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REMARKS ON THE STRUCTURE OF tt-DEGREES BASED ON CONSTRUCTIVE MEASURE THEORY

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Abstract: Based on some results in constructive measure theory, classes of sets of natural numbers being of some interest from the point of view of both constructive mathematics and recursion theory are introduced and possibility of mutual tt-reducibility of their members is studied and, moreover, an arithmetization of the Lebesgue measurability of sets of reals is proposed.

Key words: Recursion theory, constructive mathematics, measure theory, tt-reducibility, T-reducibility, constructive function of a real variable, Lebesgue measurability, B-measurability.

Classification: 03D30, 03F65

In [22] we showed that tt-reducibility of sets of natural numbers (NNs) can be studied (thanks to the well-known correspondence between sets of NNs and reals from [0,1]) with the help of $\emptyset$-uniformly continuous constructive functions of a real variable. In [7],[9],[10] and [12] constructive theory of the Lebesgue integral and Lebesgue measurability was created (for a summary and bibliography see [11]). Later, this theory was relativized and the results were used in the study of properties of Dini derivatives of constructive functions of a real variable. Some classes of reals interesting in this connection were introduced in [16]. Classes of sets of NNs corresponding to them turned out to be of some interest from the point of view of recursion theory, too. Some results on one of these classes (the class of NAP-sets studied by Kučera and Demuth) and the corresponding bibliography can be found in [20] and [21]. Other classes of such kind are introduced here. We use constructive measure theory to get a few results on mutual tt-reducibility and T-reducibility of members of these classes.

The Lebesgue measure on the class of all sets of NNs introduced by Sacks [6] is called the classical measure here. We introduce a hierarchy of relativizations of the constructive Lebesgue measure being equivalent to the classical measure. We already introduced the notion "a class of sets of NNs of B-measure zero" for any set B of NNs in [22].

We use the notation and terminology of [22]. In particular, the symbols U and V are variables for words in the alphabet Ξ ; s, t, u, v, w, x, y and z (also with subscripts) are variables for natural numbers (NNs), i and j for integers, a, b and c for rational numbers (RtNs), ρ , σ and τ for strings (of 0's and 1's), A, B and C for sets of NNs, X and Y for reals and, finally, S and T for $\emptyset$-constructive real numbers ($\emptyset$-CRNs). The set of all words in Ξ being NNs (or, as the case may be, RtNs, binary rational numbers, strings or $\emptyset^{(x)}$-CRNs) is denoted by N (or Q, Q^b, St, $D^{[x]}$, respectively). We introduce several new notions. Let $\mathfrak{D}_x$ denote the finite set with (canonical) index x [5, p. 70]. Note that $y \in \mathfrak{D}_x \Rightarrow y < x$ holds for any NNs x and y. For any set E of NNs (or, of strings) let Card(E) be its cardinality.

Let k be an NN, $2 \leq k$. $\lambda x_1 x_2 \ldots x_k \langle x_1, x_2, \ldots, x_k \rangle_k$ denotes a primitive recursive one-to-one mapping of the set of all k-tuples of NNs onto N introduced in [5], p. 64, and $\pi^k_1, \pi^k_2, \ldots, \pi^k_k$ primitive recursive functions of one variable such that $\langle \pi^k_1(z), \pi^k_2(z), \ldots, \pi^k_k(z) \rangle_k = z$ and $\pi^k_i(z) \leq z$ hold for any NNs z and i, $1 \leq i \leq k$. We shall write $\langle \ldots \rangle$ instead of $\langle \ldots \rangle_k$ wherever possible. We denote by φ_x the partial recursive function of one variable with index x, by W_x its domain and by W^s_x the finite subset of W_x enumerated after s steps. Analogously, for any set A of NNs, φ^A_x denotes the partial A-recursive function of one variable with A-index x and W^A_x the domain of φ^A_x. The notation φ^τ_x has the usual meaning (see, e.g., [20]). $W^{A,s}_x$ is defined in [22]. We denote $\varphi_x(\langle y_1, y_2, \ldots, y_k \rangle_k)$ by $\varphi_x(y_1, y_2, \ldots, y_k)$ and, analogously, $\varphi^A_x(\langle y_1, y_2, \ldots, y_k \rangle_k)$ by $\varphi^A_x(y_1, y_2, \ldots, y_k)$ for any set A of NNs. For any NNs m and n, $1 \leq m, n$, s^m_n denotes a recursive function of m+1 variables being an "s-m-n-function" for our indexing of partial recursive functions and for that of relativized partial recursive functions, i.e. fulfilling the conditional equality

$$\varphi^A_x(v_1, v_2, \ldots, v_m, y_1, y_2, \ldots, y_n) \simeq \varphi^A_{s^m_n(x, v_1, \ldots, v_m)}(y_1, y_2, \ldots, y_n)$$

and the analogical equality without superscripts A at φ for any NNs $x, v_1, \ldots \ldots, v_m, y_1, \ldots, y_n$ and any set A of NNs. Note that $\simeq$ means: both sides are defined and equal, or both are undefined.

For any sets or classes of sets E_1 and E_2 $E_1 \Delta E_2$ denotes the symmetric difference of them.

Note that $\mu(\mathfrak{L})$ denotes classical measure of $\mathfrak{L}$ for any classically measurable class $\mathfrak{L}$ of sets of NNs. There are an $\emptyset$-algorithm μ_0 of the type

($N \longrightarrow Q^b$), a recursive function sd ("symmetric difference") of two variables and an NN en ("enumeration") such that $\mu_0(x) = \mu(\langle \mathcal{D}_x \rangle^E), \langle \mathcal{D}_{sd(x,y)} \rangle$ is a set of mutually incomparable (with respect to $\subseteq$) strings,

$$A \in \langle \mathcal{D}_{sd(x,x)} \rangle^E \Longleftrightarrow A \in (\langle \mathcal{D}_x \rangle^E \Delta \langle \mathcal{D}_y \rangle^E),\ !\varphi^B_{en}(x,y) \text{ and } \mathcal{D}_{\varphi^B_{en}(x,y)} = W^{B,y}_x$$

hold for any NNs x and y, any bi-infinite set A and any set B of NNs.

Definition 1. Let A be a set of NNs. A real X is said to be

a) A-computable if it is A-recursive (i.e. $Set(X) \leq_T A$ holds);

b) weakly A-computable if there is a (fundamental) A-sequence of RtNs converging to X;

c) monotonically weakly A-computable if there is a monotone A-sequence of RtNs converging to X.

Remark 2. Let A be a set of NNs.

(i) A real is A-computable if and only if there is a canonically fundamental (or, an A-fundamental) A-sequence of $\emptyset$-CRNs (in particular, of RtNs) converging to it.

(ii) According to the relativized Limit Lemma [4], a real is weakly A-computable if and only if it is A'-computable (i.e. A'-recursive). Any A-computable real is, naturally, monotonically weakly A-computable. Moreover, a real X is A-computable if and only if there exist a non-decreasing A-sequence of $\emptyset$-CRNs and a non-increasing A-sequence of $\emptyset$-CRNs both converging to X.

(iii) We can easily construct a recursive function nds such that, for any set B of NNs, $[\varphi^B_{nds(x)}]$ is a bounded non-decreasing B-sequence of RtNs for any NN x and, in addition, a real X is monotonically weakly B-computable if and only if there is an NN y such that the B-sequence $[\varphi^B_{nds(y)}]$ converges either to X or to (- X). Thus, the class $\{C : r_C$ is a sum of a finite number of monotonically weakly B-computable reals$\}$ is, obviously, of B'-measure zero.

Sacks noted in [6] that, for any set M of reals from [0,1] and any real X, M is Lebesgue measurable and X is measure of M if and only if the class $\{A : r_A \in M\}$ of sets of NNs is classically measurable and X is its measure. We use this fact in the following definition.

Definition 3. Let B be a set of NNs.

1) A class $\mathfrak{M}$ of sets of NNs is said to be (Lebesgue) B-measurable and a real X is called B-measure of $\mathfrak{M}$ if there are a B-recursive function g and a class $\mathfrak{L}$ of sets of NNs of B-measure zero such that $\mathfrak{M}$ is B-measurable by g

and $\mathcal{E}$, i.e. $\forall xy(\mu_0(\underline{sd}(g(x),g(x+y)))\leq 2^{-x})$ and $\mathfrak{M}\,\Delta(\bigcup_{v=0}^{+\infty}\bigcap_{w=v}^{+\infty}\langle\mathcal{D}_{g(w)}\rangle^{E})\subseteq\mathcal{E}$ hold, and the canonically fundamental B-sequence $\{\mu_0(g(x))\}_x^B$ of RtNs converges to X.

2) A set M of reals from $[0,1]$ is said to be (Lebesgue) B-measurable and a real X is called B-measure of M if the class $\{A:r_A\in M\}$ is B-measurable and X is its B-measure.

Remark 4. Let B be a set of NNs. It is easy to show the following.

1) Any B-measurable class $\mathfrak{M}$ of sets of NNs is, naturally, classically measurable and $\mu(\mathfrak{M})$ is a B-computable real being a B-measure of $\mathfrak{M}$. Relativizing (to B) [12] we get results on B-measurability.

2) For any NN v, the bounded monotone B-sequence $\{\mu_0(\varphi_{en}^B(v,y))\}_y^B$ of RtNs converges (and, thus, B′-converges) to classical measure of the (evidently classically measurable) class $\langle W_v^B\rangle^E$ of sets of NNs. Consequently, the real $\mu(\langle W_v^B\rangle^E)$ is at least B′-computable and the class $\langle W_v^B\rangle^E$ is B′-measurable. It is B-measurable if and only if its classical measure is a B-computable real. These results hold uniformly (effectively) in B and v and in B, v and a B-index of a canonically fundamental B-sequence of Ø-CRNs (in particular, of RtNs) converging to $\mu(\langle W_v^B\rangle^E)$, respectively.

On the basis of relativized Specker′s example we can construct an NN v_0 such that $\mu(\langle W_{v_0}^B\rangle^E)=r_{B'}$ and, thus, $\langle W_{v_0}^B\rangle^E$ is not B-measurable.

3) Let $\mathfrak{M}$ be a classically measurable class of sets of NNs. According to well-known results on Lebesgue measurability of sets of reals and on sets of reals of the type G_δ (see, e.g., [1, p. 66]) there is a set C of NNs fulfilling $\mu(\langle(C)_{2x}\rangle^E)\leq 2^{-x}$, $\mathfrak{M}\subseteq\langle(C)_{2x+1}\rangle^E\subseteq\mathfrak{M}\cup\langle(C)_{2x}\rangle^E$ and, consequently, $\mu(\mathfrak{M})\leq\mu(\langle(C)_{2x+1}\rangle^E)\leq\mu(\mathfrak{M})+2^{-x}$ for any NN x, where, for each NN z, $(C)_z \rightleftharpoons \{y:\langle z,y\rangle\in C\}$. Using 2), we can easily show that $\mathfrak{M}$ is C′-measurable.

In Remark 4 we proved the following statement.

Theorem 5. The class $\mathfrak{M}$ of sets of NNs is classically measurable (i.e. Lebesgue measurable) and a real X is its classical measure if and only if there is a set C of NNs such that $\mathfrak{M}$ is (Lebesgue) C-measurable and X is its C-measure.

Relativizing results from [10] and [12] we get the following.

Theorem 6. Let B be a set of NNs and $\mathfrak{M}$ a B-measurable class of sets of NNs. Then there are two recursive functions g_0 and g_1 of one variable and two B-recursive functions $\bar{p}$ of one variable and $\bar{k}$ of two variables such that, for any NN v,

(a) $[\![\varphi^B_{g_1(v)}]\!]$ is a canonically fundamental B-sequence of RtNs converging to $\mu(\langle W^B_{g_0(v)}\rangle^E)$ which is less than 2^{-v};

(b) for any set A of NNs fulfilling $A \notin \langle W^B_{g_0(v)}\rangle^E$, $A \in \mathfrak{M} \Longleftrightarrow$ $\Longleftrightarrow A \in \langle \mathfrak{D}_{\bar{p}(v)}\rangle^E$ holds and, moreover, A is a point of density for $\mathfrak{M}$, if $A \in \mathfrak{M}$ holds, or a point of dispersion for $\mathfrak{M}$, if $A \notin \mathfrak{M}$, and, in both cases, the B-recursive function $\lambda x(\bar{k}(v,x))$ is a corresponding modulus.

On the basis of this theorem we can get a strengthening of [22, Theorem 4].

Theorem 7. For any string τ, any sets B and C of NNs and any B-measurable class $\mathfrak{M}$ of sets of NNs fulfilling $\mu(\mathfrak{M}) < \mu(\{\tau\}^E)$ there is a set A of NNs such that $A \in (\{\tau\}^E \setminus \mathfrak{M})$, $A \leq_T (B \oplus C)$ and $C \leq_T (B \oplus A)$ hold (in fact, we have $C \equiv_{B\text{-}tt} A$).

Remark 8. Let B be a set of NNs. According to Remark 2 and Theorem 7 the class $\{A : A \leq_T B\}$ of sets of NNs is of B'-measure zero, but any B-measurable class containing it has necessarily B-measure 1. Consequently, by Remark 4, $\{A : A \leq_T B\}$ cannot be B-measurable.

If B is non-recursive then the class $\{C : B \leq_T C\}$ is of $(B \oplus \emptyset')$-measure zero [22] and, consequently, by Remark 4 we get Sacks' result [6]: the class of all sets of NNs T-comparable with B is of classical measure zero.

Thus, any class of sets of NNs of classical measure zero can contain all sets T-comparable with a given non-recursive set, but if we have, in addition, information that the given class is of C-measure zero we can construct C-recursive sets which are not its elements.

We introduce some notation. Let $\mathrm{Tot} = \{v : \forall x(!\varphi_v(x))\}$ and let Lim be a partial $\emptyset'$-recursive function such that $\mathrm{Lim}(v) \simeq \lim_{x \to +\infty} \varphi_v(x)$ holds for any $v \in \mathrm{Tot}$. Note that $v \in \mathrm{Tot} \Rightarrow s^m_1(v, y_1, y_2, \dots, y_m) \in \mathrm{Tot}$ is valid for any NNs $m \geq 1$, $v, y_1, \dots, y_m$.

Let v be an NN such that

(1) $v \in \mathrm{Tot} \,\&\, \forall x(!\mathrm{Lim}(s^1_1(v,x)))$

and

(2) $\forall x(\mu(\langle W_{Lim(s_1^1(v,x))}\rangle^E) \leq 2^{-x})$

hold. Then, according to Remarks 2 and 4, for any NN x, the classes

$\langle W_{Lim(s_1^1(v,x))}\rangle^E$ and $\bigcup_{t=x}^{+\infty}\langle W_{Lim(s_1^1(v,t))}\rangle^E$ are $\emptyset'$-measurable and their $\emptyset'$-measures are not greater than 2^{-x} and 2^{-x+1}, respectively, and the class $\mathcal{T}_v$, where $\mathcal{T}_v \rightleftharpoons \bigcap_{w=0}^{+\infty}\bigcup_{t=w}^{+\infty}\langle W_{Lim(s_1^1(v,t))}\rangle^E$, is of $\emptyset'$-measure zero (cf. [16]).

Let, for any NNs v and w, $\hat{S}_0(v)$ denote the conjunction of (1) and (2) and let $\hat{S}(w,v)$ denote: $w,v \in Tot$,

$\forall x(Card(\{y: \varphi_v(x,y) \neq \varphi_v(x,y+1)\}) \leq \varphi_w(x))$ and (2) hold. Notice that $\hat{S}(w,v) \Rightarrow \hat{S}_0(v)$ is valid.

Definition 9. Let z be an NN. A set A of NNs is called

(a) an AP-set if there is a recursive function f such that $A \in \langle W_{f(x)}\rangle^E$ and $\mu(\langle W_{f(x)}\rangle^E) \leq 2^{-x}$ hold for any NN x (the term "effectively approximable by Σ_1^0 classes in measure" was introduced by Kučera in [2]);

(b) an NAP-set if it is not an AP-set;

(c) a z-WAP-set (z-weakly approximable ...) if there is an NN v such that $\hat{S}(z,v) \& A \in \mathcal{T}_v$;

(d) a WAP-set if it is a y-WAP-set for some NN y;

(e) an NWAP-set if it is not a WAP-set.

Let us notice that classes of arithmetical reals corresponding to these types of sets were introduced in [16]. Importance of these concepts for theory of differentiability of constructive functions of a real variable was demonstrated in [14],[15],[18] and [19]. Here, being in a situation quite analogical to that described in [16], we shall remember a few results and introduce some notation used already in [16] and [17].

Remark 10. 1) In [20, p. 74] and [21, p. 92] a characterization of the recursive function e is given. The Π_2^0 class $\bigcap_{x=0}^{+\infty}\langle W_{e(x)}\rangle^E$ (of $\emptyset'$-measure zero) is just the class of all AP-sets. For any NN x, $\langle W_{e(x)}\rangle$ is a proper covering and $\mu(\langle W_{e(x)}\rangle^E) < 2^{-x}$ holds. By definition, any class of $\emptyset$-measure zero contains AP-sets only.

2) There are NNs ψ and η such that $\hat{S}(\psi,\eta)$ holds and, for any set A

of NNs, A fulfils the formula (1) from [22, Theorem 2] if and only if A $\notin \mathcal{T}_{\psi}$ holds (we use notation from [17, Theorem 6] here).

3) a) There are recursive functions $\hat{\lambda}_0$ and $\hat{\lambda}_1$ such that for any NN $z \in$ Tot we have $\hat{S}(\hat{\lambda}_0(z), \hat{\lambda}_1(z))$ and any z-WAP-set is contained in the class $\mathcal{T}_{\hat{\lambda}_1(z)}$ of $\emptyset'$-measure zero [16, Remark 8 and p. 460].

b) There is an NN u such that $\hat{S}_0(u)$ is valid and any WAP-set is contained in the class $\mathcal{T}_u$ of $\emptyset'$-measure zero [16, Remark 8]. Thus, $\emptyset'$-almost any set of NNs is an NWAP-set. According to [16] and [17] the class of all WAP-sets is both a $\Sigma_3^{0,\emptyset'}$ class and a Σ_4^0 class being neither a Π_4^0 nor a $\Pi_3^{0,\emptyset^{(x)}}$ class for any NN x. This class and its complement are everywhere dense. According to 2) and to Theorem 2 from [22] $A' \equiv_T (A \oplus \emptyset')$ holds for any NWAP-set A.

Theorem 11. Let f be a recursive function fulfilling $\varphi_{f(v)}(x) \simeq \varphi_v(2x)$ for any NNs v and x. Let z be an NN and A and B sets of NNs such that A is both an NAP-set and a z-WAP-set and $A \leq_T B$ holds. Then B is an f(z)-WAP-set.

Proof. It is sufficient to use Theorem 18 from [21].

Theorem 12. 1) For any index y of the recursive function $\lambda x(2^{x+1})$, any set A fulfilling $\emptyset' \leq_T A$ is a y-WAP-set.

2) For any set A of NNs fulfilling $\emptyset' \leq_T A$ there are an NAP-set B and an NWAP-set C such that $C' \equiv_T B \equiv_T A$.

Proof. a) There is an NAP-set L from an r.e. tt-degree for which r_L is a monotonically weakly $\emptyset$-computable real [20, Remark 20]. L is, obviously, a v-WAP-set for any index v of the recursive function $\lambda x(2^x)$. By Theorem 11, part 1) of our theorem is valid.

b) Let A be a set of NNs, $\emptyset' \leq_T A$. By Kučera [2, Theorem 7], there is an NAP-set B such that $B \equiv_T A$. To the class $\mathcal{T}_u$ (from the part 3b) of Remark 10), the empty string and the set A we apply [22, Theorem 4]. We get an NWAP-set C fulfilling $C' \equiv_T (C \oplus \emptyset')$ (Remark 10), $C \leq_T (A \oplus \emptyset')$, $A \leq_T (C \oplus \emptyset')$ and, consequently, $C' \equiv_T A$.

In the sequel, we shall use constructive concepts and notation introduced in [22] frequently. The following statement gives us some information about connections between $\emptyset$-ucf-reducibility and (much stronger) mf-reducibility.

Theorem 13. Let F be an $\emptyset$-uniformly continuous c-function and let, for any real Y,

(3) . $J_Y^{<}$, $J_Y^{\leq}$, $J_Y^{=}$, $J_Y^{\geq}$, $J_Y^{>}$

be classes of sets of NNs, where, for any sign λ from the list $<$, $\leq$, $=$, $\geq$, $>$, $J_Y^{\lambda} \rightleftharpoons \{A:R[F](r_A) \lambda Y\}$.

1) Let Y be a real. The classes $J_Y^{\leq}$ and $J_Y^{>}$ are of the type $\langle W_t^{\mathrm{Set}(Y)}\rangle^E$ and, consequently, (Set(Y))$'$-measurable. Thus, any of the classes (3) is (Set(Y))$'$-measurable and $\mu(J_Y^{=}) = \mu(J_Y^{\leq}) - \mu(J_Y^{<})$ holds. If $\mu(J_Y^{<}) = \mu(J_Y^{\leq})$ is valid then the classes (3) are even Set(Y)-measurable and the class $J_Y^{=}$ is of Set(Y)-measure zero.

2) There are Ø-CRNs U_0 and U_1 being respectively the infimum and the supremum of the values reached by F on Ø-CRNs from $0\Delta 1$ and a non-decreasing (and, thus, Ø-uniformly continuous) c-function G for which G(0)=0, G(1)=1 and $\forall XZ(0 \leq X \leq 1 \,\&\, U_0 \leq Z \leq U_1 \Longrightarrow (R[G](X)=Z \Longleftrightarrow \mu(J_Z^{<}) \leq X \leq \mu(J_Z^{\leq})))$ hold. Thus, for any real Y from $U_0 \Delta U_1$ and any set B of NNs, we have

(Set(Y) $\leq_{\emptyset\text{-ucf}}$ B via F) $\Longleftrightarrow$ B $\in J_Y^{=}$, (Set(Y) $\leq_{mf}$ B via G) $\Longleftrightarrow$ $\mu(J_Y^{<}) \leq r_B \leq \mu(J_Y^{\leq})$

and, consequently, if $\mu(J_Y^{=}) > 0$ holds then Y is an Ø-computable real and there is a rational segment $a\Delta b$ such that $\forall X(X \in a\Delta b \Longrightarrow R[G](X)=Y)$.

3) Let Y be a real, $Y \in U_0 \Delta U_1$.

a) Let Y be Ø-computable. Then $J_Y^{=}$ is a non-empty Π_1^0 class.

If $\mu(J_Y^{=})=0$ holds then the class $J_Y^{=}$ is of Ø-measure zero and, consequently, it contains AP-sets only.

If $\mu(J_Y^{=}) > 0$ then the class $J_Y^{=}$ contains Ø$'$-recursive NWAP-sets and sets from r.e. tt-degrees being both NAP-sets and WAP-sets.

b) Let Y be not Ø-computable. Then $\mu(J_Y^{=})=0$ holds, the set A, where $A \rightleftharpoons \mathrm{Set}(\mu(J_Y^{<}))$, is non-recursive, Set(Y) $\equiv_T$ A,

(A is an AP-set) $\Longleftrightarrow$ ($J_Y^{=}$ contains AP-sets only),

$\forall z$((A is a z-WAP-set) $\Longrightarrow$ ($J_Y^{=}$ contains z-WAP-sets only)) and

(Set(Y) is weakly 1-generic) $\Longrightarrow$ ($J_Y^{=}$ is of Ø-measure zero).

The following remark will help us to prove this theorem.

Remark 14. 1) Let comp be a recursive function of two variables such that, for any NNs t and s, $\langle \mathfrak{D}_{\mathrm{comp}(t,s)}\rangle$ is a set of mutually incomparable (with respect to $\subseteq$) strings which are incomparable with strings from $\langle W_t^s\rangle$ and, in addition, the class $\langle \mathfrak{D}_{\mathrm{comp}(t,s)}\rangle^E \cup \langle W_t^s\rangle^E$ contains any set of NNs.

2) Let H be an $\emptyset$-uniformly continuous c-function. There is a recursive function f such that $\forall xzab(f(x)\leq lh(\sigma_z)\&a,b\in Seg(\sigma_z)\Rightarrow |H(b)-H(a)|\leq$ $\leq 2^{-x})$ holds. We use [22, Remark 6] and the s-m-n-theorem and construct recursive functions $\bar{m}_H$, $\bar{n}_H$, $\bar{p}_H$ and $\bar{q}_H$ of one variable and a recursive predicate P_H of three variables such that, for any NNs t, s and z, we have the following:

a) $W_{\bar{m}_H(t)}=\{x:R[H](Seg(\sigma_x))\subseteq \mathcal{S}^0(t)\}$, $W_{\bar{n}_H(t)}=\bigcup_{y\in W_t} W_{\bar{m}_H(y)}$,

$W_{\bar{p}_H(t)}=\{x:\exists y(y\in W_{\overline{in}(t)}\&H(E_l(\mathcal{S}(y)))<E_l(\mathcal{S}(x))<E_r(\mathcal{S}(x))<H(E_r(\mathcal{S}(y))))\}$, (for $\overline{in}$ see [22]), $W_{\bar{q}_H(t)}=\{x:\exists y\forall v(v\in \mathcal{D}_{comp(t,y)}\Rightarrow R[H](\sigma_v)\cap\mathcal{S}(x)=\emptyset)\}$ and, consequently,

(i) $A\in\langle W_{\bar{m}_H(t)}\rangle^E\Leftrightarrow R[H](r_A)\in\mathcal{S}^0(t)$ and $A\in\langle W_{n_H(t)}\rangle^E\Leftrightarrow$ $\Leftrightarrow R[H](r_A)\in[W_t]$ hold for any set A of NNs (for $[W_t]$ see [22]);

(ii) if H is non-decreasing then $\{B:R[H](r_B)=X\}\subseteq[W_{\overline{in}(t)}]\Leftrightarrow X\in[W_{\bar{p}_H(t)}]$ holds for any real $X\in H(0)\triangle H(1)$ and, thus, $\langle W_{\bar{n}_H(\bar{p}_H(t))}\rangle^E\subseteq\langle W_t\rangle^E$ is valid;

(iii) if $\langle W_t\rangle$ is a proper covering then, for any real X from $H(0)\triangle H(1)$, $\{B:R[H](r_B)=X\}\subseteq\langle W_t\rangle^E\Leftrightarrow X\in[W_{\bar{q}_H(t)}]$ holds and, thus, $\langle W_{\bar{n}_H(\bar{q}_H(t))}\rangle^E\subseteq\langle W_t\rangle^E$ is fulfilled;

b) P_H is used as a selector, namely, $P_H(t,s,z)$ implies

(4) $\quad\mathcal{S}(s)\subseteq\mathcal{S}^0(t)\&lh(\sigma_z)=f(x_0)$,

where

(5) $\quad x_0\rightleftharpoons\mu x(2^{-x}<\min(E_r(\mathcal{S}(t))-E_r(\mathcal{S}(s)),E_l(\mathcal{S}(s))-E_l(\mathcal{S}(t))))$;

if (4) and (5) hold, then we have: $P_H(t,s,z)\Rightarrow R[H](\sigma_z)\subseteq\mathcal{S}^0(t)$, $\neg P_H(t,s,z)\Rightarrow R[H](\sigma_z)\cap\mathcal{S}(s)=\emptyset$ and, consequently, $\langle W_{\bar{m}_H(s)}\rangle^E\subseteq\langle\{z:P_H(t,s,z)\}\rangle^E\subseteq\langle W_{\bar{m}_H(t)}\rangle^E$ is valid.

Proof of Theorem 13. Part 1) of the theorem follows immediately from Remarks 2, 4 and 14. According to [8] and [13, Lemma 1], there are $\emptyset$-CRNs U_0 and U_1, being respectively the infimum and the supremum of values reached by F on ($\emptyset$-CRNs from) $0\triangle 1$, and two $\emptyset$-sequences $\{V_y\}_y^{\emptyset}$ and $\{M_y\}_y^{\emptyset}$ of $\emptyset$-CRNs such that the first of them is everywhere dense and, for any NN x, the class $J^<_{V_x}$ is $\emptyset$-measurable and M_x is its $\emptyset$-measure. Because of monotonicity of measure and of $\emptyset$-uniform continuity of F we have $V_x \leq V_y \Rightarrow M_x \leq M_y$, $V_x \leq U_0 \Rightarrow M_x=0$, $U_1 < V_x \Rightarrow M_x=1$, $U_0 < V_x < V_y < U_1 \Rightarrow 0 < M_x < M_y < 1$ for any NNs x and y.

We construct an $\emptyset$-sequence $\{G_x\}_x^{\emptyset}$ of non-decreasing polygonal c-functions such that, for any NN x, there is an increasing finite sequence

$\{S_{x,i}\}_{i=0}^{k_x}$ of $\emptyset$-CRNs fulfilling $S_{x,0}=0$, $G_x(S_{x,0})=U_0$, $S_{x,k_x}=1$, $G_x(S_{x,k_x})=U_1$ and, for any NNs i and z, $\exists y(S_{x,i}=M_y \& G_x(S_{x,i})=G_{x+z}(S_{x,i})=V_y)$, if $0<i<k_x$ holds, G_x is linear on the segment $S_{x,i} \Delta S_{x,i+1}$ and $0 \leq G_x(S_{x,i+1})-G_x(S_{x,i}) < 2^{-x}$ is valid, if $0 \leq i < k_x$ holds. There is a non-decreasing (thus, $\emptyset$-uniformly continuous [22, Remark 6]) c-function G being a limit of the canonically uniformly fundamental $\emptyset$-sequence $\{G_x\}_x^{\emptyset}$ of c-functions (consequently, $\forall xi(0 \leq i \leq k_x \Rightarrow G(S_{x,i})=G_x(S_{x,i}))$). The described properties of G and monotonicity of measure imply 2).

According to 2) and Remark 14

(6) $\mu(\langle W_{\pi_F(x)}\rangle^E)=\mu(J^{<}_{E_r}(\mathcal{S}(x)))-\mu(J^{\leq}_{E_1}(\mathcal{S}(x)))=\mu(\langle W_{\pi_G(x)}\rangle^E)$

holds for any NN x and, thus,

(7) $\mu(\langle W_{\pi_F(t)}\rangle^E)=\mu(\langle W_{\pi_G(t)}\rangle^E)$,

(8) $\mu(\langle W_{\pi_F(\beta_G(t))}\rangle^E) \leq \mu(\langle W_t\rangle^E)$

and

(9) $Set(\mu(J_Y^{<})) \in \langle W_t\rangle^E \Rightarrow J_Y^{=} \subseteq \langle W_{\pi_F(\beta_G(t))}\rangle^E$

are valid for any NN t and any non-$\emptyset$-computable real Y from $U_0 \Delta U_1$.

Let Y be a real from $U_0 \Delta U_1$. By 1), $J_Y^{=}$ is a non-empty $\Pi_1^{0,Set(Y)}$class, $\mu(J_Y^{=})=1-\mu(\langle W_k^{Set(Y)}\rangle^E)$, where k is an NN such that $J_Y^{<} \cup J_Y^{>} = \langle W_k^{Set(Y)}\rangle^E$.

a) Let Y be $\emptyset$-computable.

If $\mu(J_Y^{=})=0$ then the Π_1^0 class $J_Y^{=}$ is of $\emptyset$-measure zero.

Let $\mu(J_Y^{=})>0$ hold. There are NNs i and j fulfilling $\mu(\langle W_k^{Set(Y)}\rangle^E) < 1-2^{-i+1}$ and $\langle W_j\rangle^E=\langle W_{e(i)}\rangle^E \cup \langle W_k^{Set(Y)}\rangle^E$. By Remark 10, $\langle W_j\rangle$ is a proper covering containing all AP-sets. We apply (i) Theorem 7 to the empty string, the $\emptyset'$-measurable class $\langle W_j\rangle^E \cup \mathcal{T}_{\mu}$ of $\emptyset'$-measure less than 1 and any $\emptyset'$-recursive set (see Remark 10) and (ii) Theorem 33 from [20] to $\langle W_j\rangle$ and any T-complete $\emptyset$-r.e. set (see Theorem 12). The proof of part 3a) is finished.

b) Let Y be non-$\emptyset$-computable and let $A \rightleftharpoons \mathrm{Set}(\mu(J_Y^{<}))$. Then $\emptyset <_T \mathrm{Set}(Y)$, $U_0 < U_1$, $R[G](r_A)=Y$ and, hence, by [22, Theorem 15], $\mathrm{Set}(Y) \equiv_T A$. On account of (8) and (9) we can limit ourselves to the following: (i) Let $J_Y^{=}$ contain AP-sets only. Then, by Remarks 10 and 14 and by (7), for any NN x, we have $J_Y^{=} \leq \langle W_{e(x)} \rangle^E$, $A \in \langle W_{\bar{n}_G(\bar{q}_F(e(x)))} \rangle^E$ and $\mu(\langle W_{\bar{n}_G(\bar{q}_F(e(x)))} \rangle^E) \leq \mu(\langle W_{e(x)} \rangle^E) < 2^{-x}$. Thus, by definition, A is an AP-set.

(ii) Let Set(Y) be weakly 1-generic [3]. We construct an $\emptyset$-sequence $\{T_x\}_x^{\emptyset}$ of $\emptyset$-CRNs contained and dense in $U_0 \triangledown U_1$ and such that no T_X is equal to a value of the c-function G in a rational point. For any NNs x and y, $\mu(J_{T_x}^{<}) = \mu(J_{T_x}^{\leq})$ and, hence, we can construct NNs $s_{x,y}$, $t_{x,y}$ and $v_{x,y}$ fulfilling $T_x \in \mathcal{S}^0(s_{x,y})$, $\mathcal{S}(s_{x,y}) \leq \mathcal{S}^0(t_{x,y})$, $\mu(\langle W_{\bar{m}_G(t_{x,y})} \rangle^E) < 2^{-x-y-1}$ and $\mathfrak{D}_{v_{x,y}} = \{z : P_F(t_{x,y}, s_{x,y}, z)\}$ (see Remark 14) and, consequently, by Remark 14 and (6), $\langle W_{\bar{m}_F(s_{x,y})} \rangle^E \leq \langle \mathfrak{D}_{v_{x,y}} \rangle^E$ and $\mu_0(v_{x,y}) < 2^{-x-y-1}$.

Thus, the open set $[\{z : \exists x(z=s_{x,y})\}]$ of reals is dense in $U_0 \triangledown U_1$, consequently, it contains Y (because of weak 1-genericity of Set(Y)), and the $\emptyset$-measurable set $\bigcup_{z=0}^{+\infty} \langle \mathfrak{D}_{v_{z,y}} \rangle^E$ of $\emptyset$-measure less than 2^{-y} contains $J_Y^{=}$ for any NN y.

Hence, $J_Y^{=}$ is of $\emptyset$-measure zero and the proof is finished.

Remark 15. 1) Let C be an SBI-set and let $\emptyset <_{tt} M \leq_{tt} C$. Then according to [22, Theorem 9] and Theorem 13 (parts 2 and 3b, where $Y \rightleftharpoons r_M$) there is a set A of NNs fulfilling $M \leq_{mf} A$ (and, thus, $M \leq_{tt} A$, if M is an SBI-set), $M \equiv_T A$, (A is an AP-set) $\Longrightarrow$ (C is an AP-set), $\forall z$((A is a z-WAP-set) $\Longrightarrow$ (C is a z-WAP-set)), (M is weakly 1-generic) $\Longrightarrow$ (C is contained in a class of $\emptyset$-measure zero (and, thus, C is an AP-set)).

2) According to [22] any NAP-set and any bi-infinite (in particular,non-recursive) set of the type $B \oplus B$ are SBI-sets.

Theorem 16. 1) No weakly 1-generic set is tt-reducible to an NAP-set.

2) If a non-recursive set B is tt-reducible to an NAP-set C then there is an NAP-set A such that $B \leq_{tt} A \leq_T B$ and $\forall z$((A is a z-WAP-set) $\Longrightarrow$ (C is a z-WAP-set)) hold.

Proof. It is sufficient to use Remark 15 and [22, Remark 8].

Definition 17. A property of sets of NNs is said to be valid for B-almost any set (or, equivalently, B-almost everywhere) if there is a class $\mathfrak{M}$

of sets of NNs of B-measure zero such that any set A of NNs fulfilling $A \notin \mathfrak{M}$ has the property.

Theorem 18. For $\emptyset'$-almost any set A of NNs we can construct an $(A \oplus \emptyset')$-recursive set B being both an NAP-set and a q-WAP-set, where q is any NN such that

(10) $$\varphi_q = \lambda x(3^{2x})$$

is valid, and fulfilling $A \leq_{\emptyset\text{-ucf}} B$ and $A \leq_{tt} B$.

Note that $\emptyset'$-almost any set of NNs is an NWAP-set (Remark 10).

Proof. Let M_x denote the set $\{z: 1 \leq z \leq 3^x\}$ and $L[x,y]$ the segment $(y-1).3^{-x} \Delta y.3^{-x}$ for any NNs x and y.

1) We construct a partial recursive function π of two variables and an $\emptyset$-uniformly continuous c-function F such that, for any NN x, (i) the set M_{2x} is the domain and M_x the range of the function $\lambda y\, \pi(x,y)$, $\pi(1,3.(v-1)+t) \simeq v$ and

$\pi(x+1,3^{2x}.(3.(v-1)+t-1)+y) \simeq (3^x.(v-2^{-1}.(1-(-1)^t))-(-1)^t.\pi(x,y)+2^{-1}.(1+(-1)^t))$

and, consequently, $\mathrm{Card}(\{z: \pi(x,z) \simeq w\}) = 3^x$ hold for any v and t from M_1, any $y \in M_{2x}$ and any $w \in M_x$;

(ii) $F(0)=0$, $F(1)=1$, $0 \leq F \leq 1$ hold and F maps the segment $L[2x,z]$ onto $L[x,\pi(x,z)]$ for any $z \in M_{2x}$.

For any NNs x and s, where $\mathfrak{D}_s \subseteq M_{2x}$, we denote by $I[x,s]$ the union $\bigcup_{w \in \mathfrak{D}_s} L[x,\pi(x,w)]$, i.e. the F-image of $\bigcup_{w \in \mathfrak{D}_s} L[2x,w]$.

2) For any NN k, let $\{a_{k,t} \Delta b_{k,t}\}_t^{\emptyset}$ be an $\emptyset$-sequence of mutually non-overlapping rational segments with binary rational end points enumerating the set $\{\mathrm{Seg}(\sigma_t): t \in W_{nol(e(2k+4))}\}$ (see [22] and Remark 10).

The predicate $v \leq w \,\&\, \sum_{t=v}^{w} |a_{u,t} \Delta b_{u,t} \cap L[2x,y]| \leq 3^{-z}.|L[2x,y]|$ of variables u,v,w,x,y, and z is denoted by P_0, the predicate $\forall s(v \leq s \Longrightarrow P_0(u,v,s, x,y,z))$ of variables u,v,x,y and z by P and the predicate $(\forall t)_{t \leq v} \exists ij(a_{u,t}=i.3^{-2x} \,\&\, b_{u,t}=j.3^{-2x})$ of variables u, v and x by R. Let us note that $P(k,0,0,1,k+2)$ holds for any NN k.

We construct a recursive function g_0 of six variables and an $\emptyset'$-recursive function g of five variables such that, for any NNs k,u,v,x,z and t, where $1 \leq v \leq 3^{2u} \,\&\, u \leq x$ holds, we have

$\mathfrak{D}_{g_0(k,u,v,x,z,t)} = \{y : y \in M_{2x} \& (\exists w)_{w \in M_x}(y = \mu s(\bar{n}(x,s) \simeq w \& L[2x,s] \subseteq L[2u,v] \&$
$\& P_0(k,0,t,x,s,z) \vee s = 3^{2x}+1))\}$ and $g(k,u,v,x,z) \simeq \lim_{s \to +\infty} g_0(k,u,v,x,z,s)$ and,

consequently, the following, where (for brevity) we replace "k,u,v,x,z" by $*$,

(i) $\mathrm{Card}(\mathfrak{D}_{g_0(*,t)}) \leq 3^{x-u}$, $\mathrm{Card}(\{s : g_0(*,s) \neq g_0(*,s+1)\}) \leq 3^{2.(x-u)}$ and

$t \in \mathfrak{D}_{g(*)} \Longleftrightarrow (\exists w)_{w \in M_x}(t \simeq \mu y(\bar{n}(x,y) \simeq w \& L[2x,y] \subseteq L[2u,v] \& P(k,0,x,y,z))$;

(ii) $\{L[x,\bar{n}(x,w)] : w \in \mathfrak{D}_{g_0(*,t)}\}$ is a set of non-overlapping segments

and, thus, $\mu(I[x,g_0(*,t)]) = 3^{-x}.\mathrm{Card}(\mathfrak{D}_{g_0(*,t)})$;

(iii) if p and r are NNs fulfilling $p \leq z \& P(k,0,u,v,p) \& P(k,r+1,u,v,2z) \&$
$\& R(k,r,x)$ then $\mathrm{Card}(\mathfrak{D}_{g(*)}) \geq 3^{x-u}.(1-3^{-p+1})$ and, hence, $\mu(I[x,g(*)]) \geq$
$\geq (1-3^{-p+1}).|L[u,\bar{n}(u,v)]|$ hold.

3) We suppose to have a fixed enumeration of all finite sequences of NNs such that any index of a finite sequence of NNs majorizes all members of the sequence. We shall construct a recursive function h of two variables. Let x be an NN. We shall distinguish two cases.

a) There are an NN m and two increasing finite sequences $\{n_i\}_{i=0}^{2m+2}$ and $\{x_j\}_{j=0}^{m}$ of NNs such that $x_m = x$ and, for any NN j, $0 \leq j \leq m$, the sequence $\{n_i\}_{i=0}^{2j+2}$ has index x_j and $R(n_0,n_{2j+1},n_{2j+2}) \& (j < m \Rightarrow x_j < n_{2j+4})$ holds. For any NN t, we construct a finite sequence $\{s_{t,j}\}_{j=0}^{m}$ of NNs fulfilling $s_{t,0} =$
$= g_0(n_0,0,1,x_0,n_0+3,t)$ and

$$\mathfrak{D}_{s_{t,j+1}} = \bigcup_{w \in \mathfrak{D}_{s_{t,j}}} \mathfrak{D}_{g_0(n_0,x_j,w,x_{j+1},n_0+j+4,t)}$$

for any NN j, $0 \leq j < m$, and we put $h(x,t) = s_{t,m}$.

b) In the other case we define $h(x,t)=0$ for any NN t ($\mathfrak{D}_0 = \emptyset$).

Thus, $\mathrm{Card}(\{t : h(x,t) \neq h(x,t+1)\}) \leq 3^{2x}$ holds for any NN x. Let
$H \rightleftharpoons \lambda x(\lim_{t \to +\infty} h(x,t))$. Then H is an $\emptyset'$-recursive function and according to
2) $\mu(\bigcup_{w \in \mathfrak{D}_{H(x)}} L[2x,w]) \leq 3^{-x}$ holds for any NN x. Hence, for any NN q and any set B of NNs such that (10) is valid and the set $\{x : r_B \in \bigcup_{w \in \mathfrak{D}_{H(x)}} L[2x,w]\}$ is infinite, B is, by definition, a q-WAP-set.

4) Let k be an NN. We construct increasing $\emptyset'$-sequences $\{n_{k,t}\}_t^{\emptyset'}$ and and $\{x_{k,t}\}_t^{\emptyset'}$ of NNs such that $n_{k,0}=k$ and $n_{k,1}=\mu w(k<w \,\&\, P(k,w+1,0,1,2(k+3)))$ is valid, and, for any NN s, $n_{k,2s+2}=\mu z(n_{k,2s+1}<z \,\&\, R(k,n_{k,2s+1},z))$ holds, the finite sequence $\{n_{k,t}\}_{t=0}^{2s+2}$ has index $x_{k,s}$ and $n_{k,2s+3}=\mu w(x_{k,s}<w \,\& \,\& \,\forall y(y\in \mathcal{D}_{H(x_{k,s})} \Rightarrow P(k,w+1,x_{k,s},y,2(k+s+4))))$ is valid. Then, according to 2) and 3), for any NN s,

$$(11) \qquad \forall y(y\in \mathcal{D}_{H(x_{k,s})} \Rightarrow P(k,0,x_{k,s},y,k+s+3))$$

holds, $I[x_{k,0},H(x_{k,0})]$ is contained in $0\Delta 1$ and its measure is at least $(1-3^{-k-1})$, $I[x_{k,s+1},H(x_{k,s+1})]\subseteq I[x_{k,s},H(x_{k,s})]$ and $\mu(I[x_{k,s+1},H(x_{k,s+1})])\geq(1-3^{-k-s-2})\cdot\mu(I[x_{k,s},H(x_{k,s})])$ hold. Consequently, the class

$$(12) \qquad \bigcap_{s=0}^{+\infty} I[x_{k,s},H(x_{k,s})]$$

of reals being of the type $0\Delta 1\setminus[W_t^{\emptyset'}]$ is contained in $0\Delta 1$, its measure is at least $(1-3^{-k})$ and $\emptyset'$-computable and, hence, (12) is $\emptyset'$-measurable.

5) Let k be an NN and A a non-recursive SBI-set such that r_A is in (12). Then, according to 2), for any NN s there is just one NN w_s fulfilling

$$(13) \qquad w_s\in \mathcal{D}_{H(x_{k,s})} \,\&\, r_A\in L[x_{k,s},\bar{n}(x_{k,s},w_s)]$$

and, thus, $L[2x_{k,s+1},w_{s+1}]\subseteq L[2x_{k,s},w_s]$ holds for any NN s. Moreover, $\lambda s(w_s)$ is, obviously, an $(A\oplus\emptyset')$-recursive function. Hence, there is an $(A\oplus\emptyset')$-recursive set B fulfilling $r_B\in L[2x_{k,s},w_s]$ for any NN s. This together with 3) and validity of (11) and (13) for any NN s gives us: B is a q-WAP-set, where (10) holds, $A\leq_{\emptyset\text{-ucf}}B$ by F and $B\notin\langle W_{e(2k+4)}\rangle^E$ hold. To finish the proof it is sufficient to use Remark 10, [22, Theorem 9] and to notice that $\emptyset'$-almost any set is a non-recursive SBI-set (Remark 8 and [22, Remark 8]).

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