Commentationes Mathematicae Universitatis Carolinae

Petronije S. Milojević

Solvability of semilinear equations with strong nonlinearities and applications to elliptic boundary value problems

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 4, 735--750

Persistent URL: http://dml.cz/dmlcz/106585

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,4 (1987)

SOLVABILITY OF SEMILINEAR EQUATIONS WITH STRONG NONLINEARITIES AND APPLICATIONS TO ELLIPTIC BOUNDARY VALUE PROBLEMS

P.S. MILOJEVIĆ

Abstract. Solvability of two classes of semilinear equations involving strongly nonlinear perturbations of type (M) with respect to two Banach spaces is established. An application to elliptic BV problems is also given.

Key words: Semilinear equations, noncoercive, nonlinear operators of type (M), strong nonlinearities, boundary value problems, elliptic equations.

AMS (MOS) Classification Numbers: 47H05,47H15,35J60

1. INTRODUCTION

Many problems in analysis reduce to solving operator equations of the form

$$\lambda Cx - Ax - Nx = f, \tag{1}$$

where f is a given element in a Hilbert space H, $\lambda \in R$, A is linear, C and N are nonlinear mappings. Motivated by applications to strongly nonlinear elliptic problems, we shall study Eq. (1) in the following setting.

(i) There is a pair $\{V, V^*\}$ of Banach spaces in duality with $V \subset H \subset V^*$, i.e., there is a nondegenerate continuous bilinear form $<,>$ on $V \times V^*$. (V^* need not be the dual of V in the usual sense.) Suppose that V is reflexive and compactly embedded in H, $| < x, y > | \leq ||x||_V ||y||_{V^*}$ on $V \times V^*$ and the duality $<,>$ is compatible with the inner product $(\, , \,)$, i.e., $< x, y >= (x, y)$ for $(x, y) \in V \times H$.

(ii) Let $\{U, U^*\}$ be another pair of Banach spaces in duality compatible with $(,)$ such that U is separable, $U \subset V$ and $V^* \subset U^*$ and the injections are continuous and dense.

(iii) $A:V \to V^*$ is a continuous "variational extension" of a closed linear mapping $A_1:D(A_1) \subset H \to H$ such that $U \subset D(A_1) \subset V$ and $< Ax, y >= (A_1x, y)$ for $x \in D(A_1)$ and $y \in V$. Moreover, let C, $N:D(N) \subset V \to U^*$ be such that $N - C$ is of type (M) relative to (U, V) with $U \subset D(N)$ and $(N - C)(U) \subset H$ (see Definition 1 below).

Under some additional conditions, we shall prove that Eq. (1) is solvable for each $\lambda \in R$ and each $f \in H$. If a is the quasinorm of C (i.e., $a = \limsup_{\|x\| \to \infty} \|Cx\|/\|x\|$) and λ_1 is the first eigenvalue of A_1, then the problem is not coercive when $|\lambda|a \geq \lambda_1$.

The above idea of using two pairs of Banach spaces with compatible dualities for studying (locally) coercive operator equations (with f of small norm) is due to Kato [10]. Earlier, Hess [9] has also studied operator equations in a less general setting under a global coercivity condition. One importance of studying operator equations in such a setting lies in the fact that certain differential equations, which have been successfully handled earlier only by the method of Nash-Moser type (cf. Moser [15] and Rabinowitz [16]),reduce to them, and the problem of "loss of derivatives" is not present [10]. Another importance of this setting is demonstrated in the paper by an application to a class of (noncoercive) semilinear elliptic equations with strong nonlinearities (cf. also Hess [9]). Earlier, coercive quasilinear elliptic equations with strong nonlinearities have been studied by many authors using either truncation techniques and/or approximation results of Hedberg's type and generalized degree theories (e.g. [5,7,8,9,12,17]).

The second abstract problem we treat is the solvability of

$$(2) \qquad Kx - \lambda Lx + Mx = f, \qquad (x \in D(M), \quad f \in H),$$

where $L:H \to H$ is linear symmetric and compact and $K, M:D(M) \subset H \to H$ are nonlinear with $K + M$ of type (M) relative to (U, H). It is an extension of the problem studied by Kesavan [11] when $M:H \to H$ is completely continuous (i.e. $Mx_n \to Mx$ if $x_n \rightharpoonup x$ (weakly)) and K is the identity.

2. SOLVABILITY OF EQ. (1) WITH $|\lambda|a < \lambda_1$

Our basic assumptions on A_1 and A are:

(3) *A_1 is symmetric and for some positive $c \notin \sigma(A_1)$, the spectrum of A_1, $B_c = A_1 + cI$ is positive, i.e., $(B_cx, x) > 0$ for $0 \neq x \in D(A_1)$ and $B_c^{-1} : H \to H$ is*

compact.

(4) *There are constants* $c_1 > 0$ *and* $c_2 \geq 0$ *such that*

$$< Ax, x > \geq c_1 \| x \|_V^2 - c_2 \| x \|^2 \text{ for all } x \in V.$$

Let $\lambda_1 \leq \lambda_2 \leq \ldots, \lambda_k \to \infty$, be the sequence of eigenvalues of A_1 and $\{e_k\}_1^\infty$ be the corresponding system of orthonormal eigenvectors complete in U and H. Set $H_n = \text{lin.sp.}\{e_1, \ldots . e_n\}$ and let $P_n : H \to H_n$ be the orthogonal projection onto H_n for each n. Since $\{\mu_k = \lambda_k + c\}$ and $\{e_k\}$ are the eigenvalues and eigenvectors of B_c, we have by the variational characterization of $\{\mu_k\}$:

$$(5) \quad (B_c x, x) \geq \mu_1 \| x \|^2 \text{ and } (B_c(I-P_k)x, (I-P_k)x) \geq \mu_{k+1} \| (I-P_k)x \|^2,$$

$$\forall x \in D(A_1).$$

Now we define the class of permissible nonlinearities.

Definition 1. (cf. [9]) Let $U \subset D(N) \subset V$ and $N : D(N) \to U^*$. Then N is said to be of *type* (M) *relative to* (U, V) if (i) N is continuous from each finite-dimensional subspace of U into the weak topology of U^* and (ii) whenever $\{x_n\} \subset U, x_n \rightharpoonup x$ in V, $Nx_n \rightharpoonup y$ in U^* with $y \in V^*$ and $\limsup < Nx_n, x_n > \leq < y, x >$, then $x \in D(N)$ and $Nx = y$. If y in (ii) is given in advance, we say that N is of type (M) at y relative to (U, V).

Recall that $N : D(N) \to U^*$ is *quasibounded* if, whenever $\{x_n\} \subset U$ is bounded in V and $< Nx_n, x_n > \leq const. \| x_n \|_V$, then $\{Nx_n\}$ is bounded in U^*. We say that C has a linear growth if there are positive constants a, b and ρ such that

$$(6) \qquad \|Cx\| \leq a\|x\| + b \quad \text{for all } \|x\| \geq \rho, \ x \in U.$$

Our first result is:

THEOREM 1 (cf. [14]). *Let* $|\lambda| a < \lambda_1$, *(3),(4), and (6) hold,* $(N - \lambda C)(U) \subset H$, $(Nx, x) \geq 0$ *for* $x \in U$, N *be quasibounded and* $N - \lambda C$ *be of type (M) relative to* (U, V) *and* $A : V \to V^*$ *be linear and continuous. Then Eq (1) is solvable in* V *for each* $f \in H$.

Proof. Let $f \in H$ be fixed and choose an $r \geq \rho$ such that $\| f \| + |\lambda| b < r(\lambda_1 - |\lambda| a)$. Then, for each $x \in \partial B(0,r) \cap H_n, n \geq 1$, we have

$$(\lambda P_n C x - A_1 x - P_n N x - P_n f, x) = (\lambda C x - A_1 x - N x - f, x)$$

$$\leq (|\lambda| a - \lambda_1) \| x \|^2 + (\| f \| + |\lambda| b) \| x \| < 0.$$

Hence, the homotopy $H_n(t,x) = t(\lambda P_n C x - A_1 x - P_n N x - P_n f) - (1-t)x \neq 0$ on $[0,1] \times \partial B(0,r) \cap H_n$, and therefore the Brouwer degree deg $(\lambda P_n C - A_1 - P_n N - P_n f, B \cap H_n, 0) \neq 0$ for each $n \geq 1$. Thus, there is an $x_n \in B(0,r) \cap H_n$ such that $\lambda P_n C x_n - A_1 x_n - P_n N x_n = P_n f$, $n \geq 1$. Moreover, (4) implies that

$$c_1 \| x_n \|_V^2 - c_2 \| x_n \|^2 \leq (A_1 x_n, x_n)$$

$$\leq a \mid \lambda \mid \| x_n \|^2 + (\| f \| + |\lambda| b) \| x_n \|,$$

and consequently, $\{x_n\}$ is bounded in V. Next,

$$< N x_n, x_n > = (N x_n, x_n) = (P_n N x_n, x_n) = (\lambda P_n C x_n - A_1 x_n - P_n f, x_n)$$

$$\leq a \mid \lambda \mid \| x_n \|^2 + (\| f \| + |\lambda| b) \| x_n \| - < A x_n, x_n >$$

$$\leq a \mid \lambda \mid \| x_n \|^2 + (\| f \| + |\lambda| b) \| x_n \| + \| A \| \| x_n \|_V^2 \leq const. \| x_n \|_V,$$

and therefore, $\{N x_n\}$ is bounded in U^* by the quasiboundedness of N. Thus, we may assume that $x_n \rightharpoonup x$ in V, $A x_n \rightharpoonup A x$ and $(N - \lambda C) x_n \rightharpoonup y$ in U^*. Moreover, for each $u \in H_n$, $< (N - \lambda C) x_n, u > = -(A_1 x_n + P_n f, u)$. Then, for each $u \in \cup_{n \geq 1} H_n$, $u \in H_k$ for some k and for each $n \geq k$,

$$< (N - \lambda C) x_n, u > = - < A x_n + f, u > \rightarrow - < A x + f, u > .$$

Since $\overline{\cup H_n} = U$, it follows that $< (N - \lambda C) x_n, u > \rightarrow - < A x + f, u >$ for each $u \in U$, and therefore $y = -A x - f$. Moreover,

$$< A x_n, x_n - x > \geq < A x, x_n - x > - c_2 \| x_n - x \|^2$$

imples that $< A x, x > \leq \liminf < A x_n, x_n >$ and consequently,

$$\limsup < (N - \lambda C) x_n, x_n > = \limsup [(-f, x_n) - < A x_n, x_n >]$$

$$\leq - < Ax + f, x > .$$

Hence, $x \in D(N)$ and $\lambda Cx - Ax - Nx = f$ by property (M).∎

Remark 1. When $\lambda = 0$ $(< \lambda_1)$, Theorem 1 is a global analogue of the result of T. Kato [10] for mappings of the form $T = A + N$ (compare also with Hess [9]).

3. THE CASE $|\lambda| a \geq \lambda_1$

This is a noncoercive case and a major additional difficulty is to show that the set

$$S_\lambda(f) = \{x \in H_n \mid \lambda P_n Cx - A_1 x - P_n(N_1 + N_2)x = P_n f,\ n = 1, 2, \ldots\}$$

is bounded in H, where now $N = N_1 + N_2 : D(N) \subset V \to U^*$.

PROPOSITION 1. *Let (3) and (6) hold, N be such that $N_i(U) \subset H$, i=1,2, N_1 be of type (M) at 0 relative to (U,H) and*
(7) $(N_i x, x) \geq 0$ for $x \in U$, $i = 1, 2$, and $x = 0$ if $N_1 x = 0$.
(8) If $(N_1 x_n, x_n) \to 0$ for some $\{x_n\} \subset U$ bounded in H, then $N_1 x_n \rightharpoonup 0$ in U^*.
(9) There is a $\delta > 1$ such that $N_1(tx) = t^\delta N_1(x)$ for all $x \in U$, $t \geq 0$.
(10) There are positive constants a_1, b_1, and $\delta_1 < \delta$ such that

$$\| N_2 x \| \leq a_1 \| x \|^{\delta_1} + b_1 \text{ for all } x \in U \text{ with } \| x \| \text{ large.}$$

Then $S_\lambda(f)$ is bounded in H for each λ with $|\lambda| a \geq \lambda_1$ and each $f \in H$.

Proof. Let $|\lambda| a \geq \lambda_1$ be fixed and suppose that $S_\lambda(f)$ is not bounded in H for some $f \in H$. Let $x_{n_k} \in S_\lambda(f)$ be such that $\| x_{n_k} \| \to \infty$ as $k \to \infty$, and set $u_n = \frac{x_{n_k}}{\| x_{n_k} \|}$. Then

$$(11) \qquad (N_1 u_{n_k}, u_{n_k}) = \frac{1}{\| x_{n_k} \|^{\delta - 1}} [c \| u_{n_k} \|^2$$

$$- (B_c u_{n_k}, u_{n_k}) - \|x_{n_k}\|^{-1}((N_2 - \lambda c)x_{n_k} - f, u_{n_k})] \to 0 \text{ as } k \to \infty$$

and $N_1 u_{n_k} \rightharpoonup 0$ in U^* by (8). Since we may assume that $u_{n_k} \rightharpoonup u$ in H, the (M)-property of N_1 implies that $u \in D(N_1)$ and $N_1 u = 0$. Hence, $u = 0$ by (7).

Next, let $\alpha \in (0,1)$ and $\epsilon > 0$ small be fixed, $\bar{a} = a + \epsilon$ and $m \geq 1$ be such that $\lambda_{m+1} - |\lambda|\bar{a} > \alpha$ and $\| (I - P_m)f \| \leq \alpha$. Then, for each $n_k > m$ large and fixed, (6) and (7) imply that

$$(|\lambda|\bar{a} + c)(\| P_m x_{n_k} \|^2 + \| (I - P_m)x_{n_k} \|^2) \geq ((\lambda P_{n_k} C + c)x_{n_k}, x_{n_k})$$

$$= (B_c x_{n_k}, x_{n_k}) + (P_{n_k}(N_1 + N_2)x_{n_k}, x_{n_k}) + (P_{n_k} f, x_{n_k})$$

$$\geq (B_c P_m x_{n_k}, P_m x_{n_k}) + (B_c(I - P_m)x_{n_k}, (I - P_m)x_{n_k}) + (P_m f, P_m x_{n_k})$$

$$+((I - P_m)f, (I - P_m)x_{n_k}) \geq \mu_1 \| P_m x_{n_k} \|^2 + \mu_{m+1} \| (I - P_m)x_{n_k} \|^2$$

$$- \| P_m f \| \| P_m x_{n_k} \| - \| (I - P_m)f \| \| (I - P_m)x_{n_k} \|,$$

or after rearranging,

$$(\lambda_{m+1} - |\lambda|\bar{a}) \| (I - P_m)x_{n_k} \|^2 - \| (I - P_m)f \| \| (I - P_m)x_{n_k} \|$$

$$\leq (|\lambda|\bar{a} - \lambda_1) \| P_m x_{n_k} \|^2 + \| f \| \| P_m x_{n_k} \| .$$

Since $\| (I - P_m)f \| \leq \alpha$, we get, after dividing by $\| x_{n_k} \|^2$,

$$(\lambda_{m+1} - |\lambda|\bar{a}) \| (I - P_m)u_{n_k} \|^2 - \alpha \| x_{n_k} \|^{-1} \| (I - P_m)u_{n_k} \|$$

$$\leq (|\lambda|\bar{a} - \lambda_1) \| P_m u_{n_k} \|^2 + \| x_{n_k} \|^{-1} \| f \| \| P_m u_{n_k} \|,$$

or

$$(\lambda_{m+1} - |\lambda|\bar{a}) \| (I - P_m)u_{n_k} \|^2 - \alpha \| (I - P_m)u_{n_k} \| \tag{12}$$

$$\leq (|\lambda|\bar{a} - \lambda_1) \| P_m u_{n_k} \|^2 + \| f \| \| P_m u_{n_k} \| .$$

On the other hand, we may assume that $P_m u_{n_k} \to v_0 \in H_m$ as $k \to \infty$ and $(I - P_m) u_{n_k} \rightharpoonup -v_0 \in H_m^{\perp}$. Hence, $v_0 = 0$ and $\| (I - P_m)u_{n_k} \| \to 1$ as $k \to \infty$ since

$$1 = \| u_{n_k} \|^2 = \| P_m u_{n_k} \|^2 + \| (I - P_m) u_{n_k} \|^2 .$$

Finally, passing to the limit in (12) we obtain $\lambda_{m+1} - |\lambda|\bar{a} \leq \alpha$, which contradicts our choices of α and m. Hence, $S_\lambda(f)$ is bounded in H for all λ with $|\lambda|a \geq \lambda_1$ and $f \in H$. ∎

Our basic result in this case is:

THEOREM 2 (cf. [14]). *Let $|\lambda|a \geq \lambda_1$, (3)–(4) hold, $N = N_1 + N_2$ be such that $N_i(U) \subset H$, i=1,2, N_1 be of type (M) at 0 relative to (U, H), $u = 0$ if $(N_1 u, u) = 0$ and (6)–(10) hold. Suppose that $N : D \to U^*$ is quasibounded, $N - \lambda C$ is of type (M) relative to (U, V) and $A : V \to V^*$ is continuous. Then Eq. (1) is solvable in V for each $f \in H$.*

Proof. Let $f \in H$ be fixed. We will show first that each finite dimensional equation in $S_\lambda(f)$ is solvable. For each $n \geq 1$, we claim that there is a constant $c_n > 0$ such that

$$(N_1 x, x) \geq c_n \| x \|^{1+\delta} \text{ for each } x \in H_n. \tag{13}$$

If not, then there is a sequence $\{x_k\} \subset H_n$ for some n such that

$$(N_1 x_k, x_k) \leq \frac{1}{k} \| x_k \|^{1+\delta} \text{ for each } k,$$

and, setting $u_k = \frac{x_k}{\|x_k\|}$, we get

$$0 \leq (N_1 u_k, u_k) \leq \frac{1}{k} \to 0 \text{ as } k \to \infty. \tag{14}$$

We may assume that $u_k \to u$ in H_n and, passing to the limit in (14), we get $(N_1 u, u) = 0$. Hence, $u = 0$ in contradiction to $\| u \| = 1$, and therefore (13) holds for each n and some $c_n > 0$.

Next, we choose $r_n \geq \rho$ such that $\frac{\|f\| + |\lambda| b}{r_n} < \lambda_1 - |\lambda|a + c_n r_n^{\delta - 1}$ and note that for each $x \in \partial B(0, r_n) \cap H_n$,

$$(\lambda C x - A_1 x - N_1 x - N_2 x - f, x) \leq (|\lambda|a - \lambda_1 - c_n r_n^{\delta - 1} + \frac{\| f \| + |\lambda| b}{r_n}) r_n^2 < 0.$$

Hence, as before, there is an $x_n \in H_n$ such that $\lambda Cx_n - A_1x_n - P_n(N_1 + N_2)x_n = P_nf$ for each $n \geq 1$. Moreover, $S_\lambda(f)$ is bounded in H by Proposition 1, and is also bounded in V by (4). Finally, the completion of the theorem can be carried out as in Theorem 1.∎

4. SOLVABILITY OF EQ. (2)

We assume that $K : D(M) \subset H \to H$ has a linear growth and is coercive, i.e.,

(15) There are positive constants a, b, c, and $\rho \geq 0$ such that
(i) $\|Kx\| \leq a\|x\| + b$ *for all* $\|x\| \geq \rho$,
(ii) $(Kx, x) \geq c\|x\|^2$, *for all* $x \in D(M)$.

Again, the noncoercive case is harder and a result analogous to Proposition 1 holds.

PROPOSITION 2. *Let $L : H \to H$ be a linear, symmetric, positive and compact mapping, $Le_k = \lambda_k e_k$ for $k \geq 1$ with $\{e_k\} \subset U$ and complete in H, and $\{H_n, P_n\}$ as before. Suppose that $M = M_1 + M_2 : D(M) \subset H \to H$ is such that M_1 is quasibounded and of type (M) at 0 relative to (U,H), M_1, M_2, and K satisfy (7), (9), (10), and (15) on U, respectively. Then, for each $\lambda \geq c\lambda_1^{-1}$ and each $f \in H$, the set $S_\lambda(f) = \{x \in H_n \mid P_nKx - \lambda Lx + P_nMx = P_nf, n = 1, 2, \ldots\}$ is bounded in H.*

Proof. Let $\lambda \geq c\lambda_1^{-1}$ be fixed and suppose that $S_\lambda(f)$ is not bounded in H for some $f \in H$. Let $x_{n_k} \in S_\lambda(f)$ be such that $\| x_{n_k} \| \to \infty$ and $u_{n_k} = \frac{x_{n_k}}{\|x_{n_k}\|}$. Then, $(M_1u_{n_k}, u_{n_k}) \to 0$ as in (11), and therefore $\{M_1u_{n_k}\}$ is bounded in H by the quasiboundedness of M_1. Thus, we may assume that $u_{n_k} \rightharpoonup u$ and $M_1u_{n_k} \rightharpoonup y$ in H with $y = 0$, since L is injective and

$$L(\frac{x_{n_k}}{\| x_{n_k} \|^\delta}) = \lambda^{-1}\frac{P_{n_k}Kx_{n_k}}{\| x_{n_k} \|^\delta} - P_{n_k}M_1u_{n_k} - \frac{P_{n_k}(M_2x_{n_k} - f)}{\| x_{n_k} \|^\delta} \rightharpoonup y.$$

Moreover, $M_1u = 0$ since M_1 is of type (M) at 0, and consequently $u = 0$.

Next, let $\alpha \in (0,1)$ be fixed and $m \geq 1$ be such that $\| (I - P_m)f \| \leq \alpha$ and $c - \lambda\lambda_{m+1} > \alpha$. Then, using the variational characterization of the eigenvalues of L:

$$(Lx,x) \leq \lambda_1 \| x \|^2 \text{ and } (L(I-P_n)x,(I-P_n)x) \leq \lambda_{n+1} \| (I-P_n)x \|^2, \ x \in H,$$

we obtain, as in the proof of Proposition 1, that for each $n_k > m$

$$(c - \lambda\lambda_{m+1}) \| (I - P_m)u_{n_k} \|^2 - \alpha \| (I - P_m)u_{n_k} \|$$
$$\leq (\lambda\lambda_1 - c) \| P_m u_{n_k} \|^2 - \| f \| \| P_m u_{n_k} \| .$$

Again, $\| (I - P_m)u_{n_k} \| \to 1$ and $\| P_m u_{n_k} \| \to 0$ as $k \to \infty$, and therefore passing to the limit in the last inequality we get that $c - \lambda\lambda_{m+1} \leq \alpha$, which contradicts our choices of m and α. Hence, $S_\lambda(f)$ is bounded in H.∎

Our main solvability result for Eq. (2) reads:

THEOREM 3. (cf. [14]) *Let $L : H \to H$ be linear, symmetric, positive, and compact, $\{H_n, P_n\}$ be as in Proposition 2, K, $M = M_1 + M_2 : D(M) \subset H \to H$ be such that (15) holds and $K + M$ is of type (M) relative to (U, H).*
(a) If M is quasibounded and $(Mx, x) \geq 0$ for $x \in D(M)$, then Eq. (2) is solvable for each $f \in H$ and each $\lambda < c\lambda_1^{-1}$.
(b) If M_1 is quasibounded and of type (M) at 0 relative to (U, H), M_1 and M_2 satisfy (7), (9), and (10) on U, respectively, and $u = 0$ if $(M_1 u, u) = 0$, then Eq. (2) is solvable for each $f \in H$ and each $\lambda \geq c\lambda_1^{-1}$.

Proof. Let $f \in H$ be fixed. We will show first that each equation $P_nKx - \lambda Lx + P_nMx = P_nf$ is solvable in H_n. Suppose that $\lambda < c\lambda_1^{-1}$. If $\lambda > 0$, then choosing $r > 0$ such that $\| f \| < (c - \lambda\lambda_1)r$, we get that for $x \in B(0,r) \cap H_n$,

$$(P_nKx - \lambda Lx + P_nMx - P_nf, x) \geq (c - \lambda\lambda_1) \| x \|^2 - \| x \| \| f \| > 0.$$

If $\lambda < 0$, then taking $r > 0$ with $||f|| < cr$, we get that for $x \in B(0,r) \cap H_n$

$$(P_nKx - \lambda Lx + P_nMx - P_nf, x) \geq c||x||^2 - ||f||||x|| > 0.$$

Hence, using the homotopy $H_n(t,x) = t(P_nKx - \lambda Lx + P_nMx - P_nf) + (1-t)x$ on $[0,1] \times \bar{B}(0,r) \cap H_n$, we get that $\deg(P_nK - \lambda L + P_nM, B \cap H_n, P_nf) \neq 0$

for each $n \geq 1$. Thus, there is an $x_n \in B(0,r) \cap H_n$ such that $P_nKx_n - \lambda Lx_n + P_nMx_n = P_nf$ with $n \geq 1$.

Next, if $\lambda \geq c\lambda_1^{-1}$, then (13) holds for M_1 and each n. Now, we choose $r_n > 0$ such that $\frac{\|f\|}{r} < c - \lambda\lambda_1 + c_n r_n^{\delta-1}$, and note that for $x \in \partial B(0,r_n) \cap H_n$

$$\begin{aligned}&(P_nKx - \lambda Lx + P_nMx - P_nf, x)\\ &\geq (c-\lambda\lambda_1) \parallel x \parallel^2 + c_n \parallel x \parallel^{1+\delta} - \parallel f \parallel \parallel x \parallel > 0.\end{aligned}$$

Hence, as above, $P_nKx_n - \lambda Lx_n + P_nMx_n = P_nf$ for some $x_n \in B(0,r_n) \cap H_n$ and each n, and $\{x_n\}$ is bounded in H by Proposition 2.

Now, since $\{x_n\}$ is bounded in either case, some subsequence $x_{n_k} \rightharpoonup x$ in H. It remains to show that $Kx - \lambda Lx + Mx = f$. Since M is quasibounded in either case and

$$\begin{aligned}(Mx_n, x_n) = (P_nMx_n, x_n) &\leq -c \parallel x_n \parallel^2 + \lambda(Lx_n, x_n) + (f, x_n)\\ &\leq \ const. \parallel x_n \parallel,\end{aligned}$$

it follows that $\{Mx_n\}$ is bounded and a subsequence $(K+M)x_{n_k} \rightharpoonup y$. Moreover,

$$P_{n_k}(K+M)x_{n_k} = P_{n_k}f + \lambda Lx_{n_k} \rightharpoonup f + \lambda Lx = y$$

and

$$\limsup((K+M)x_{n_k}, x_{n_k}) \leq (\lambda Lx + f, x) = (y, x).$$

Hence, $x \in D(M)$ and $(K+M)x = y$ by property (M), and therefore, $Kx - \lambda Lx + Mx = f$.∎

Remark 2. Analyzing the above proof we see that $x_{n_k} \to x$ if either $K+M$ is of type (S_+) (i.e. $x_n \to x$ if $x_n \rightharpoonup x$ and $\limsup((K+M)x_n, x_n - x) \leq 0$), or $K+M$ is compact on H. When M_1 and M_2 are completely continuous on H, and $K = I$, Theorem 3 has been proved by Kesavan [11] using different type of arguments.

5. AN APPLICATION

Let $Q \subset R^n$ be a bounded domain with the smooth boundary ∂Q, $H = L_2(Q)$ and $W_2^k(Q)$ be the usual real Sobolev space with norm $\parallel . \parallel_k$, $k \geq 1$ an integer.

Let $F = F_1 + F_2, G : Q \times R \to R$ be Carathéodory functions and V be a closed subspace of $W_2^m(Q)$ containing $\overset{\bullet}{W}{}_2^m(Q)$.

In this section we shall establish the weak solvabilty in V of the semilinear elliptic equation

(16)

$$\sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)D^\beta u(x)) + F(x, u(x)) - \lambda G(x, u(x)) = f(x), \quad x \in Q$$

where the coefficients $a_{\alpha\beta}(x) = a_{\beta\alpha}(x)$ are real valued, smooth and bounded, $f \in L_2$, $\lambda \in R$, F is strongly nonlinear, and G has linear growth.

We begin by specifying conditions on the linear part.

(H1) The bilinear form $a(u, v) = \sum_{|\alpha|,|\beta|\leq m}(D^\alpha u, a_{\alpha\beta} D^\beta v)_{L_2}$ is coercive on V, i.e., there are constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$a(u, u) \geq c_1 \parallel u \parallel_m^2 - c_2 \parallel u \parallel^2, \text{ for } u \in V.$$

Using the Lax-Milgram theorem, one can show (see, e.g., [2]) that $a(u, v)$ generates a linear, closed, and densily defined mapping $A_1 : D(A_1) \subset L_2 \to L_2$, with compact resolvent, characterized by $D(A_1) = \{u \in V \mid \textit{for some } h \in L_2, a(u, v) = (h, v) \textit{ for all } v \in V\} \subset W_2^{2m}$ and $a(u, v) = (A_1 u, v)$ for $u \in D(A_1)$ and $v \in V$. Let $\{B_j\}_1^m$ be boundary differential operators of orders $m_j \leq 2m, 1 \leq j \leq m$, such that the problem

$$\sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^\alpha(a_{\alpha\beta}(x)D^\beta u) = f(x) \quad \text{in } Q$$

$$B_j u(x) = \sum_{|\alpha|\leq m_j} b_{j\alpha}(x) D^\alpha u(x) = 0 \quad \text{on } \partial Q$$

is regularly elliptic (cf., e.g., [2]). Set $\widetilde{W_2^{2m}} = \{u \in W_2^{2m}(Q) \mid B_j u = 0 \textit{ on } \partial Q, \ j = 1, \ldots m\}$. We assume (cf. [1]):

(H2) V is such that $D(A_1) = \widetilde{W_2^{2m}}$, A_1 is symmetric in L_2 and possesses an orthonormal basis of eigenfunctions $\{u_k\}$ in L_2; $A_1 u_k = \lambda_k u_k$.

Let $H_n = lin.sp.\{u_1, \ldots u_n\}, W = W_2^k(Q) \cap \widetilde{W_2^{2m}}$ with $k \geq \max\{1 + [\frac{n}{2}], 2m\}$ and note that $W \subset C(\bar{Q})$ by the Sobolev embedding theorem. If

$a_{\alpha\beta}, B_{j\alpha}$ and ∂Q are sufficiently smooth, then $\overline{\cup H_n} = U$ for some closed subspace U of W. Indeed, write $k = 2m + 2rm + s$ for some $r \geq 0$ and $0 \leq s < 2m$, and note that $B_c : W_2^{2m+2im+s}(Q) \cap \widetilde{W_2^{2m}} \to W_2^{2im+s}(Q)$ is a homeomorphism for each integer $i \in [0, r]$. Let $i = 0$ and note that $\overline{\cup H_n} = \widetilde{W_2^s}$ since $\widetilde{W_2^{2m}}$ is dense in $\widetilde{W_2^s}$ and $\overline{\cup H_n} = \widetilde{W_2^{2m}}$ (cf. [1]). Since $\widetilde{W_2^s}$ is a closed subspace of W_2^s, $U_0 = B_c^{-1}(\widetilde{W_2^s})$ is closed subspace of $W_2^{2m+s} \cap \widetilde{W_2^{2m}}$ and $\overline{\cup H_n} = U_0$. To see this, let $f \in U_0, g = B_c f \in \widetilde{W_2^s}$ and $g_n \in H_n$ be such that $g_n \to g$ in $\widetilde{W_2^s}$. Then, $B_c^{-1} g_n \to f$ in U_0 with $B_c^{-1} g_n \in H_n$, and therefore $\overline{\cup H_n} = U_0$. Next, let $i = 1$ and note that $U_1 = B_c^{-1}(U_0)$ is closed in $W_2^{4m+s}(Q) \cap \widetilde{W_2^{2m}}$ and $\overline{\cup H_n} = U_1$ as above. Proceeding in this way, we get that $U = U_r$ is a closed subspace of W with $\overline{\cup H_n} = U$.

Now, denote by $<,>$ the usual duality between V and its dual V^* or U and U^* and note that $<,>$ is compatible with the inner product $(\ ,\ )$ on H in either case. Since $a(u, .)$ is a continuous linear functional on V for each $u \in V$, it defines a continuous linear mapping $A : V \to V^*$ such that $a(u, v) = < Au, v >$ for $u, v \in V$, and $< Au, v > = (A_1 u, v)$ for $u \in D(A_1), v \in V$.

Regarding the nonlinear part, we assume:

(F1) $F_1(x, 0) = 0$ *and* $F_1(x, .)$ *is increasing in a neighborhood of 0 for a.e.* $x \in Q$, *and for each* $s \geq 0$ *there is a function* $h_s \in L_2$ *such that*

$$\sup_{|t| \leq s} | F_1(x, t) | \leq h_s(x) \text{ and } F_1(x, t)t \geq 0 \text{ for a.e.} x \in Q,\ t \in R.$$

(F2) $| F_2(x, t) | \leq a(x) + b(x) | t |$ *for a.e.* $x \in Q, t \in R$ *and some* $a, b \in L_2$.
(F3) $s = 0$ *if* $F_1(x, s) = 0$ *for some* $x \in Q$, *and* $F_1(x, st) = s^\delta F_1(x, t)$ *for a.e.* $x \in Q, t \in R, s \geq 0$ *and some* $\delta > 1$.
(G1) $|G(x, t)| \leq c(x) + d(x)|t|$ *for a.e.* $x \in Q$, $t \in R$ *and some* $c, d \in L_2$.

Let $D(N_1) = \{u \in V \mid F_1(x, u) \text{ and } F_1(x, u)u \text{ are in } L_1\}$, and $C, N = N_1 + N_2 : D(N_1) \to U^*$ be defined by $< Cu, v > = (G(x, u), v)$ and $< N_1 u + N_2 u, v > = (F_1(x, u) + F_2(x, u), v)$ for $u \in D(N_1)$ and $v \in U$. By (F1), $U \subset D(N_1)$, N is well defined and $N(U) \subset H$. Moreover, (6) holds for some constants a and b, by $(G1)$.

PROPOSITION 3. *(a) If (F1) holds, then $N_1 : D(N_1) \to U^*$ is of type (M) at 0 relative to (U, L_2) and (8) holds.*
(b) If (F1), (F2), and (G1) hold, then $N : D(N_1) \to U^*$ is quasibounded and $N - \lambda C$ is of type (M) relative to (U, V).

Proof. (a) Suppose that $\{u_n\} \subset U, u_n \rightharpoonup u$ in $L_2, N_1 u_n \rightharpoonup 0$ in U^* and $\limsup(N_1 u_n, u_n) \leq 0$. Then Fatou's lemma and (F1) imply that $(N_1 u_n, u_n) \to 0$, and therefore we may assume that $F_1(x, u_n(x))u_n(x) \to 0$ a.e. in Q. Since $F_1(x,t)t$ is also increasing in t in a neighborhood of zero for a.e. $x \in Q$, it follows that $u_n(x) \to 0$ a.e. in Q. To show that $u_n \to 0$ in L_1, let $\epsilon > 0$ be fixed and, for any $n \geq 1$, define $Q_1 = \{x \in Q \mid\mid u_n(x) \mid \leq \frac{1}{\epsilon}\}$ and $Q_2 = Q \backslash Q_1$. Then, for any measurable subset $A \subset Q$,

$$\int_A \mid u_n(x) \mid dx \leq \int_{A \cap Q_1} \mid u_n(x) \mid dx + \epsilon \int_{A \cap Q_2} u_n^2(x) dx \leq \frac{m(A)}{\epsilon} + const.\epsilon.$$

Hence, $u_n \to 0$ in L_1 by Vitali's theorem, and $u = 0$ with $N_1 0 = 0$ since $u_n \rightharpoonup u$ in L_1.

To see that (8) holds, let $\{u_n\} \subset U$ be bounded in L_2 and $(N_1 u_n, u_n) \to 0$ as $n \to \infty$. We get, as above, that $u_n \rightharpoonup u$ in $L_2, u_n \to 0$ in L_1 and therefore $u = 0$. On the other hand, for any $\epsilon > 0$,

$$\mid F_1(x, u_n(x)) \mid \leq \sup_{|t| \leq \frac{1}{\epsilon}} \mid F_1(x,t) \mid + \epsilon F_1(x, u_n(x)) u_n(x) \tag{17}$$

and, for any measurable subset $A \subset Q$,

$$\int_A \mid F_1(x, u_n(x)) \mid dx < \| h_{\frac{1}{\epsilon}} \|_{L_1(A)} + const.\epsilon.$$

Hence, by Vitali's theorem, $F_1(., u_n) \to F_1(., u) = 0$ in L_1, and therefore $N_1 u_n \rightharpoonup 0$ in U^*.
(b) Note first that $C, N_2 : V \to L_2$ are completely continuous since V is compactly embedded in L_2. Let $i : U \to V$ be the natural injection. Next, let $\{u_n\} \subset U, u_n \rightharpoonup u$ in $V, (N - \lambda C)u_n \rightharpoonup i^* v$ in U^* for some $v \in V^*$ and $\limsup < (N - \lambda C)u_n, u_n > \leq < v, u >$. Hence, in view of (17), Vitali's theorem and Fatou's lemma imply that $F_1(., u_n) \to F_1(., u)$ in L_1 and

$$\int_Q F_1(x, u) u dx \leq \liminf \int_Q F_1(x, u_n) u_n dx \leq const.$$

Thus, $u \in D(N_1), N_1 u_n \rightharpoonup N_1 u$ in U^* and $(N - \lambda C)u_n = N_1 u_n + N_2 u_n - \lambda C u_n \rightharpoonup N_1 u + N_2 u - \lambda C u = (N - \lambda C)u = i^* v$, proving that $(N - \lambda C) : D(N_1) \to U^*$ is of type (M) relative to (U, V). Moreover, using (17) as above, we see that N_1 is quasibounded and therefore such is $N = N_1 + N_2$ by the boundedness of N_2.∎

Now, let $\lambda \in R$ and $f \in L_2$. We are looking for a solution u of the following variational problem:

$$(18) \quad \begin{cases} a(u,v) + \int_Q F(x,u)v dx - \lambda \int_Q G(x,u)v dx = (f,v) \ \forall v \in W_2^k \cap V, \\ u \in D(N_1) \subset W_2^m \end{cases}$$

which can be considered as weak formulation of Eq. (16). We have:

THEOREM 4. *Let $a_{\alpha\beta}, b_{j\alpha}$ and ∂Q be sufficiently smooth, (H1), (H2), (F1), (F2), and (G1) hold. Then BVP (18) has a solution for each $|\lambda| a < \lambda_1$ and each $f \in L_2$. If, in addition, (F3) holds, then the same conclusion is also valid for $|\lambda| a \geq \lambda_1$.*

Proof. Let $i : U \to V$ be the natural injection and $i^* : V^* \to U^*$ be its dual mapping. Define a bilinear form on $V \times i^*(V^*)$ by $< u, i^* v > = < u, v >$ for $u \in V, v \in V^*$, and note that $< i^* A u, v > = < Au, v >$ for $u, v \in V$. Since BVP (18) is equivalent to the operator equation $\lambda i^* C u - i^* A u - N u = -i^* f$, the conclusions of the theorem follow, in view of Proposition 3, from Theorems 1 and 2 with $V^*, \lambda C - A$ and f replaced by $i^*(V^*), i^*(\lambda C - A)$ and $i^* f$, respectively.∎

For the sake of comparison, cons' the BVP

$$(19\pm) \quad \begin{cases} -\Delta u = \pm \mid u \mid^{p-1} u + \lambda u + f \ in \ Q \subset R^n \\ u = 0 \ on \ \partial Q. \end{cases}$$

Theorem 4 implies that BVP (19_-) has a weak solution for each $\lambda \in R, f \in L_2$ and $p > 1$. However, the situation is quite different for BVP (19+) and has been studied by many authors. Many exsistence results on positive solutions of (19+) with $p < \frac{n+2}{n-2}$ are known (see the review article by P.L. Lions [13] and the references in there). In the critical case, when $p = \frac{n+2}{n-2}$, Brezis-Nirenberg [6] have shown that BVP (19+), with $f = 0$, has a positive solution only for $\lambda \in (0, \lambda_1)$ provided $n \geq 4$ and Q is starshaped. If, in addition, Q is not contractable and $n \geq 3$, Bahri-Coron [3] have established this fact also for $\lambda = 0$ (using the methods of algebraic topology). For the exsistence of infinitely many solutions of (19+) with $\lambda = 0$, we refer to Bahri-Lions [4] and the references therein.

Remark 8. When $1 < p < \frac{n+2}{n-2}$ ($p > 1$ if $n \leq 2$), the weak solvability of (19_-) was proved by Kesavan [11] using different methods. When $F_2 = 0, \lambda = 0$ and A is coercive, Theorem 4 is contained in Hess [9] with $m = 1$, and in Webb [17] and Brezis-Browder [5] (under an additional condition on F) with $m > 1$. For an application of Theorem 3, with $M : H \to H$ completely continuous, to the Von Kármán Equations, we refer to [11].

The author would like to thank the referee for pointing out some inaccuracies in the paper.

References

[1] S. Agmon, On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, *Comm. Pure App. Math.*, **15**(1962), 119–147.

[2] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Princeton, NJ, Van Nostrand, 1965.

[3] A. Bahri and J.M. Coron, Sur une équation elliptique nonlinéaire avec l'exposant critique de Sobolev, *C.R. Acad. Sc. Paris, t.* **301**, *Serie I*, (1985), 345–348.

[4] A. Bahri and P.L. Lions, Remarques sur la théorie variationnelle des points critiques et applications, *C.R. Acad. Sc. Paris, t.* **301**, *Serie I*, (1985), 145–147.

[5] H. Brezis and F.E. Browder, Some properties of higher order Sobolev spaces, *J. Math. Pures Appl.*, **61**(1982), 245–259.

[6] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Comm. Pure Appl. Math.* **36**(1983), 437–477.

[7] F.E. Browder, Exsistence theory for boundary-value problems for quasilinear elliptic systems with strongly nonlinear lower order terms, *Proc. Symp. Pure Math.*, **23**(1971), 269–286, AMS, Providence, 1973.

[8] F.E. Browder, Degree of mapping for nonlinear mappings of monotone

type, *Proc. Nat. Acad. Sci. USA*, **180**(1983), 2408–2409.

[9] P. Hess, On nonlinear mappings of monotone type with respect to two Banach spaces, *J. Math. Pures Appl.*, **52**(1973), 13–16.

[10] T. Kato, Locally coercive nonlinear equations with applications to some periodic solutions, *Duke Math. J.*, **51**(1984), 923–936.

[11] S. Kesavan, Existence of solutions by the Galerkin method for a class of nonlinear problems, *Applicable Anal.*, **16**(1983), 279–290.

[12] R. Landes and V. Mustonen, Boundary value problems of strongly nonlinear second order elliptic equations, *Bollettino U.M.I.*, **(6) 4–B** (1985), 33–55.

[13] P.L. Lions, On the existence of positive solutions of semilinear elliptic equations, *SIAM Review*, **24**(1982), 441–467.

[14] P.S. Milojević, Solvability of some semilinear equations with strong nonlinearities-I, II *Abstracts AMS*, **85T–47–16**, (January 1985); **85T–47–264** (October 1985).

[15] J. Moser, A rapidly convergent iteration method and non-linear partial differential equations-I, *Ann. Scouola Normale Supp. Pisa*, **20**(1966), 265–315.

[16] P.H. Rabinowitz, A rapid convergence method for a singular perturbation problem, *Ann. Inst. Henri Poincaré, Analyse nonlinéaire*, **1**(1984), 1–17.

[17] J.R.L. Webb, Boundary value problems of strongly nonlinear elliptic equations, *J. London Math. Soc.*, **(2) 21**(1980), 123–132.

Department of Mathematics,
New Jersey Institute of Technology,
Newark, New Jersey 07102
USA

(Oblatum 6.6. 1986, revisum 10.8. 1987)