Commentationes Mathematicae Universitatis Carolinae

Karel Horák; Vladimír Müller

On the structure of commuting isometries

Commentationes Mathematicae Universitatis Carolinae, Vol. 28 (1987), No. 1, 165--171

Persistent URL: `http://dml.cz/dmlcz/106519`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
28,1 (1987)

ON THE STRUCTURE OF COMMUTING ISOMETRIES

Karel HORÁK and Vladimír MULLER

Abstract: We give two examples disproving a Slociński's conjecture about the structure of two commuting isometries.

Key words: Commuting isometries, Wold decomposition, unilateral shift.

Classification: 47D05

Let V be an isometry acting on a separable (complex) Hilbert space H. By the well-known Wold theorem H can be decomposed into the orthogonal sum $H = H_1 \oplus H_2$ where H_1 and H_2 reduce V, $V|H_1$ is unitary, and $V|H_2$ is a unilateral shift. For a pair of commuting isometries the situation is much more complicated. This was studied in a series of papers [6], [9], [12], [1], [11], [7], [2] but satisfactory results were obtained only in the case of a pair $V_1, V_2 \in B(H)$ of doubly commuting isometries ($V_1V_2 = V_2V_1$, $V_1V_2^* = V_2^*V_1$, see [10], [7]). In this case space H can be decomposed into the orthogonal sum of four subspaces

(1) $$H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$$

such that all the summands reduce both V_1 and V_2, $V_1|H_{uu}\oplus H_{us}$ and $V_2|H_{uu}\oplus H_{su}$ are unitary, $V_1|H_{su}\oplus H_{ss}$ and $V_2|H_{us}\oplus H_{ss}$ are unilateral shifts.

The more detailed structure of these subspaces is described in [2].

In [11], Slociński suggested to study pairs of commuting isometries satisfying the following property (we call such isometries compatible).

Definition. Let V_1, V_2 be commuting isometries on a separable Hilbert space H. We say that V_1 and V_2 are compatible if $P_1(m)$ commutes with $P_2(n)$ for every positive integers m, n, where $P_i(m)$ is the orthogonal projection onto the range of V_i^m $(i = 1,2)$.

From further description of summands in the Wold-type decomposition (1) of a pair of doubly commuting isometries it is easy to see that

$$P_1(m)\ P_2(n) = P_2(n)\ P_1(m) = P(m,n)$$

for any positive integers m, n, where P(m,n) is the orthogonal projection onto the range of $V_1^m V_2^n$. This means that any two doubly commuting isometries are compatible but the converse is not true:

Example 1. Let $S \subset Z \times Z$ be a non-void set of pairs of integers such that $(i,j) \in S$ implies $(i+1,j) \in S$ and $(i,j+1) \in S$. Let H_S be a Hilbert space with an orthonormal basis $\{e_s: s \in S\}$. Define isometries $V_1(S)$, $V_2(S) \in B(H_S)$ by $V_1(S)e_{ij} = e_{i+1,j}$, $V_2(S)e_{ij} = e_{i,j+1}$.

Clearly, $V_1(S)$ and $V_2(S)$ are compatible isometries but in general they are not doubly commuting. If for example $(0,1) \in S$, $(1,0) \in S$ and $(0,0) \notin S$ then $V_2(S)^* V_1(S)e_{01} = e_{10}$ and $V_1(S)V_2(S)^* e_{01} = 0$.

As the property of compatibility means some sort of orthogonality, the preceeding example suggests the possibility of some model for compatible isometries. In [11] Slociński conjecured that for any two compatible isometries V_1, $V_2 \in B(H)$ the space H can be decomposed into the orthogonal sum $H = \bigoplus_{i=1}^{\infty} H_i$ of subspaces reducing both V_1 and V_2 such that $V_1|H_i$ and $V_2|H_i$ are unitarily equivalent to a pair $V_1(S)$, $V_2(S)$ for some S (see Example 1).

The aim of this note is to disprove the Slociński's conjecture. We exhibit two examples showing difficulties which arise in the study of compatible isometries. Although the Slociński's conjecture is not true we hope that these two examples will enable to construct some canonical model for compatible isometries similar to the theory of multiplicity for normal operators.

Let S, $V_1(S)$ and $V_2(S)$ be as in Example 1. Suppose that both $V_1(S)$ and $V_2(S)$ are unilateral shifts, i.e. they contain no unitary part. Let $(i,j) \in S$. Note that $x = e_{ij}$ has the following properties:

(2) For every $k \geq 0$ there exists $n_k \geq 0$ such that

$$V_2(S)^k x \in V_1(S)^{n_k} H_S \ominus V_1(S)^{n_k+1} H_S$$

(in fact n_k is the integer satisfying $(i-n_k, j+k) \in S$ and $(i-n_k-1, j+k) \notin S$),

(3) if $V_2(S)^k x \in V_1(S)^r H_S$ for some $k > 0$, $r \geq 0$ then

$$(x, V_1(S)^{*r} V_2(S)^k x) = 0.$$

Analogously for $V_1(S)^k x \in V_2(S)^r H_S$. These properties will be used later.

<u>Example 2</u>. Let M be a separable Hilbert space and $U \in B(M)$ be a unitary operator which contains no bilateral shift (i.e. there is no subspace which reduces U to a bilateral shift). Put $H = \bigoplus_{i=0}^{\infty} M_i$, $M_i = M$ $(i \geq 0)$, and define isometries V_1, $V_2 \in B(H)$ by

$$V_1(x_0, x_1, \dots) = (0, x_0, x_1, \dots),$$

$$V_2(x_0,x_1,\dots) = (0,Ux_0,Ux_1,\dots).$$

Clearly, V_1 and V_2 are commuting unilateral shifts which are compatible as $V_1^n H = V_2^n H$ for every $n \geqq 0$.

Suppose that there exists a subspace $H' \subset H$ reducing both V_1 and V_2 such that the pair $(V_1|H', V_2|H')$ is unitarily equivalent to $(V_1(S), V_2(S))$ for some S. Then there exists $x \in H' \subset H$, $x \neq 0$, with property (3). In particular, $(x, V_2^{*n} V_1^n x) = 0$ and $(x, V_1^{*n} V_2^n x) = 0$ for every $n > 0$. Taking $H'' = \bigvee\{\dots, V_2^* V_1 x, x, V_1^* V_2 x, V_1^{*2} V_2^2 x, \dots\}$ and using the relations

$$V_1 V_1^* = V_2 V_2^*, \; V_1^* V_2 V_1^* = V_1^{*2} V_2$$

we find that $V_1^* V_2 | H''$ is a bilateral shift. On the other hand, $V_1^* V_2(x_0,x_1,\dots) = (Ux_0, Ux_1,\dots)$, hence $V_1^* V_2$ is an orthogonal sum of countably many copies of U which was supposed not to contain a bilateral shift. By theory of multiplicity (see [3]) $V_1^* V_2$ does not contain a bilateral shift as well, a contradiction.

<u>Example 3</u>. Let $S = \{(i,j) \in Z \times Z : j \geqq 0\}$. For $x \in \langle 0,1)$ let $d_k(x)$ be the binary digits of $x = \sum_{k=1}^{\infty} d_k(x) 2^{-k}$ (for the sake of uniqueness we exclude the case $d_n(x) = d_{n+1}(x) = \dots = 1$ for some n). For $(i,j) \in S$ define

$$B_{ij} = \{x \in \langle 0,1) : i + \sum_{k=1}^{j} d_k(x) \geqq 0\}.$$

Let H be the Hilbert space of all matrices $f = (f_{ij})_{(i,j) \in S}$ of functions $f_{ij} \in L^2(\langle 0,1))$, $\operatorname{supp} f_{ij} \subset B_{ij}$, with the norm $|f|^2 = \sum_{(i,j) \in S} |f_{ij}|^2$. As usual, we identify functions which differ only on a set of zero Lebesgue measure m, and all the inclusions are to be understood in this way (for example $\operatorname{supp} f_{ij} \subset B_{ij}$ means that $m(\{x \in \langle 0,1) : x \notin B_{ij}, f(x) \neq 0\}) = 0$). For $f = (f_{ij}) \in H$ define

$$(V_1f)_{ij} = f_{i-1,j}, \quad (V_2f)_{ij} = f_{i,j-1}.$$

Obviously, V_1 and V_2 are commuting isometries. Further

$$(4) \qquad V_1^nH = \{(f_{ij})_{(i,j)\in S}: \text{supp } f_{ij} \subset B_{i-n,j}\},$$
$$V_2^mH = \{(f_{ij})_{(i,j)\in S}: \text{supp } f_{ij} \subset B_{i,j-m}\},$$

which easily gives that V_1 and V_2 are compatible, and

$$(5) \qquad V_1^nH \ominus V_1^{n+1}H = \{(f_{ij}): \text{supp } f_{ij} \subset B_{i-n,j} - B_{i-n-1,j}\}$$

where

$$B_{i-n,j} - B_{i-n-1,j} = \{x \in \langle 0,1): i - n + \sum_{r=1}^{j} d_r(x) = 0\}.$$

Suppose that there exists a subspace $H' \subset H$ reducing both V_1 and V_2 such that the pair $(V_1|H', V_2|H')$ is unitarily equivalent to $(V_1(S), V_2(S))$ for some S. Then there exists $x \in H' \subset H$, $x \neq 0$, with property (2). Let $x = (f_{ij})_{(i,j)\in S}$ and i', j' be fixed indices such that $f_{i'j'} \neq 0$. For $k \geq 1$ let n_k be such that

$$V_2^kx \in V_1^{n_k}H \ominus V_1^{n_k+1}H$$

(see (2)). Then (5) gives

$$\text{supp } f_{i'j'} \subset B_{i'-n_k, j'+k} - B_{i'-n_k-1, j'+k} =$$
$$= \{x \in \langle 0,1): i' - n_k + \sum_{r=1}^{j'+k} d_r(x) = 0\}.$$

This inclusion with the analogical condition for $k + 1$

$$\text{supp } f_{i'j'} \subset \{x \in \langle 0,1): i' - n_{k+1} + \sum_{r=1}^{j'+k+1} d_r(x) = 0\}$$

gives the inclusion

$$\text{supp } f_{i'j'} \subset \{x \in \langle 0,1): d_{j'+k+1}(x) = n_{k+1} - n_k\}.$$

Therefore

$$\text{supp } f_{i'j'} \subset \bigcap_{k=1}^{\infty} \{x \in \langle 0,1): d_{j'+k+1}(x) = n_{k+1} - n_k\}$$

and $m(\text{supp } f_{i'j'}) = 0$, hence $f_{i'j'} = 0$ a.e., a contradiction.

R e f e r e n c e s

1 D. GAŞPAR, N. SUCIU: On the structure of isometric semigroups, Operator Theory: Adv. and Appl. 14, Birkhäuser Verlag Basel, 1984, 125-139.

2 D. GAŞPAR, N. SUCIU: Intertwinings of isometric semigroups and Wold type decompositions, Sem. de operatori liniari şi analiză armonică, No.3, 1985.

3 P.R. HALMOS: Introduction to Hilbert space and the theory of spectral multiplicity, Chelsea Publishing Comp. New York, 1951.

4 P.R. HALMOS: Shifts on Hilbert spaces, J. reine und angew. Math. 208(1961), 102-112.

5 H. HELSON: Lectures on invariant subspaces, Academic Press, New York and London, 1964.

6 H. HELSON, D. LOWDENSLAGER: Prediction theory and Fourier series in several variables II, Acta Math. 106(1961), 175-213.

7 G. KALLIAMPUR, V. MANDREKAR: Nondeterministic random fields and Wold and Halmos decompositions for commuting isometries, Prediction Theory and Harmonic Analysis, North Holland Publ. Comp., 1983, 165-190.

8 M. KOSIEK: Lebesgue-type decompositions of a pair of commuting Hilbert space operators, Bull. Acad. Polon. Sci., Ser. Math. 27(1979), 583-589.

9 P.S. MUHLY: A structure theory for isometric representations of a class of semigroups, J. Reine und angew. Math. 225(1972), 135-154.

10 M. SLOCIŃSKI: On the Wold-type decomposition of a pair of commuting isometries, Annales Polonici Math. 37(1980), 255-262.

11 M. SLOCIŃSKI: Models of commuting contractions and isometries, Report of the 11th Conference on Operator Theory Bucharest 1986 (to appear).

12 I. SUCIU: On the semigroups of isometries, Studia Math. 30

(1968), 101-110.

Matematický ústav ČSAV
Žitná 25, 115 67 Praha 1
Czechoslovakia

(Oblatum 20.10. 1986)