_{\mu\in I}B(x_\mu,r_\mu)$ is nonempty, then so is the intersection $\bigcap_{\mu\in I}B(x_\mu',r_\mu)$.

Theorem 4. Let $T:X\to B_R$ be a ρ_1-nonexpansive mapping of a subset X of B_R into B_R. There exists a ρ_1-nonexpansive mapping $\tilde{T}:B_R\to B_R$ such that its restriction to X is identical with T.

As we know for every nonexpansive mapping $T:B\to B$ with a fixed point we can construct nonexpansive mappings

$$S_{1t}=(1-t)I+tT,$$
$$S_{2t}=(1-t)I\oplus tT,$$

where $0<t<1$ and $p=(1-t)x\oplus ty$ denotes the unique point of geodesic segment $[x,y]$ satisfying

$$\rho_1(x,p)=t\,\rho_1(x,y)\ \text{ and }\ \rho_1(y,p)=(1-t)\,\rho_1(x,y).$$

These mappings have the same fixed point set as the mapping T and their iterations tend weakly to fixed points of T ([10]).

Now we show that in general we cannot replace the weak convergence by the strong one. The example given below is a modification of the Genel-Lindenstrauss example ([2]).

Example 1. Let H_R be l_2 with the orthogonal basis $\{e_k\}$. First we define inductively sequences $\{x_i\}$ and $\{Tx_i\}$ which satisfy

$$x_i = \frac{x_{i-1}+Tx_{i-1}}{2}$$

for $i=2,3,\dots$. We start the construction of the sequence $\{x_i\}$ by picking $x_1=\frac{1}{2}e_1$. Let n_1 and φ_1 satisfy conditions

$$\mathbf{N} \ni n_1 > 10,$$

$$\varphi_1 = \frac{\pi}{3(n_1-1)},$$

$$\frac{1}{2}(\cos\varphi_1)^{n_1} > \frac{3}{8} = \frac{\frac{1}{4}+\frac{1}{2}}{2}.$$

The points x_i, $i=1,\dots,n_1$ and Tx_i, $i=1,\dots,n_1-1$ will be chosen in the plane $P_1=\mathrm{lin}(0,e_1,e_2)$ according to the following rules:

$$\|x_i\| = \|Tx_i\|,\ i=1,2,\dots,n_1-1$$

$$(x_i,Tx_i)=\|x_i\|^2\cos(2\varphi_1),\ i=1,2,\dots,n_1-1$$

$$x_{i+1}=\frac{x_i+Tx_i}{2},\ i=1,2,\dots,n_1-1.$$

It is clear that for every $1\le i,\ j\le n_1-1$ we have

$$\rho_1(Tx_iTx_j)=\rho_1(x_ix_j).$$

In this place we must modify the Genel-Lindenstrauss example. We define the point Tx_{n_1} in the following way. Let y_1 be the next point after x_{n_1} (in the plane P_1) chosen according to the above rules. It means that

$$\|y_1\| = \|x_{n_1}\| \text{ and } (x_{n_1},y_1)=\|x_{n_1}\|^2\cos(2\varphi_1).$$

We set

$$Tx_{n_1}=z_1+\|x_{n_1}\|\sin(\varphi_1)e_3,$$

where

$$z_1=\frac{y_1+x_{n_1}}{2}.$$

Then we have

$$\|Tx_{n_1}\| = \|x_{n_1}\| \text{ and } \rho_1(Tx_{n_1},Tx_1) < \rho_1(x_{n_1},x_i)$$

for $i=1,2,\dots,n_1-1$, since

$$\cos((n_1-i)\varphi_1)\ \cos(\varphi_1)\cos((n_1-i-1)\varphi_1)$$

and

$$\sigma(Tx_{n_1},Tx_i)=\frac{(1-\|Tx_{n_1}\|^2)(1-\|Tx_i\|^2)}{[1-\|x_{n_1}\|\,\|x_i\|\cos(\varphi_1)\cos(n_1-i-1)\varphi_1)]^2}>$$

$$>\frac{(1-\|x_{n_1}\|^2)(1-\|x_i\|^2)}{[1-\|x_{n_1}\|\,\|x_i\|\cos((n_1-1)\varphi_1)]^2}=\sigma(x_{n_1},x_i)$$

for $i=1,2,\dots,n_1-1$.

As usual we put

$$x_{n_1+1}=\frac{x_{n_1}+Tx_{n_1}}{2}.$$

The point x_{n_1+1} belongs to $P_2=\mathrm{lin}(X_{n_1+1},e_3)$ (and so will all points x_i, $i=n_1+2,\dots,n_2$, which we will construct next) and

$$\|x_{n_1+1}\|\geq\|z_1\|\geq\frac{1}{2}\,\frac{3}{4}=\frac{3}{8}.$$

Since the angle between halfplanes

$$\{\lambda x_{n_1+1}+\mu(Tx_{n_1}-x_{n_1}):\lambda\in R,\ \mu>0\}$$

and

$$Q_2=\{\lambda x_{n_1+1}+\mu e_3:\lambda\in R,\ \mu>0\}$$

is acute , the orthogonal projections of TX_{n_1} and x_{n_1} on P_2 show that there exists the angle $\psi_2>0$ such that for every $u\in Q_2$ with $\|u\|=\|x_{n_1+1}\|$ and $(u,x_{n_1+1})>\|x_{n_1+1}\|^2\cos(\psi_2)$ we have

$$\rho_1(u,Tx_{n_1})<\rho_1(x_{n_1+1},x_{n_1}).$$

Similarly, applying the orthogonal projection of

$$u'\in\{u\in\bar{Q}_2:(u,x_{n_1+1})\geq\|u\|\,\|x_{n_1+1}\|\cos\tfrac{\pi}{3},\ \tfrac{5}{16}=\frac{\frac{1}{4}+\frac{3}{8}}{2}\leq\|u\|\leq\tfrac{1}{2}\}$$

on $\mathrm{lin}(x_{n_1+1},Tx_{n_1}-x_{n_1})$ we get

$$\rho_1(u',Tx_{n_1})<\rho_1(u',x_{n_1}).$$

Taking $w_1=\dfrac{z_1+x_{n_1}}{2}$ we obtain

$$\rho_1(\lambda w_1, Tx_i) < \rho_1(\lambda w_1, x_i)$$

for $0 < \|\lambda w_1\| < 1$, $\lambda > 0$, $i=1,2,\dots,n_1-1$, since the angle between w_1 and Tx_i is less than the angle between w_1 and x_i. Hence for every $u \in \{\lambda w_1 + \mu e_3 : \lambda > 0, \mu \geq 0\}$ with $\|u\| < 1$ we have

$$\rho_1(u, Tx_i) < \rho_1(u, x_i)$$

$(i=1,2,\dots,n_1-1)$ and therefore the number

$$\varepsilon_2 = \min\{\rho_1(u', x_i) - \rho_1(u', Tx_u) : u' \in R_2,\ 1 \leq i \leq n_1-1\}$$

where

$$R_2 = \{u = \lambda w_1 + \mu e_3 : \mu \geq 0,\ \tfrac{5}{16} \leq \|u\| \leq \tfrac{1}{2},\ (u, w_1) \geq \|u\|\,\|w_1\| \cos \tfrac{2}{5}\pi\},$$

is positive. Now it is clear that we can find n_2 and φ_2 which satisfy

$$n_2 - n_1 > 10,$$

$$\varphi_2 = \frac{\pi}{3(n_2-n_1-1)} < \frac{\psi_2}{2},$$

$$\frac{3}{8}(\cos\varphi_2)^{n_2-n_1} > \frac{\frac{1}{4}+\frac{3}{8}}{2} = \frac{5}{16},$$

$$\tanh^{-1}\left(1 - \frac{(\frac{3}{4})^2}{[1 - \frac{1}{4}\cos(2\varphi_2)]^2}\right) < \varepsilon_2.$$

By this way we can repeat the procedure used for constructing x_i, $i=1,\dots,n_1$ by starting with x_{n_1+1} and rotating always in the plane P_2 by a fixed angle $2\varphi_2$.

We must check whether T has been nonexpansive on its domain of definition until now, i.e. whether

$$\rho_1(Tx_i, Tx_j) \leq \rho_1(x_i, x_j)$$

for $1 \leq i, j \leq n_2$. For $n_1+1 \leq i, j < n_2$ we have it by the same reason as in the first case. Applying the orthogonal projection of Tx_{n_1} and x_{n_1} on P_2 we obtain

$$\rho_1(Tx_{n_1}, Tx_i) < \rho_1(x_{n_1}, x_i)$$

for $n+2 \leq i < n_2$. By the choice of ψ_2 and φ_2 we also have

$$\rho_1(Tx_{n_1}, Tx_{n_1+1}) < \rho_1(x_{n_1}, x_{n_1+1}).$$

For $1 \leq i \leq n_1-1$ and $n_1+1 \leq j < n_2$ we get

$$\rho_1(Tx_i,Tx_j) \leq \rho_1(Tx_i,x_j)+\rho_1(x_j,Tx_j) < \rho_1(Tx_i,x_j)+\varepsilon_2 \leq$$
$$\leq \rho_1(x_i,x_j)-\varepsilon_2+\varepsilon_2=\rho_1(x_i,x_j).$$

All other cases were considered earlier.

Now it is clear how to continue the inductive definition of $\{x_i\}$ and $\{Tx_i\}$. The sequence $\{x_i\}$ is ρ_1-bounded by $\tanh^{-1}\frac{1}{2}$ and also ρ_1-bounded from below by $\tanh^{-1}\frac{1}{4}$. The sequence does not converge strongly, however, $\{x_i\}$ tends weakly to 0.

Next we use the extension property of $(B_{\mathbf{R}},\rho_1)$ and we obtain a nonexpansive mapping $T:B_{\mathbf{R}}\to B_{\mathbf{R}}$. It is easy to see that $S^i_{1,1/2}(x_1)$ and $S^i_{2,1/2}(x_1)$ tend weakly to 0 only. Since we have a nonexpansive retraction of B on $B_{\mathbf{R}}$ the analogous example can be constructed in (B,ρ_1). In this example T is not holomorphic.

Now we consider $B^n_{\mathbf{R}}$ $(n\geq 2)$ furnished with the following metric which is also called hyperbolic ([1]):

$$\rho_n((x_1,\dots,x_n),(y_1,\dots,y_n))=\max_{1\leq k\leq n}\rho_1(x_k,y_k)$$

for $(x_1,\dots,x_n).(y_1,\dots,y_n)\in B^n_{\mathbf{R}}$.

For n=2 and $H_{\mathbf{R}}=\mathbf{R}^2$ we have the example which shows that the Theorem 3 is false in this case.

Example 2. If $a_1=(0,0,0,0)$, $a_2=(\mu,0,0,0)$, $a_3=(0,0,\mu,0)$, $b_1=a_1$, $b_2=a_2$,

$$b_3=\left(\frac{1-(1-\mu^2)^{1/2}}{\mu},\ \frac{[\mu^4-(1-(1-\mu^2)^{1/2})^2]^{1/2}}{\mu},0,0\right).$$

$r=\frac{1}{2}\tanh^{-1}\mu$ and $0<\mu<1$, then $\rho_2(a_i,a_j)=\rho_2(b_i,b_j)$ for i,j=1,2,3,
$\bigcap_{i=1}^{3}B(a_i,r)\neq\emptyset$, but $\bigcap_{i=1}^{3}B(b_i,r)=\emptyset$.

The case $H_{\mathbf{R}}=\mathbf{R}$ and $B^n_{\mathbf{R}}=(-1,1)^n$ is different from the above one.

Lemma 1. Let $x_1,\dots,x_m$ be real numbers from $(-1,1)\subset\mathbf{R}$. If $r_1,\dots,r_m$ are positive numbers and $\rho_1(x_i,x_j)\leq r_i+r_j$ for $i,j=1,2,\dots,m$, then

$$\bigcap_{i=1}^{m}B(x_i,r_i)\neq\emptyset.$$

Proof: Let us notice that for any pair (i,j), $1\leq i<j\leq n$ we have $B(a_i,r_i)\cap B(a_j,r_j)\neq\emptyset$. Now it is sufficient to apply the Helly's Theorem ([6]).

Theorem 5. If $H_R=R$ and $B_R=(-1,1)\subset H_R$ then every nonexpansive mapping $T:X\longrightarrow B_R^n$ has a nonexpansive extenstion on the whole B_R^n.

Proof: It is sufficient to prove, by the Helly's Theorem, that for every points $x_1,\dots,x_{n+1},y_1,\dots,y_{n+1}\in B_R^n$ and positive numbers $r_1,\dots,r_{n+1}$ with $\rho_n(y_i,y_j)\le\rho_n(x_i,x_j)$ $(i,j=1,2,\dots,n+1)$ and $\bigcap_{i=1}^{n+1} B(x_i,r_i)\neq\emptyset$ we also have $\bigcap_{i=1}^{n+1} B(y_i,r_i)\neq\emptyset$.

But then for every $k=1,2,\dots,n$ we obtain $\rho_1(y_{ki},y_{kj})\le r_i+r_j$ $(1\le i,j\le\le n+1)$ where $y_i=(y_{1i},\dots,y_{ni})$ and we apply the Lemma 1.

References

[1] T. FRANZONI and E. VESENTINI: Holomorphic maps and invariant distances, North-Holland, Amsterdam, 1980.

[2] A. GENEL and J. LINDENSTRAUSS: An example concerning fixed points, Israel J. Math. 22(1975), 81-85.

[3] K. GOEBEL, T.SĘKOWSKI and A. STACHURA: Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball, Nonlinear Analysis 4(1980), 1011-1021.

[4] K. GOEBEL and W.A. KIRK: Iteration processes for nonexpansive mappings, Contemporary Mathematics 21(1983), 115-123.

[5] T.L. HAYDEN and T.J. SUFFRIDGE: Biholomorphic maps in Hilbert space have a fixed point, Pacif. J. Math. 38(1971), 419-422.

[6] E. HELLY: Über Mengen konvexer Körper mit gemeinschaftlichen Pubkten, Jber. Deutsch. Math. Verein 32(1923), 175-176.

[7] S. KOBAYASHI: Invariant distances for projective structures, Istituto Nazionale di Alta Matematica Francesco Severi, XXVI (1982), 153-161.

[8] T. KUCZUMOW: Fixed points of holomorphic mappings in the Hilbert ball, Colloq. Math., in print.

[9] T. KUCZUMOW and A. STACHURA: Extensions of nonexpansive mappings in the Hilbert ball with the hyperbolic metric. Part I, Comment. Math. Univ. Carolinae 29(1988), 399-402.

[10] S. REICH: Averaged mappings in the Hilbert ball, J. Math. Anal. Appl. 109(1985), 199-206.

[11] I.J. SCHOENBERG: On a theorem of Kirszbraun and Valentine, Amer. Math. Monthly 60(1953), 620-622.

[12] T.J. SUFFRIDGE: Common fixed points of commuting holomorphic mappings, The Michigan Math. J. 21(1975), 309-314.

Instytut Matematyki UMCS, Pl.Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland

(Oblatum 8.2. 1988)