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Georg Hetzer

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FORCED PERIODIC OSCILLATIONS IN THE CLIMATE SYSTEM VIA AN ENERGY BALANCE MODEL

Georg HETZER x)

Abstract: This note deals with the existence of periodic solutions via shooting method for a class of semilinear diffusion equations arising from highly idealized climate models. Sub- and supersolutions come from an associated stationary diffusion problem studied in [6]. The procedure closely follows [2]. From the climatic viewpoint it is important that our approach allows us to estimate the amplitude of these periodic solutions by means of a forthcoming numerical analysis of the stationary diffusion problem which is based on [8,9].

Key words: Semilinear diffusion equation, compact oriented surface, Legendre type operator, periodic solutions.

Classification: Primary 35K57

Secondary 58G99

Introduction. A class of highly idealized climate models, so-called energy balance models, relies on a climatically averaged sea level atmospheric temperature T as the only climatic indicator (cf. [6] and the references therein). T is determined from the energy budget by

$$(1)\quad c(x)\frac{\partial T}{\partial t}(x,t)-(\mathcal{A}T)(x,t)=Q(x,t)[1-\alpha(x,T(x,t))]-R_e(x,T(x,t)),$$

where x varies in M and M means either S^2 or $[-1,1]$ depending on whether longitudinal asymmetries are taken into account or not. c denotes the heat capacity, Q the incoming solar radiation, R_e the outgoing terrestrial radiation, and α the albedo. $\mathcal{A}$ replaces the mean horizontal heat flux, which is very roughly modelled either by a diffusion operator

$$(2)\quad \mathcal{A}w=-\operatorname{div}(k\operatorname{grad}w)$$

on S^2 or by a Legendre operator

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(3) $(\mathcal{A}w)(x) = -\frac{d}{dx}[(1-x^2)k(x)w'(x)]$

on $[-1,1]$, where the diffusion coefficient $k \in C^2(M)$ is positive. Unlike in [6], here Q depends periodically on t in accordance with our purpose to get insight into the response of the climate system (seasonally independent energetic part) to a slow periodic forcing with regard to the incoming radiation. By the way, our result also extends to approaches including a seasonal cycle in such a way as it is done in [10], where some computer simulations are made thereto.

Our interest in periodic forcing is partly motivated by the astronomical theory of ice-ages due to Milankovitch (cf. [3,7] e.g.). This theory relates the variation of the earth's climate between glacials and interglacials to corresponding variations of the earth's orbit, namely of its eccentricity, obliquity and position of the perihelion. There are well made doubts that annually averaged energy balance models might be able to reproduce these cycles because of the extremely small differences in yearly energy caused thereof.

In order to underpin (or object against) them, one should be able to make use of a forthcoming numerical analysis of

(4) $(\mathcal{A}w)(x) = \mu Q_0(x)[1-\alpha(x,w(x))] - R_e(x,w(x))$

$(x \in M)$ by means of an algorithm due to Jarausch and Mackens [8,9]. Q_0 denotes the actual annual mean of the incoming radiation and $\mu \in \mathbf{R}_+$ is the so-called solar constant, which serves for structural sensitivity studies in this context. Therefore we establish here that given solutions (μ_1, w_1), (μ_2, w_2) of (4) satisfying $w_1 \leq w_2$ and $\mu_1 Q_0(x) \leq Q(x,t) \leq \mu_2 Q_0(x)$ for all $x \in M$ and $t \in \mathbf{R}$, there is a periodic solution T of (1) with $w_1(x) \leq T(x,t) \leq$ $\leq w_2(x)$ for all $x \in M$, $t \in \mathbf{R}$. Moreover, it turns out that every periodic solution T, which satisfies the last inequality for one $t \in \mathbf{R}$, fulfils it for every $t \in \mathbf{R}$ such that $\|w_1 - w_2\|_\infty$ is an upper bound of the amplitude for the climatically relevant oscillations.

As in [2] our proof relies on writing (1) as an evolution equation and observing that the Poincaré operator associated with (1), i.e. the operator assigning to every initial value of the order interval $[w_1, w_2]$ the solution of (1) after one period is compact and pointwise increasing.

The required solution pairs of (4) are obtained for concrete setups by the numerical studies mentioned above, and their existence can also be discussed by virtue of the S-structure of the principal solution branch of (4), which [6] deals with.

The main result. Throughout we deal with (1) under the following setting.

(H1) M is either the compact interval $[-1,1]$ or a connected, compact, oriented, 2-dimensional Riemannian manifold with Riemannian metric g. We write m for the Lebesgue measure on $[-1,1]$ or for the measure induced by g on the Borel subsets of M.

(H2) $c,k \in C^2(M)$ positive.

(H3) $Q \in C^2(M \times \mathbf{R})$ bounded, $\inf Q > 0$, $\omega \in (0,\infty)$, $Q(x,\cdot)$ ω-periodic for $x \in M$.

(H4) $\alpha \in C^2(M \times \mathbf{R})$, $0 < \inf \alpha$, $\sup \alpha < 1$, $q \in [1,\infty)$, $\sup \{ |(\partial_2 \alpha)(x,y)|/(1+|y|^q) : x \in M, y \in \mathbf{R} \} < \infty$,

(H5) $R_e \in C^2(M \times \mathbf{R})$, $R_e(x,\cdot)$ odd for $x \in M$, $R_e(x,y) > 0$ for $x \in M$, $y \in (0,\infty)$, $r_1 \in (0,\infty)$, $r_2 \in \mathbf{R}_+$: $R_e(x,y) \geq r_1 y - r_2$ for $x \in M$, $y \in \mathbf{R}_+$. $\sup \{ |(\partial_2 R_e^*)(x,y)|/(1+|y|^q) : x \in M, y \in \mathbf{R} \} < \infty$.

We refer to [6] for a discussion of the climatic background of this setting.

We write X for the weighted Sobolev space $W^{2,2}((-1,1);p^2k^2)$ (cf. [12]), $p(x) := 1-x^2$ for $x \in [-1,1]$, or for the Sobolev space $W^{2,2}(M)$, if M is a 2-dimensional manifold. By a solution of (1) we understand a function $v \in C(M \times \mathbf{R}_+) \cap C^1(m \times (0,\infty))$ with $v(\cdot,t) \in X$ for $t \in (0,\infty)$, which satisfies (1) pointwise m-almost everywhere.

Actually, regularity theory yields κ , $\sigma \in (0,1)$ with

$$v(x,\cdot) \in C^{1,\kappa}_{loc}(0,\infty)$$

for $x \in M$, and

$$v(\cdot,t) \in \begin{cases} C^{1,\sigma}(M) \cap C^2((-1,1)) & M = [-1,1] \\ C^{2,\sigma}(M) & \text{otherwise} \end{cases}$$

for $t \in (0,\infty)$. This follows similarly to [6; 3.1.3].

Finally, we assume:

(H6) There are positive functions $Q_1, Q_2 \in C^2(M)$ satisfying $Q_1(x) \leq Q(x,t) \leq Q_2(x)$ for $x \in M$, $t \in \mathbf{R}$, and $w_1, w_2 \in X$ with $w_1 \leq w_2$ such that for $j=1,2$ holds:

(5) $(\mathcal{A} w_j)(x) = Q_j(x)\,[1-\alpha(x,w_j(x))] - R_e(x,w_j(x))$.

Clearly, choosing solution pairs (μ_1,w_1), (μ_2,w_2) to (4) as indicated in the introduction is the most natural special case of (H6) ($Q_j = \mu_j Q_o$) from the climatic viewpoint. Moreover, μ_1 and μ_2 will differ from 1 by at most a small percentage in dealing with the ice-age problem.

Theorem. If (H1) - (H6) are satisfied, there exists an ω-periodic solution v of (1) with $w_1(x) \leq v(x,t) \leq w_2(x)$ for $x \in M$ and $t \in \mathbf{R}_+$.

More precisely, let P be the Poincaré operator induced by (1), then, depending on j=1 or 2, the Picard iteration $(P^k w_j)_{k \in \mathbf{N}}$ converges uniformly to the initial values of the smallest respectively greatest periodic solution of (1) in the order interval $[w_1, w_2]$ of C(M). Moreover, the sequence is pointwise increasing for j=1, decreasing for j=2. Of course, there may be only one periodic solution in $[w_1, w_2]$.

Proof. In the following, we write H for $L_2(M)$, $\langle \, , \rangle$ for its standard inner product, and $|| \quad ||$ for the norm associated with $\langle \, , \rangle$. We introduce a second inner product on H by

$$\langle \varphi, \psi \rangle_C := \int_M \varphi \psi c \, d\mu$$

for $\varphi, \psi \in H$, and denote the induced norm by $|| \quad ||_C$, which is obviously equivalent to $|| \quad ||$. As in [6], we take advantage of the fact that selfadjointness depends on the given inner product. We associate with $\mathcal{L}$ two linear operators, namely A given by $Aw = \mathcal{L} w$ for $w \in X$, and A_C given by $(A_C \, w)(x) = \frac{1}{c(x)} (\mathcal{L} w)(x)$ for $w \in X$, $x \in M$. A is selfadjoint concerning $\langle \, , \rangle$, A_C concerning $\langle \, , \rangle_C$, and the graph norms of both are equivalent to a once for all fixed natural Sobolev norm on X. Moreover, we have $\langle Aw, w \rangle \geq 0$ and $\langle A_C w, w \rangle_C \geq 0$ for $w \in X$ such that the fractional powers A^γ and A_C^γ are defined, say, for $\gamma \in \mathbf{R}_+$. For $\gamma \in [0,1]$, interpolation theory shows that both operators have the same domain called X_γ in the following. We consider X_γ under the norm $|| \quad ||_\gamma$ given by $||\psi||_\gamma = ||(A+Id)^\gamma \psi||$ for $\psi \in X_\gamma$, and under $|| \quad ||_{\gamma;C}$ given by $||\psi||_{\gamma;C} = ||(A_C + Id)^\gamma \psi||_C$ for $\psi \in X_\gamma$. Both norms are equivalent. We list the following imbedding properties:

(6) X_β is compactly imbedded into X_γ for $0 \leq \gamma < \beta \leq 1$,

(7) $X_{1/2}$ is continuously imbedded into $L_r(M)$ for $r \in [1, \infty)$,

(8) X_γ is continuously imbedded into $X^{0,\sigma}(M)$ for $\gamma \in (\frac{1}{2}, 1]$, and $\sigma \in [0, \gamma - 1/2)$, if $M = [-1,1]$, respectively $\sigma \in [0, 2\gamma - 1)$, otherwise.

Assuming (H3) - (H5) we obtain a continuously Fréchet-differentiable mapping $\mathcal{R} : \mathbf{R} \times L_{2q+2}(M) \to H$ by

$$\mathcal{R}(t,w)(x) = \frac{1}{c(x)} \left(Q(x,t) \, [1 - \alpha(x, w(x))] - R_e(x, w(x)) \right)$$

for $t \in \mathbf{R}$, $w \in L_{2q+2}(M)$, and $x \in M$. Moreover, the Fréchet-derivative of $\mathcal{R}$ is bounded on bounded subsets of $\mathbf{R} \times L_{2q+2}(M)$, which guarantees in view of (6) and (7) that $\mathcal{R}$ can be considered as a locally Lipschitz continuous map from $\mathbf{R} \times X_\gamma$ into H for $\gamma \geq 1/2$, and hence the evolution equation

$$\dot{u}+A_c u= \mathcal{R}(t,u) \tag{9}$$

associated with (1) in H falls into the scope of [5]. In particular, this yields for each $\psi \in X_{1/2}$ that the initial value problem (9), $u(0)=\psi$, possesses exactly one maximal solution $u(\cdot;\psi)$: $[0,\tau_\psi) \rightarrow H(t_\psi \in (0,\infty])$ in the sense of [5; 3.3.1]. Using the smoothing action of the semigroup generated by A_c+Id and regularity properties of A which obviously carry over to A_c one gets quite similarly to [6; 3.1]:

Lemma 1. Suppose that (H1) - (H5) are fulfilled, $\gamma \in [1/2,1)$ and $\psi \in X_\gamma$, then we have:

1. $u(\cdot;\psi) \in C([0,\tau_\psi),X_\gamma) \cap C^{1,\kappa}_{loc}((0,\tau_\psi),X_\beta)$ for $\beta \in [0,1)$ and $\kappa=(1-\beta)/2$,
2. $u \in C^{1,\kappa}_{loc}((0,\tau_\psi), C^{0,\sigma}(M))$ for $\sigma \in [0,1/2)$ and $\kappa<(1-2\sigma)/4$,
3. $u(t) \in C^{1,\sigma}(M)$ for $t \in (0,\tau_\psi)$ and $\sigma \in (0,1/2)$, and it belongs to $C^{2,\sigma}(M)$, if M is 2-dim.

In order to establish global solvability as well as the fact that the Poincaré operator is increasing, we need the following comparison assertion:

Lemma 2. Let (H1) and (H2) be satisfied, $b \in (0,\infty)$, and $f \in C(M \times [0,b) \times \mathbf{R})$. Assume that $\partial_3 f$ exists and is continuous on $M \times [0,b) \times \mathbf{R}$. For j=1,2 let $v_j \in C(M \times [0,b)) \cap C^1(M \times (0,b))$ with $v_j(\cdot,t) \in X$ and either $(\mathcal{A} v_j)(\cdot,t)$ being continuous on $[-1,1]$ for $t \in (0,b)$, if $M=[-1,1]$, or $v_j(\cdot,t) \in C^2(M)$ for $t \in (0,b)$ otherwise. Finally, suppose

$$v_1(x,0) \geq v_2(x,0) \text{ for } x \in M \tag{10}$$

and

$$\begin{aligned} &c(x) \frac{\partial}{\partial t} v_1(x,t)+(\mathcal{A}v_1)(x,t)-f(x,t,v_1(x,t)) \geq \\ &\geq c(x) \frac{\partial}{\partial t} v_2(x,t)+(\mathcal{A}v_2)(x,t)-f(x,t,v_2(x,t)) \end{aligned} \tag{11}$$

for $x \in M$ and $t \in (0,b)$.

Then we have $v_1(x,t) \geq v_2(x,t)$ for all $x \in M$, $t \in (0,b)$.

It is straightforward (cf. [4,11] e.g.) to reduce this assertion to the case where $v_1 \equiv 0$, $f(x,t,y)=h(x,t)y$ for $x \in M$, $t \in (0,b)$, $y \in \mathbf{R}$ with

$h \in C(M\times(0,b))$, and strict inequality holds in (10) and (11) and in the conclusion. Now, if M is 2-dimensional, it is a matter of standard technique in reasoning by contradiction to establish this special case. We refer to [13], especially, with regard to the continuity assumptions which are somewhat weaker than currently used. The same works for $M=[-1,1]$, too, except for the case where 0 is attained for the first time at a boundary point. Then one observes that for $g\in C([-1,1])$ and $w\in X\cap C^1([-1,1])$ with $\mathcal{B}w=g$ we have

$$(12) \qquad 2k(\pm 1)w'(\pm 1)=\pm g(\pm 1).$$

Thus, let e.g. $\bar{t}\in(0,b)$ with $v_2(-1,\bar{t})=0$ and $v_2(x,t)<0$ for $x\in[-1,1]$, $t\in[0,\bar{t})$, then (11) and (12) yield

$$c(-1)\frac{\partial}{\partial t}v_2(-1,\bar{t})-2k(-1)\frac{\partial}{\partial x}v_2(-1,\bar{t})<0,$$

which contradicts $\bar{t}$ being the maximum of $v_2(-1,\cdot)$ on $(0,\bar{t}]$ and -1 being the maximum of $v_2(\cdot,\bar{t})$ on $[-1,1]$.

Now we are able to establish global existence and the bounds we need:

Lemma 3. Let (H1) - (H6) be satisfied, $\gamma\in(1/2,1)$, and $\beta\in[0,1)$, then we have:

1. If $\psi\in X_\gamma$ fulfils $w_1(x)\leq\psi(x)\leq w_2(x)$ for $x\in M$, then $w_1(x)\leq u(t;\psi)(x)\leq w_2(x)$ for all $x\in M$, $t\in(0,\tau_\psi)$.
2. $\tau_\psi=\infty$ for $\psi\in X_\gamma$.
3. Given $a>0$, there is some $r>0$ with $\|u(t;\psi)\|_\beta\leq r$ for all $t\in[a,\infty)$ and $\psi\in X_\gamma$, satisfying $w_1(x)\leq\psi(x)\leq w_2(x)$ for all $x\in M$.

Proof. 1.: We apply Lemma 2 twice, on the one hand with $v_1=u(\cdot;\psi)$ and $v_2\equiv w_1$, and on the other hand with $v_1\equiv w_2$ and $v_2=u(\cdot;\psi)$. (8) and Lemma 1.1 immediately yield $(x,t)\rightarrow u(t)x$ to be continuous on $M\times(0,\psi)$. Observing that $(A+Id)^{-1}$ is a bounded mapping from $C^{0,\sigma}(M)$ into $C^{1,\sigma}(M)$, we conclude because of

$$u(t;\psi)=(A+Id)^{-1}[c\mathcal{R}(t,u(t;\psi))+u(t;\psi)-c\dot{u}(t;\psi)]$$

and Lemma 1.2 that $u(\cdot;\psi)$ maps compact subsets of $(0,\tau_\psi)$ into bounded ones. This together with $u(\cdot;\psi)\in C^1((0,\tau_\psi),\ C(M))\cap C((0,\tau_\psi),\ W^{1,2}(M))$ according to Lemma 1 implies that $u(\cdot;\psi)$ can be represented by a C^1-function on $M\times(0,\tau_\psi)$. Moreover, Lemma 1.3 delivers $u(t;\psi)\in C^2(M)$ for $t\in(0,\tau_\psi)$ in case that $M\neq[-1,1]$, whereas $\mathcal{B}u(t;\psi)$ continuous on $[-1,1]$ follows from (9), Lemma 1 and (H3) - (H5). (10) and (11) are obviously satisfied, and hence Lemma 2 yields 1.

In order to establish (2) and (3), we first note that there are $\rho_1, \rho_2 \in \mathbb{R}_+$ with $||u(t;\psi)||_\infty \leq \rho_1$ and $||\mathcal{R}(t,u(t;\psi))||_\infty \leq \rho_2$ for all $t \in (0,\tau_\psi)$ and all $\psi \in X_\gamma$ satisfying $w_1 \leq \psi \leq w_2$. Moreover, since A_c+Id generates an analytic semigroup, we find some $\tilde{r}_\beta \in \mathbb{R}_+$ with

$$||(A_c+Id)^\beta \circ \exp(-t(A_c+Id))||_{[c]} \leq \tilde{r}_\beta t^{-\beta} e^{-t}$$

for $t > 0$, $|| \; ||_{[c]}$ means the operator norm coming from $|| \; ||_c$. Using [5;3.3.2] we get for $t > 0$:

$$||u(t;\psi)||_{\beta;c} = ||(A_c+Id)^\beta(u(t;\psi))||_c \leq \tilde{r}_\beta t^{-\beta} e^{-t} ||\psi||_c + \tilde{r}_\beta(\rho_1+\rho_2)(\int_M c\,d\mu)^{1/2} \Gamma(1-\beta).$$

Thus given $a \in (0,\tau_\psi)$, we have $||u(\cdot,\psi)||_{\beta;c}$ bounded on $[a,\tau_\psi)$, which yields $\tau_\psi = \infty$ in view of [5;3.3.4], i.e. Assertion 2. Furthermore, 3 is evident.

Remark 1. Clearly, global existence already follows under (H1) - (H5) for every $\psi \in X_{1/2}$.
In order to see this, let first $\psi \in X_\gamma$ ($\gamma \in (1/2,1)$). One sets $Q_1 \equiv \inf Q$, $Q_2 \equiv \sup Q$ and denotes the mapping $\mathcal{R}$ with Q_j in place of Q for j=1,2, by $\mathcal{R}_j$. The global solutions of $\dot{u}+A_c u = \mathcal{R}_j u, u(0) = \psi$, which exist and are bounded accordingly to [6;3.1], take over the role of w_1 and w_2 in the comparison arguments of the previous lemma. But, having established the case $\gamma > 1/2$ one simply observes $u(t;\psi) \in X_\gamma$ for $t > 0$ in case that $\psi \in X_{1/2}$, and uniqueness in order to get the full assertion.

Now we finish the proof of the theorem.

Let $\gamma \in (1/2,1)$, then X_γ is an ordered Banach space with regard to the pointwise ordering, since it can be continuously imbedded into C(M) according to (8). In the following, $[w_1,w_2]$ means the corresponding order interval in X_γ, which is unbounded unless $w_1 = w_2$. We define the Poincaré operator P to (9) on $[w_1,w_2]$ by

$$P\psi := u(\omega;\psi)$$

for all $\psi \in [w_1,w_2]$. Lemma 3.1 guarantees $P([w_1,w_2]) \subset [w_1,w_2]$. P being continuous follows from the "continuous dependence on initial values" stated in [5;3.4], e.g. Applying Lemma 2 with $v_1 = u(\cdot;\varphi)$ and $v_2 = u(\cdot;\psi)$, we get $P\varphi \leq P\psi$ for all $\varphi,\psi \in [w_1,w_2]$, $\varphi \leq \psi$. Finally, $P([\psi_1,\psi_2])$ is bounded in X_β for $\beta \in (\gamma,1)$ according to Lemma 3.3, hence relatively compact in view of (6).

Therefore, we can apply the fixed point theorem for compact, order increasing mappings on a closed order interval (cf. [1; Cor. 6.2]), and get the existence of a fixed point of P in $[w_1,w_2]$. More precisely, we obtain that the Picard iteration $(P^k w_1)_{k\in\mathbb{N}}$ and $(P^k w_2)_{k\in\mathbb{N}}$ converge in X_γ to the minimal and maximal fixed point, respectively, and moreover, the first sequence is increasing, the second one decreasing with regard to the pointwise ordering.

Clearly, each fixed point of P is the initial value to an ω-periodic solution of (9), which stays according to Lemma 3 at $[w_1,w_2]$ for all times. Combining Lemma 1 and the observation we made with regard to regularity in the proof of Lemma 3 we further see that each periodic solution of (9) can be considered as a solution of (1) in the sense introduced here.

Remark 2. Clearly, a periodic solution can be extended to **R**, and $C^{1,x}_{loc}((0,\tau_\psi),\dots)$ can be substituted by $C^{1,x}(\mathbf{R},\dots)$ for such a solution in Lemma 1.

Moreover, the proof also shows that a periodic solution of (1) staying at one point of time in $[w_1,w_2]$ is one of the solutions in $[w_1,w_2]$, and hence stays between the minimal and maximal periodic solution of $[w_1,w_2]$.

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Department of Foundations, Analysis and Topology, 228 Parker Hall, Auburn University, AL 36849, U.S.A.

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