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Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 2, 221--226

Persistent URL: http://dml.cz/dmlcz/106534

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,2(1987)

ISOMORPHISM DOES NOT IMPLY LOWER BASE-ISOMORPHISM FOR NON-REGULAR OR NON-LOCALLY FINITE-DIMENSIONAL CYLINDRIC ALGEBRAS [1)]

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Abstract: This paper deals with Serény's theorem giving sufficient conditions for two cylindric set algebras to be lower base-isomorphic, a cylindric algebra version of Vaught's theorem on the existence of prime models of atomic theories in countable languages: it is proved that Serény's theorem requires all the conditions given in its statement, and a model-theoretical corollary of this dependence is stated, too.

Key words: Cylindric set algebra, (lower) base-isomorphism, regular, Lf, Dc, base-minimal.

Classification: 03G15

In this note we deal with base-isomorphism of cylindric set algebras (Cs's). We follow the terminology of [HMT], [HMT 81] and [AN 81]. The only exception is that instead of Gothic letters doubly (or sometimes simply) underlined Latin letters are used. Concepts that we do not define here exactly, can usually be found in the monograph [HMT]. An isomorphism between two Cs's is called base-isomorphism iff it is induced by a bijection between their bases. (See Def. 3.1.37 of [HMT], Def. I.3.5 of [HMT 81] or Def. 3.1 of [AN 81].) The following question is investigated in several papers: Under which condition is it true that an isomorphism between two Cs_α's is necessarily a base-isomorphism? (See Thm. I.3.6 of [HMT 81], Lemma 4.3 of [L 85],[N 83], Propositions 3.4 and 3.5 and Problems 3 and 4 of [AN 81].) The above implication is hardly ever true; hence the notion of lower base-isomorphism is introduced in Definition 3.1 of [AN 81]. The notion of lower base-isomorphism is a generalization of the

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1) Research supported by the Hungarian National Scientific Research grant No. 1810.

notion of base-isomorphism. Roughly speaking, two Cs_α's are lower base-isomorphic if omitting the "superfluous" elements of their bases they become base-isomorphic. More precisely, a lower base-isomorphism is a composition of a strong ext-isomorphism, a base-isomorphism and a strong sub-isomorphism. (See Definition 3.1 (vi) of [AN 81].) In [S 83] the following sufficient condition under which two Cs_α's are lower base-isomorphic, is stated. The following Proposition 1 is an algebraic version of Vaught's theorem (Thm. 2.3.4 of [ChK 73]) on the existence of prime models of atomic theories.

Proposition 1. If $\alpha \geq \omega$, $\mathfrak{A}_0$ and $\mathfrak{A}_1 \in Cs_\alpha^{reg} \cap Lf_\alpha$, $\mathfrak{A}_0 \cong \mathfrak{A}_1$, for every $n \in \omega$ $\mathfrak{Nr}_n \mathfrak{A}_0$ is atomic, and $\mathfrak{A}_0$ (and $\mathfrak{A}_1$) are countably generated then $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are lower base-isomorphic; moreover, every isomorphism from $\mathfrak{A}_0$ onto $\mathfrak{A}_1$ is a lower base-isomorphism.

It is asked in [N 83] whether some of the conditions of Proposition 1 can be omitted. We claim that the answer is negative. In more detail, it is proved in [BSh 85] that the condition "$\mathfrak{A}_0$ is countably generated" cannot be omitted even from the first part of Proposition 1, it follows from Theorem 1 of [B 85] that the same holds for the condition "for every $n \in \omega$ $\mathfrak{Nr}_n \mathfrak{A}_0$ is atomic", finally it is proved in [B 86], that the same holds for the condition "$\alpha \geq \omega$". Herein we prove that the same holds for the condition "$\mathfrak{A}_0$ and $\mathfrak{A}_1$ are regular and Lf algebras"; moreover, the condition Lf cannot be replaced by Dc. That is, in Theorem 2 below for every $\alpha \geq \omega$ a pair $(\mathfrak{A}_0, \mathfrak{A}_1)$ of base-minimal $Cs_\alpha \cap Dc_\alpha$'s are constructed such that $\mathfrak{A}_0$ and $\mathfrak{A}_1$ satisfy all the conditions of Proposition 1 except for locally- finiteness, but they are not base-isomorphic, while in Theorem 5 for every $\alpha \geq \omega$ a pair of base-minimal Cs_α's $(\mathfrak{B}_0, \mathfrak{B}_1)$ is constructed such that they satisfy all the conditions but the regularity of Proposition 1 but they are not base-isomorphic. Since the algebras mentioned above are base-minimal and not base-isomorphic, they cannot be lower base-isomorphic. (A Cs_α is called base-minimal iff no Cs_α is strongly sub-isomorphic to it except itself. That is, we cannot obtain an isomorphic Cs_α whenever we omit some elements of its base. See Definition 3.3 (ii) of [AN 81].)

Theorem 2. For every $\alpha \geq \omega$ there exist base-minimal

Cs_α's $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that

(i) $\mathfrak{A}_0 \cong \mathfrak{A}_1$,

(ii) for every $n \in \omega$ $\mathfrak{Nr}_n\mathfrak{A}_0$ is atomic,

(iii) $\mathfrak{A}_0$ and $\mathfrak{A}_1 \in Cs_\alpha^{reg} \cap Dc_\alpha$,

(iv) $\mathfrak{A}_0$ and $\mathfrak{A}_1$ are not base-isomorphic.

Proof. For the sake of simplicity we assume now that $\alpha = \omega\cdot 2$. For other infinite ordinals the construction and proofs are similar.

Set $p_0 = \omega \times 1$, the constant function on ω with range $\{0\}$ and $\mathfrak{C} = \mathfrak{Sb}\,{}^\alpha 3$, the full Cs_α with base 3. Define X_0 and Y as follows: Let

$X_0 = \{f \in {}^\alpha 3 : \omega 1 f \in {}^\omega 2^{(p(0))}\}$ and $Y = \{f \in {}^\alpha 3 : \omega 1 f \in {}^\omega 2\}$.

We let $\mathfrak{A}_0 = \mathfrak{Sg}^{(\mathfrak{C})}\{X_0\}$ and $\mathfrak{A}_1 = \mathfrak{Sg}^{(\mathfrak{C})}\{Y\}$, the cylindric set algebras of dimension α generated by the elements X_0 and Y respectively. We note that the above algebras were constructed in the proof of Proposition 3.5 (iv) in [AN 81]. It was also proved there that they satisfy the above Conditions (i), (iii) and (iv). They are also base-minimal since they are infinite-dimensional Cs's with a finite base. (See the remark after Definition 2.4.61 of [HMT].) So we have to prove only that Condition (ii) holds. First we recall a definition and a lemma from [AN 81] and [HMT].

Definition 3. (See Definitions 1.2 and 1.3.1 of [AN 81] or Definition 3.1.56 of [HMT].)

(i) Let $\mathfrak{A} \in CA_\alpha$. An element $z \in A$ is small iff for every infinite $\Gamma \subseteq \Delta z$ and every finite $\Delta \subseteq \alpha$ there is a finite $\Theta \subseteq \Gamma$ such that

$$c^{\partial}_{(\Theta)} c_{(\Delta)} z = 0.$$

(ii) We set $Dm_\Gamma(\mathfrak{A}) = \{z \in A : \Delta z \sim \Gamma \text{ is finite}\}$.

Lemma 4. (See Lemma 1.3.3 of [AN 81] or Lemma 3.1.59 of [HMT].) Let $\mathfrak{A} \in CA_\alpha$ be generated by a set Z of small elements and suppose that $\Gamma \subseteq \alpha$. Then

$$Dm_\Gamma(\mathfrak{A}) = Sg(Z \cap Dm_\Gamma(\mathfrak{A})).$$

It is proved in the proof of Lemma 3.5 (iv) in [AN 81] that X_0 is small. On the other hand it is easy to see that $\Delta X_0 = \omega$. Hence applying Lemma 4 with $\Gamma = 0$ and $Z = \{X_0\}$ we obtain that each element of $\mathfrak{A}_0$ of finite dimension belongs to the minimal subalgebra of $\underline{\underline{A}}_0$. In other respects it is easy to see that

(1) the minimal subalgebra of an arbitrary Cs of finite dimension is atomic.

These facts prove that (ii) is true. q.e.d. Theorem 2.

Theorem 5. For every $\alpha \geq \omega$ there exist base-minimal Cs_α's $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$ such that they satisfy Conditions (i),(ii) and (iv) of Theorem 2 and Condition

(v) $\underline{\underline{B}}_0 \in Lf_\alpha$.

Proof. The proof of Theorem 5 is similar to that of Theorem 2. We again suppose that $\alpha = \omega \cdot 2$. Let p_0 be the function that was defined in the proof of Theorem 2, let $p_1 = (\alpha \sim \omega) \times 1$ and let $\underline{\underline{D}} = \underline{\underline{Sb}}(^\alpha 2)$. Set

$Z_0 = \{ f \in {}^\alpha 2 : \omega 1 f \in {}^\omega 2^{(p0)} \}$ and

$Z_1 = \{ f \in {}^\alpha 2 : (\alpha \sim \omega) 1 f \in {}^{\alpha \sim \omega} 2^{(p1)} \}$

and finally for $i < 2$ let

$\underline{\underline{B}}_i = \underline{\underline{Sg}}^{(\underline{\underline{D}})} \{ Z_i \}$.

Throughout let i=0 or i=1. It also can be proved that $\underline{\underline{B}}_i$ is base-minimal, since it is also infinite-dimensional Cs with a finite base. (See the remark after Definition 2.4.61 of [HMT].) It is easy to see that

(1) $\Delta Z_i = 0$.

Hence, by Theorem 2.1.5 (i) of [HMT], (v) holds. Set $\underline{\underline{M}}$ to be the minimal subalgebra of $\underline{\underline{D}}$. By (1)

(2) $B_i = \{ m_0 + m_1 \cdot Z_i + m_2 \cdot -Z_i : m_0, m_1 \text{ and } m_2 \in M \}$.

Now we prove the atomicity of the finite-dimensional neat-reducts of $\underline{\underline{B}}_i$. We have

(3) If p, q and $r \in M$ and $x = p + q \cdot Z_i + r \cdot -Z_i = 0$ then $p=q=r=0$.

In fact, suppose the hypotheses of (3). Then, by the theory of Boolean algebras we have $p = q \cdot Z_i = r \cdot -Z_i = 0$. Suppose $q \neq 0$. Then, by 2.1.16, 3.1.71 and 2.3.14 of [HMT], and (1) we have $0 = c_{(\Delta q)} 0 = c_{(\Delta q)} (q \cdot Z_i) = Z_i$. This contradiction proves q=0. Similarly r=0. (3) implies that whenever $m_0 + m_1 \cdot Z_i + m_2 \cdot -Z_i = n_0 + n_1 \cdot Z_i + n_2 \cdot -Z_i$ with m_0, m_1, m_2, n_0, n_1 and $n_2 \in M$ then $m_j = n_j$ for $j < 3$. Hence

(4) if $X = m_0 + m_1 \cdot Z_i + m_2 \cdot -Z_i$ with m_0, m_1 and $m_2 \in M$ then $\Delta X = \Delta m_0 \cup \Delta m_1 \cup \Delta m_2$.

Now, let $n \in \omega$ and $x \in Nr_n\underline{\underline{B}}_0 \sim \{0\}$. Then, by (2) and (4), $x=m_0+m_1 \cdot Z_0+$ $+m_2 \cdot -Z_0$ for some m_0, m_1 and $m_2 \in Nr_n\underline{\underline{M}}$. We have either $m_0 \neq 0$ or $m_1 \neq 0$ or $m_2 \neq 0$. Suppose $m_1 \neq 0$. By the statement (1) of the proof of Theorem 2 take $a \in AtNr_n\underline{\underline{M}}$ such that $a \leq m_1$. Then $a \cdot Z_0 \in AtNr_n\underline{\underline{B}}_0$ and $a \cdot Z_0 \leq$ $\leq x$. The other two cases can be treated similarly. Hence (ii) holds. By (1), (2) and (3)

$$\{(m_0+m_1 \cdot Z_0+m_2 \cdot -Z_0,\ m_0+m_1 \cdot Z_1+m_2 \cdot -Z_1) : m_0, m_1 \text{ and } m_2 \in M\}$$

is an isomorphism from $\underline{\underline{B}}_0$ onto $\underline{\underline{B}}_1$. Thus (i) holds. Finally, $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$ are not base-isomorphic since neither $2 \upharpoonright Id$ nor the permutation (12) induces an isomorphism between $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$.

q.e.d. Theorem 5.

Corollary 6. Neither of the conditions "$\underline{\underline{A}}_0$ and $\underline{\underline{A}}_1$ are regular" and "$\underline{\underline{A}}_0$ is Lf" can be omitted even from the first part of Proposition 1; moreover, Lf cannot be replaced by Dc there.

Proof: By Theorems 2 and 5 and the remark before Theorem 2.

q.e.d. Corollary 6.

Corollary 7. Vaught's theorem on prime models cannot be extended to infinitary languages.

Proof By Corollary 6 herein and Section 4.3 of [HMT].

q.e.d. Corollary 7.

Acknowledgement

We thank H. Andréka and R. Thompson for their precious help in preparing the paper.

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(Oblatum 12.1. 1987)