Commentationes Mathematicae Universitatis Carolinae

Jan Pelant; Svatopluk Poljak; Daniel Turzík

Limit behaviour of trajectories involving subgradients of convex functions

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 3, 457--466

Persistent URL: http://dml.cz/dmlcz/106560

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,3 (1987)

LIMIT BEHAVIOUR OF TRAJECTORIES INVOLVING SUBGRADIENTS OF CONVEX FUNCTIONS

Jan PELANT, Svatopluk POLJAK, Daniel TURZÍK

Abstract: We investigate trajectories $\{y_i\}_{i=1}^{\infty}$ of mappings $h=f\circ g$ such that $y_{i+1}=fg(y_i,\dots,y_{i-q+1})$ where $q\geq 1$, $f:R^m\to R^m$ is cyclically monotone and g is one of the following

(a) $g=g(y_t,\dots,y_{t-q+1})=\sum_{k=1}^{q} A_k y_{t-k+1}$ where $q\geq 1$ and $A_{k-q+1}=A_k^T$ (the transposed matrix) for $k=1,\dots,q$.

(b) $g=g(y_t)$ where g is cyclically monotone (for q=1).

We show that there is an integer r such that

(*) $\lim_{i\to\infty}\|y_{i+r}-y_i\|=0$ provided the trajectory is bounded.
(Namely, it is $r=q+1$ in case (a) and $r=1$ in case (b).

The paper is motivated by the study of cellular automata.

Key words: Convex function, subgradient, trajectory.

Classification: 94C10, 33A70, 26B25

We investigate trajectories $\{y_i\}_{i=1}^{\infty}$ of mappings h=fg such that $y_{i+1}=fg(y_i,\dots,y_{i-q+1})$, where $q\geq 1$, $f:R^m\to R^m$ is cyclically monotone and g is some other mapping. We give some partial answers when there is some r for which

$$\lim_{i\to\infty}\|y_{i+r}-y_i\|=0$$

The considered question arises in the study of cellular automata (cf. [3]), used e.g. for modeling of neural networks ([1],[5]) or social influence ([2],[8],[11]). For better understanding we will use the following interpretation.

Consider a group of p members. The opinions of members are represented by vectors from R^m. The members change their opinions simultaneously in discrete steps according to mutual influences. The new opinion y_{t+1}^i of the i-th member at time t+1 depends on q previous opinions of other members so that $y_{t+1}^i=f^i g^i(y_t,\dots,y_{t-g+1})$ where $y_t=(y_t^1,\dots,y_t^p)$ is a concatenation of opinions of all members, g^i is an influence mapping, and f^i is an evaluation mapping for $i=1,\dots,p$. (One may object that considering the transition mapping as a composition of two mappings is superfluous. But this can be natural from the point of view of some application, and moreover, it enables us to study the behaviour of the group according to properties of f^i and g^i.) Now, if we put $f=(f^1,\dots,f^p)$ and $g=(g^1,\dots,g^p)$, we get the problem formulated in the beginning of the section. Answering this question, we describe the "limit" opinion of the group.

We assume that the evaluation mapping f is always cyclically monotone. Let us remark that if all f^i are cyclically monotone then f is as well. With respect to the influence mapping g we consider two particular cases.

(a) $g=g(y_t,\dots,y_{t-q+1})=\sum_{k=1}^{q} A_k y_{t-k+1}$ where $q\geq 1$ and $A_{k-q+1}=A_k^T$ (the transposed matrix) for $k=1,\dots,q$;

(b) $g=g(y_t)$ where g is cyclically monotone (for $q=1$).

We show there is an integer r such that

(✱) $\lim_{i\to\infty} \|y_{i+r}-y_i\|=0$ provided the trajectory is bounded.

Namely, it is $r=q+1$ in case (a) and $r=1$ in case (b).

In particular, if $h=fA$ then $r=2$ for A symmetric and $r=1$ for A positive definite matrix. Presented results generalize [3],[4],[6],[7],[8] where f of finite range (i.e. a system with a finite number of states) was considered. In this case, (✱) means the existence of a period on the trajectory of fA.

1. Some facts on cyclically monotone mappings. The following definitions and results are due to Rockafellar [10] . Let M be a subset of m-dimensional Euclidean space R^m. A multivalued mapping $F:M\longrightarrow \exp R^m$ is cyclically monotone (abbreviated as c.m.) on M if

$$\forall n\,\forall x_1,\dots,x_n\in M \quad \forall y_1\in F(x_1),\dots,y_n\in F(x_n):$$
$$\sum_{i=1}^{n}(x_i-x_{[i+1]})y_i\geq 0$$

where $[n+1]=1$ and $[i+1]=i+1$ for $i<n$. If there is a unique value $y\in F(x)$ for every x, we put $f(x)=y$ and we say that f is c.m. if F is.

A c.m. mapping F is maximal if there is no c.m. mapping $G\neq F$ satisfying $G(x)\supseteq F(x)$ for all $x\in R^m$.

A real function $u:M\longrightarrow R$ is a potential of F on M if

(1) $u(z)-u(x)\geq(z-x)y$ for each $z,x\in M$ and $y\in F(x)$.

It was proved by Rockafellar [10] that

(i) F is c.m. on M if and only if F has some potential on M;

(ii) F is maximal on R^m if and only if F has a unique potential.

For a potential u of F define

$$\bar{u}(z)=\sup\{u(x)+(z-x)y:\ x\in M \text{ and } y\in F(z)\}.$$

It follows from (1) that

(a) $\bar{u}$ is a convex function with finite values on convex closure of M;

(b) $\bar{u}(x)=u(x)$ for $x\in M$;

(c) F(x) is a subset of subgradients of $\bar{u}$ at x.

It was observed in [7] that the statement (i) remains true if the notion of a c.m. mapping is defined with respect to an arbitrary binary operation instead of the scalar product.

2. Limit behaviour of trajectories. In this section we give some results concerning the limit behaviour of bounded trajectories of discrete influence models.

A sequence $\{y_i\}_{i=1}^{\infty}$ of points of R^m is called bounded if there is a $K\geq 0$ such that $\|y_i\|\leq K$ for every i (by $\|y\|$ we denote a norm of y). We adopt the following notation. If r, s, r< s are fixed integers then [i] is an integer satisfying $[i]=i \bmod (s-r)$ and $r\leq [i]< s$ for every integer i.

Lemma 1. Let $\{x_i\}_{i=1}^{\infty}$ be a bounded sequence in R^m and $f:R^m\to R^m$ be a continuous c.m. mapping. Assume that for every $\varepsilon>0$ there exists an infinite set S of integers such that

(2) $0\leq \sum_{i=r}^{s-1} f(x_i)(x_i-x_{[i+1]})<\varepsilon$ for every $r,s\in S$, $r<s-1$.

Then $\lim_{i\to\infty}\|f(x_i)-f(x_{i-1})\|=0$.

Proof. Let $u:R^m\to R$ be a convex function such that the gradient $\nabla u=f$. Choose an $\varepsilon>0$, and let S be an infinite set given by the assumptions of the lemma. Then the following holds for every $i>r=\min S$.

(3) $u(x_{i+1})-u(x_i)\leq (x_{i+1}-x_i)f(x_i)+\varepsilon$.

(If not, take $s\in S$, $s>i$. Summing the inequalities

$$u(x_{[j+1]})-u(x_j)\geq (x_{[j+1]}-x_j)f(x_j)$$

for $j=r,\dots,s-1$, $j\neq i$, and the inequality

$$u(x_{i+1})-u(x_i)>(x_{i+1}-x_i)f(x_i)+\varepsilon,$$

we get a contradiction with (2).)

Inequality (3) says that $f(x_i)$ is an ε-subgradient of u at x_{i+1}. (For the definition of ε-subradient see [10].) The lemma follows from the following claim.

Claim. Let $M \subseteq R^m$ be a compact set. Then for each $\delta > 0$ there is $\varepsilon > 0$ such that the following holds for every $x \in M$.

If v is an ε-subgradient of u at x, then $\|f(x)-v\| < \delta$.

Proof of Claim. Choose $\delta > 0$. For every $x \in M$ define $\varphi(x)$ and $F(x)$ by $\varphi(x) = \inf\{\varepsilon > 0$: there is an ε-subgradient σ of u at x such that $\|\sigma - f(x)\| > \delta\}$, and $F(x) = \inf\{\liminf \{(v_n)\} : \{v_n\}$ converges to $x\}$. F is clearly lower semicontinuous and it is easy to see that $F(x) > 0$ for all $x \in M$. Hence the compactness of M applies. □

Lemma 2. Let $\{y_i\}_{i=1}^{\infty}$ be a bounded sequence in R^m and q be a positive integer. Then for every $\delta > 0$ there exists an infinite set S of integers satisfying:

(i) $\|y_{r-j} - y_{s-j}\| < \delta$ for every $r,s \in S$, $j=0,\dots,q$,

(ii) $r \equiv s \mod(q+1)$ for every $r,s \in S$.

Proof. Let $M \subseteq R^m$ be a compact set containing all y, and denote by M^{ω} the cartesian product of countably many copies of M. Define a mapping $s: M^{\omega} \to M^{\omega}$ which assigns $s(z) = \{z_2, z_3, \dots\}$ to every sequence $z = \{z_1, z_2, \dots\}$ (the shift to the left). Put $t^0 = \{y_i\}_{i=1}^{\infty}$ and denote $t^{p+1} = s(t^p)$, $p=1,2,\dots$. By the compactness of M^{ω} there is an accumulation point $\bar{t} \in M^{\omega}$ of the sequence $\{t_i\}_{i=0}^{\infty}$. Consider the following basic neighbourhood

$$\mathcal{U} = \{\{z_i\} \in M : \|z_i - \bar{t}_i\| < \delta/2 \text{ for } i=1,2,\dots,q\}$$

of $\bar{t}$. The set $S = \{n : t_n \in \mathcal{U}\}$ is infinite and satisfies (i). Clearly there is an infinite set $S' \subseteq S$ satisfying also (ii). □

Theorem 1. Let $q \geq 1$ be an integer and $A_1, \dots, A_q$ be square matrices of order m such that $A_k^T = A_{q-k+1}$ for $k=1,\dots,q$. Let $f: R^m \to R^m$ be a continuous c.m. mapping and let $\{y_i\}_{i=1}^{\infty}$ be a bounded sequence such that

$$y_{i+1} = f\Big(\sum_{k=1}^{q} A_k y_{i-k+1}\Big) \text{ for } i \geq q.$$

Then $\lim_{i\to\infty} \|y_{i+q+1} - y_i\| = 0$.

Proof. Let K be an upper bound of $\|y_i\|$. Denote a_{ij}^k the (i,j)-th entry of A_k and put $M = \max\{a_{ij}^k : i,j=1,\dots,m,\ k=1,\dots,q\}$. Choose an $\varepsilon > 0$. Put $\delta = (2m^2 MK(q^2+q))^{-1}$. Let S be an infinite set which exists by Lemma 2 for the given sequence $\{y_i\}_{i=1}^{\infty}$, q, and δ. Let r and s, $r < s-1$, be arbitrary elements of S. Put $x_i = \sum_{k=1}^{q} A^k y_{i-k+1}$ for $i \geq q$. Let us define the expressions W and V by

$$W=\sum_{i=r}^{s-1}(f(x_i)-f(x_{[i-q-1]}))x_i$$

and

$$V=\sum_{i=r}^{s-1}((y_{[i+1]}-y_{[i-q]})\sum_{k=1}^{q}A_k y_{[i-k+1]}).$$

We have $W\geq 0$ as f is c.m., and it is easy to check that V=0 as $A_k^T=A_{q-k+1}$. If we substitute $y_{i+1}=f(x_i)$ and $x_i=\sum_{k=1}^{q}A_k y_{i-k+1}$ into W, we get

$$W=\sum_{i=r}^{s-1}((y_{i+1}-y_{[i-q-1]+1})\sum_{k=1}^{q}A_k y_{i-k+1}).$$

Let us compute the difference W-V. It is a sum of the following four terms.

$$(y_s-y_r)\sum_{k=1}^{q}A_k y_{s-k}+\sum_{i=r}^{s-2}(y_{i+1}\sum_{k=1}^{q}A_k(y_{i-k+1}-y_{[i-k+1]}))+$$

$$+(y_s-y_r)\sum_{k=1}^{q}A_k y_{r+q+1-k}+\sum_{\substack{i=r\\ i\neq r+q}}^{s-1}(y_{[i-q]}\sum_{k=1}^{q}A_k(y_{[i-k+1]}-y_{i-k+1})).$$

The first and the third terms can be estimated each by δm^2qMK. The second and the fourth terms each by δq^2m^2MK. As V=0, we get $0\leq W=W-V<\varepsilon$. As s-r is divisible by q+1, it is s-r=t(q+1) for some $t>0$. We can write

$$0\leq\sum_{i=r}^{s-1}(f(x_i)-f(x_{[i-q-1]})x_i=$$

$$=\sum_{j=0}^{q}\sum_{i=0}^{t-1}(f(x_{r+i(q+1)+j})-f(x_{[r+(i-1)(q+1)+j]}))x_{r+i(q+1)+j}<\varepsilon .$$

As f is c.m., it holds

$$0\leq\sum_{i=0}^{t-1}(f(x_{r+i(q+1)+j})-f(x_{[r+(i-1)(q+1)+j]}))x_{r+i(q+1)+j}<\varepsilon$$

for each j=0,...,q. Hence we can apply Lemma 1 to each sequence $\{x_{i(q+1)+j}\}_{i=1}^{\infty}$, j=0,...,q, and the statement follows. □

<u>Theorem 2.</u> Let $G:R^m\to \exp R^m$ be a c.m. multivalued mapping, $f:R^m\to R^m$ be a continuous c.m. mapping, and $\{y_i\}_{i=1}^{\infty}$ be a bounded sequence such that for each i $y_{i+1}=f(x_i)$ for some $x_i\in G(y_i)$. Then $\lim_{i\to\infty}\|y_i-y_{i-1}\|=0$.

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be the sequence of points defined in the theorem. As $\{y_i\}_{i=1}^{\infty}$ is bounded, $\{x_i\}$ is bounded as well. (Subgradients on a compact subset are bounded.)

Let K be such that $\|x_i\|\leq K$ for all i. Let x be an accumulation point of $\{x_i\}$. Choose $\varepsilon\geq 0$ and define $S=\{i:\|y_i-f(x)\|<\varepsilon\}$. Let r and s, $r<s-1$, be arbitrary elements of S. Define the expressions W and V as follows.

$$W=\sum_{i=r}^{s-1}(f(x_i)-f(x_{[i-1]}))x_i$$

and

$$V=\sum_{i=r}^{s-1}(y_i-y_{[i+1]})x_i.$$

As f and G are c.m., both W and V are nonnegative, and hence

$$0 \leq W \leq W+V=(y_r-y_{s-1})(x_r-x_s) < 4\varepsilon k.$$

We apply Lemma 1 to the sequence $\{x_i\}_{i=1}^{\infty}$ and the statement follows. □

<u>Theorem 3.</u> Let $\{y_i\}_{i=1}^{\infty}$ be a bounded sequence, A be a positive definite matrix of order m, and $F:R^m \rightarrow \exp R^m$ be a c.m. multivalued mapping, such that the sequence satisfies $y_{i+1} \in F(Ay_i)$ for every i. Then $\lim_{i\to\infty} \|y_i-y_{i-1}\| = 0$.

Proof. Let K be such that $\|y_i\| \leq K$ for all i. Choose an $\varepsilon > 0$. As the sequence $\{y_i\}$ is bounded, there exists an infinite set S such that $|(y_s-y_r)Av| < \varepsilon$ holds for every $s,r \in S$ and every vector v with $\|v\| \leq 2K$. Let r and s, $r<s-1$, be arbitrary elements of S. Define the expressions W and V by

$$W= \sum_{i=r}^{s-1} y_{i+1}(Ay_i-Ay_{[i+1]})$$

and

$$V= \sum_{i=r}^{s-1} y_{[i+1]}(Ay_{[i+1]}-Ay_i)= \frac{1}{2} \sum_{i=r}^{s-1} (y_{[i+1]}-y_i)A(y_{[i+1]}-y_i) \geq 0.$$

The first equality holds as A is symmetric, V is nonnegative as A is positive definite. As F is c.m., we have $W \geq 0$. For the sum V+W we get the inequality

$$0 \leq V \leq V+W=(y_s-y_r)A(y_{s-1}-y_r) < \varepsilon .$$

This gives

$$\sum_{i=r}^{s-1} (y_{[i+1]}-y_i)A(y_{[i+1]}-y_i) < 2\varepsilon$$

and the statement immediately follows. □

3. <u>Connections to finite models.</u> Theorems 1, 2 and 3 have a common pattern: under some assumptions on the sequence $\{y_i\}_{i=1}^{\infty}$ there is some r such that $\lim_{i\to\infty} \|y_{i+r}-y_r\| = 0$. It follows that accumulation points of any of sequences $\{y_{ir+k}\}_{i=1}^{\infty}$ form a connected closed set in R^m. This connected set degenerates to one-point set provided that the corresponding discrete influence model has a finite number of states. This finite case was solved in [7]. In this section we show:

(a) how the results of the present paper generalize the results for finite models;

(b) an example of a strongly cyclically monotonous mapping for which the sequence $\{y_{ir}\}_{i=1}^{\infty}$ is not convergent.

We say that a mapping $f:R^m \rightarrow R^m$ is <u>strongly</u> cyclically monotone if it is cyclically monotone and

$f(x_1)=\ldots=f(x_n)$ whenever

$$\sum_{i=1}^{m}(x_i-x_{[i+1]})f(x_i)=0.$$

Mappings with this property were used in [6] and [7] for describing periods of systems with a finite number of states. (The period of a sequence $\{y_i\}$ is the least positive integer r such that $y_{i+r}=y_i$ for all i greater than some n_0.) These results are covered by the theorems of Section 2 as the following holds.

Theorem 4. Let M be a finite subset of R^m, and $f:M \to R^m$ be a mapping. Then f can be extended to some continuous c.m. mapping $\bar{f}:R^m \to R^m$ if and only if f is strongly c.m.

Proof. The "only if part" follows from the following observations.

(i) To each convex function $u:R^m \to R$ there is some strongly c.m. mapping $g:R^m \to R^m$ such that g(x) is a subgradient of u of x for each x.

(ii) If $g:R^m \to R^m$ is a continuous c.m. mapping, then g is maximal.

Proof of (i). Take some good ordering $\prec$ of R^m and define $g(x)=\min_{\prec}\{y:$ y is a subgradient of u at $x\}$.

Proof of (ii). It is easy to see that (ii) holds for m=1. When m>1, we can use the following reduction. Let $G:R^m \to \exp R^m$ be a c.m. multivalued mapping. For each i=1,...,m and each $x=(x_1,\ldots,x_m)\in R^m$ let us define a multivalued mapping $G_i^x:R \to \exp R$ by $G_i^x(t)=\{z:$ z is the i-th component of some $y\in G(x_1,\ldots,x_{i-1},t,x_{i+1},\ldots,x_m)\}$. Then the mappings G_i^x are c.m.

Now, if $\bar{f}$ is a continuous c.m. mapping, then it is maximal by (ii), and thus $\bar{f}$ coincides with the strongly c.m. mapping g given by (i) (for the potential u of $\bar{f}$).

The "if part". We show that there is a potential u^* of f satisfying

(4) $u^*(x)-u^*(y)=(x-y)f(y)$ iff $f(x)=f(y)$.

For a potential u of f define a number $N(u)=|\{(x,y):u(x)-u(y)=(x-y)f(y)$ and $f(x)\neq f(y)\}|$. Clearly (4) is equivalent to $N(u^*)=0$. Let u be a potential with N(u) minimal. Define a digraph G with the set of vertices M and with the directed edges $(x,y)\in E(G)$ if $u(x)-u(y)=(x-y)f(y)$. Consider a partition of G into components of strong connectivity. If N(u)>0 then there exist some edges between distinct components. Let S be a component for which there is some edge from S to M-S but no edge from M-S to S. Put $\varepsilon=\min\{u(y)-u(x)-(x-y)f(x):\ x\in S,\ y\in M-S\}$. According to the choice of S we have $\varepsilon>0$. Define a potential u' by

$$u'(x)=\begin{cases} u(x)+\frac{1}{2}\varepsilon & \text{for } x\in S,\\ u(x) & \text{for } x\in M-S.\end{cases}$$

Obviously, it holds $N(u') < N(u)$ which contradicts the choice of u. Thus, let u^* be a potential satisfying (4). Define $\bar{u}: R^m \to R$ by

$$\bar{u}(x) = \max\{u^*(y) + (x-y)f(y) : y \in M\}$$

(i.e. $\bar{u}$ is a maximum of a finite number of linear functions). Clearly, $\bar{u}$ is a convex function on R^m, and $\bar{u}(x) = u^*(x)$ for each $x \in M$. It follows from (4) that for each $x \in M$ there is a neighbourhood U_x in which $\bar{u}$ is linear. It holds that there is a differentiable convex function u such that $u(y) = \bar{u}(y)$ for $y \in U_x$, $x \in M$, and $u(y) \geq \bar{u}(y)$ for all $y \in R^m$.

Let M be a finite subset of R^m and $f: R^m \to M$ be strongly c.m. . Then we get the following corollaries.

Corollary 2. Let $A_1, \ldots, A_q$ be square matrices of order m such that $A_j^T = A_{q-j+1}$ for $j=1,\ldots,q$. Then the period of the sequence $\{y_i\}_{i=1}^{\infty}$ where $y_{i+1} = f(\sum_{j=1}^{q} A_j y_{i-j+1})$ for $i \geq q$ and $y_1, \ldots, y_q$ arbitrary, is a divisor of q+1. □

Corollary 2. Let $G: M \to \exp R^m$ be a c.m. multivalued mapping. Then the sequence $\{y_i\}_{i=1}^{\infty}$ where $y_{i+1} = f(x_i)$ for some $x_i \in G(y_i)$, is eventually constant. □

An important case of Corollary 1 is when q=1 and A is a symmetric matrix (the period is then 1 or 2); moreover, if A is positive definite then the period is 1 by Corollary 2.

Remark 1. We give an example of a c.m. mapping f for which the trajectory

$$\{f^i(A_1)\}_{i=1}^{\infty} \tag{5}$$

where $f^{i+1}(A_1) = f(f^i(A_1))$, is not convergent ($A_1$ is the starting point of the trajectory).

We will construct f on a discrete subset of R^2 only. Let us define $A_n = (x_n, y_n)$, n=1,2,..., where $x_n = 1/(2^{n-1})$, $y_n = 0$ for n=4k+2 or 4k+3, and $y_n = 1$ otherwise. Put $t_n = A_{n+1} - A_n$ and choose integers $p_n > 2^{3n+3} \|t_n\|^2$. Set $C(n,j) = A_n + \frac{j}{p_n} t$ for $j=0,\ldots,p_n-1$, n=1,2,... . Then the mapping f defined by

$f(C(n,j)) = A_n + \frac{j+1}{p_n} t_n$ (i.e. $= C(n+1,0)$ for $j = p_n - 1$ and $= C(n,j+1)$ for $j < p_n - 1$) is strongly c.m. and the set of accumulation points of the trajectory (5) is the set $\{(0,y) : 0 \leq y \leq 1\}$.

For the proof it is sufficient to check that a function u defined by

$$u(C(n,j)) = \frac{1}{2}\|C(n,j)\|^2 + \sum_{i=1}^{n-1} \|t_i\|^2/(2p_i) + j\|t_n\|2/(2p_n^2)$$

is a potential of f. We omit the proof.

Remark 2. The following construction of some strongly c.m. mappings was given in [6].

Let $g_1,\dots,g_m$ be nondecreasing real functions such that $g_1(x)\geq g_2(x)\geq \geq \dots \geq g_m(x)$ for all $x\in R$. We define $f(x_1,\dots,x_m)=(y_1,\dots,y_m)$ so that every $y_i=g_j(x_i)$ for some j, and every g_j is applied to one component. The rule is that g_1 acts on the greatest x_i, g_2 acts on the greatest component but one, etc. Ties are solved so that $y_i\geq y_j$ provided $x_i=x_j$ and $i<j$.

Mappings constructed as above are often used as "social welfare functions" (see e.g. [9]).

Remark 3. The problem of extensions of cyclically monotone mappings related to Theorem 4 is further studied in [12].

Remark 4. The results in the present paper can be extended to the infinite dimensional domain.

References

[1] E.R. CAIANELLO, De LUCA, I. RICCIARDI: Reverbations and control of neural networks, Kybernetik 4(1967).

[2] J.R.P. FRENCH: A formal theory of social power, Psych. Review 63(1956), 181-194.

[3] E. GOLES, J. OLIVOS: Compertement iteratif des fonctions a multiseuil, Information and Control 45(1980), 300-313.

[4] E. GOLES, J. OLIVOS: Compertement periodique des fonctions a seul binaires et applications, Disc. Applied Math. 3(1981).

[5] T. KITAGAWA: Dynamical systems and operators associated with a single neuronic equation, Math. Biosci. 18(1973).

[6] S. POLJAK, D. TURZÍK: On systems, periods and semipositive mappings, Comment. Math. Univ. Carolinae 25(1984), 597-614.

[7] S. POLJAK, D. TURZÍK: An application of convexity to discrete systrems, Discrete Appl. Math, 13(1986), 27-32.

[8] S. POLJAK, M. SŮRA: On periodical behaviour in societies with symmetric influences, Combinatorica 3(1983), 119-121.

[9] F.S. ROBERTS: Discrete Mathematical Models with Application to Social, Biological and Environmental Problems, Prentice-Hall, Englewood Cliffs, N.J. 1976.

[10] R.T. ROCKAFELLAR: Convex Analysis, Princeton University Press, Princeton N.J., 1970.

[11] S. POLJAK, D. TURZÍK: Social influence models with ranking alternatives and local election rules, Math. Soc. Sciences 10(1985), 189-198.

[12] J. PELANT, S. POLJAK: Extensions of cyclically monotone mappings, to appear in Proc. of the Winter School 1985 in Rend. Circ. Mat. Palermo.

J. Pelant: Mathematical Institute, Žitná 25, 115 67 Praha 1, Czechoslovakia

S. Poljak: Dept. of Applied Math., Charles University, Malostranské nám. 25, 118 00 Praha 1, Czechoslovakia

D. Turzík: Institute of Chemical Technology, Dept. of Mathematics, Suchbátarova 3, 166 28 Praha 6, Czechoslovakia

(Oblatum 3.11. 1986)