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RANDOM SEMINORMED SPACES

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Abstract: This article deals with a very important case of statistical linear space. Properties of this special case lead to the definition of a random seminormed space.

Key words: Statistical metric space, statistical linear space, random seminorm, ε-η-topology, t-norm.

Classification: 60B99

Basic properties of statistical linear spaces in the sense of Menger (SLM-spaces) are given in [1], [2]. These spaces are special cases of statistical metric spaces introduced by Sklar and Schweizer in [3]. For simplicity we present the definition of a statistical linear space (SLM-space) here too. First we need the notion of a t-norm. A function $T: \langle 0,1\rangle \times \langle 0,1\rangle \to \langle 0,1\rangle$ satisfying

(a) $T(a,b) = T(b,a)$; $T(a,1) = a$ for $a > 0$

(b) $T(a,b) \le T(c,d)$ for $a \le c$, $b \le d$

(c) $T(T(a,b),c) = T(a,T(b,c))$

(d) $T(0,0) = 0$

will be called a t-norm.

Definition 1. Let S be a real linear space, let $\mathcal{F}$ be the set of all probability distribution functions defined on the real line R_1. Let $\mathcal{G}: S \to \mathcal{F}$ be a given mapping. For every $x \in S$ let us denote $\mathcal{G}(x) = F_x \in \mathcal{F}$ and we demand that $\mathcal{G}$ satisfies:

1. $x = 0 \Leftrightarrow F_x = H$ where $H(u) = 0$ $u \le 0$; $H(u) = 1$ $u > 0$
2. $F_{\lambda x}(u) = F_x(u/|\lambda|)$ for every $x \in S$ and every $\lambda \ne 0$.
3. $F_x(u) = 0$ for every $u \le 0$ and every $x \in S$.
4. $T(F_x(u), F_y(v)) \le F_{x+y}(u+v)$ for every $u, v \in R_1$ and every pair $x, y \in S$ where T is t-norm satisfying (a), (b), (c), (d).

Under these conditions the triple (S,G,T) is called a linear statistical space in the Menger sense (SLM-space). We shall start with construction of a special SLM-space. Let S be a linear space of all real sequences $x = \{x_i\}_{i=1}^{\infty}$ where linear operations of addition and scalar multiplication are defined coordinatewise, let $a = \{a_i\}_{i=1}^{\infty}$ be a sequence of positive reals such that $\sum_1^{\infty} a_i = 1$. Let F be the set of all probability distribution functions on reals. Let us define a mapping $\mathcal{V}: S \to F$ in the following way:

if $x = \{x_i\}_{i=1}^{\infty}$ then we put

$$\mathcal{V}_x(u) = 0 \qquad \text{for } u \le |x_1|$$

$$\mathcal{V}_x(u) = a_1 \qquad \text{for } |x_1| < u \le |x_1| + |x_2|$$

$$\vdots$$

$$\mathcal{V}_x(u) = \sum_{i=1}^{n} a_i \ \text{ for } \sum_{i=1}^{n} |x_i| < u \le \sum_{i=1}^{n+1} |x_i| .$$

In case $\sum^{\infty} |x_i| < \infty$ we must distinguish two possibilities:

a) $\sum_{i=1}^{\infty} |x_i|$ contains infinitely many non-zero elements then

$$\mathcal{V}_x(u) = 1 \text{ for } u \ge \sum_1^{\infty} |x_i|$$

b) $\sum_{i=1}^{\infty} |x_i|$ contains finitely many non-zero elements only then

$$\mathcal{V}_x(u) = 1 \text{ for } u > \sum_1^{\infty} |x_i| .$$

<u>Theorem 1.</u> The triple $(S,\mathcal{V},\min)$ is SLM-space with the t-norm $T(a,b) = \min(a,b)$.

Proof. See [1].

Now, using the mapping $\mathcal{V}$ and the norm min we can introduce a linear topology into S, the ε-η-topology, and as it is shown in [1], [2] $(S,\mathcal{V},\min)$ is a locally convex metrizable linear topological space with the base of neighborhoods of the null element

$$\{O(\varepsilon,\eta) = \{x \in S: \mathcal{V}_x(\eta) > 1 - \varepsilon\},\ 0<\varepsilon\le 1,\ \eta>0\}.$$

<u>Theorem 2.</u> In SLM-space $(S,\mathcal{V},\min)$ the ε-η-topology is equivalent to the coordinatewise convergence.

Proof. Let $x_n \to 0$ in the ε-η-topology, $x_n = \{x_{ni}\}_{i=1}^{\infty}$; it means $(\forall \varepsilon \in (0,1> \forall \eta>0\ \exists n_0 \in N\ \forall n \ge n_0) \Rightarrow (x_n \in O(\varepsilon,\eta)) \Leftrightarrow (\mathcal{V}_{x_n}(\eta) > 1 - \varepsilon)$. As $\sum_1^{\infty} a_i = 1$ then for every $n_1 \in N$ there exists $0 < \varepsilon < 1$ such that $1 - \varepsilon > \sum_{i=1}^{n_1} a_i$ and hence $\mathcal{V}_{x_n}(\eta) > \sum_1^{n_1} a_i$. It follows from the construction of $\mathcal{V}$ that

$$\sum_{i=1}^{n} |x_{ni}| < \eta$$

for every $n \geq n_0$. But this inequality says that

$$x_{ni} \underset{n\to\infty}{\to} 0 \text{ for every } i \in N.$$

Conversely, let $x_n \to 0$ coordinatewise , i. e. $\lim_{n\to\infty} x_{ni} = 0$ for every $i \in N$. Let fix $i_0 \in N$ and let choose for arbitrarily chosen $\varepsilon > 0$ such a number $n_0(i_0) \in N$ that for every $n \geq n_0(i_0)$ $|x_{ni}| < \varepsilon$ and hence

$$\sum_{i=1}^{i_0} |x_{ni}| < \varepsilon\, i_0 \text{ for every } n \geq \max_{1\leq i\leq i_0} \{n_0(i)\}.$$

According to the construction of V we obtain that

$$Vx_n(i_0\varepsilon) > \sum_{i=1}^{i_0} a_i \qquad \text{for } n \geq \max_{1\leq i\leq i_0} \{n_0(i)\}.$$

With respect to the arbitrariness of ε and i_0 it implies that $x_n \to 0$ in the ε-η-topology. Q.E.D.

Further,we can remind without proofs that SLM-space $(S,V,\min)$ is a complete topological space and the subset $S^* \subset S$ formed by all sequences with finite length, i. e.

$$S^* = \{x \in S:\ x = \{x_1,x_2,x_3,\dots,x_n,0,0,0,\dots\}\},$$

can be identified with the topological dual space. The construction of the mapping V is based on choice of a sequence $\{a_n\}_{n=1}^{\infty}$. The equivalence between the ε-η-topology and the coordinatewise convergence yields that the choice of $\{a_n\}_{n=1}^{\infty}$ is not so important because all ε-η-topologies generated by all possible sequences $a = \{a_n\}_1^{\infty}$ are mutually equivalent.

Let $x = \{x_n\}_{n=1}^{\infty}$ be an arbitrary point of S, let $n \in N$, we can write

$$x = \sum_{i=1}^{n} x_i e_i + x^{(n)}$$

where $e_i = (0,0,\dots,0,1,0,\dots) \in S$. Unfortunately, we cannot write

$$x = \sum_1^{\infty} x_i e_i$$

because we do not know in which sense the convergence of this series could be understood. So, we can understand this convergence in a probabilistic sense, namely, instead of the basic vectors $\{e_i\}_{i=1}^{\infty}$ we shall consider random variables $\{\xi_i\}_{i=1}^{\infty}$ such that for every $x \in S$ the series $\sum_1^{\infty} \xi_i x_i$ will be absolutely convergent in the sense almost surely.

<u>Theorem 3.</u> Let SLM-space $(S,V,\min)$ be given. We can construct a sequence $\{\xi_i\}_{i=1}^{\infty}$ of random variables such that for every $x = \{x_i\}_{i=1}^{\infty} \in S$ the series

$$\xi_x = \sum_1^\infty x_i \xi_i$$

is absolutely convergent a. s. and

$$\Psi_x(u) = P\{\omega: \sum_1^\infty |x_i| \xi_i(\omega) < u\}$$

for every real u.

Proof. Let us consider the sequence of vectors $\{e_i\}_{i=1}^\infty$ where $e_1 = (1,0,0,\dots)$, $e_2 = (0,1,0,\dots)$, $e_3 = (0,0,1,0,\dots),\dots$ The mapping Ψ assigns to e_n the probability distribution function F_n of the form

for $n = 1$	$F_1(u) = 0$	for $u \le 1$
	$F_1(u) = 1$	for $u > 1$
for $n \ge 2$	$F_n(u) = 0$	for $u \le 0$
	$F_n(u) = \sum_{j=1}^{n-1} a_j$	for $0 < u \le 1$
	$F_n(u) = 1$	for $u > 1$.

Further, for every k-tuple $(e_{n_1}, e_{n_2}, \dots, e_{n_k})$ we shall define the common probability distribution function by

$$F_{n_1,n_2,\dots,n_k}(u_1,u_2,\dots,u_k) = \min_{1\le j\le k} \{F_{n_j}(u_j)\}.$$

This system of probability distribution functions satisfies Kolmogorov's consistence conditions and hence we can construct a sequence $\{\xi_n\}_{n=1}^\infty$ of random variables which satisfy (for every $n \in N$)

$$P\{\omega: \bigcap_{i=1}^n \{\omega: \xi_i(\omega) < u_i\}\} = F_{1,2,\dots,n}(u_1,u_2,\dots,u_n) = \min_{1\le i\le n} F_i(u_i).$$

Let $x = \{x_i\}_{i=1}^\infty \in S$ be quite arbitrary and let us consider random variables

$$\sum_{i=1}^n x_i \xi_i(\omega), \qquad \sum_{i=1}^n |x_i| \xi_i(\omega).$$

We shall prove that the sequence $\{\sum_{i=1}^n |x_i| \xi_i(\omega)\}_{n=1}^\infty$ is fundamental in probability. Surely,

$$P\{\omega: |\sum_{i=1}^{k+n} |x_i| \xi_i(\omega) - \sum_{i=1}^n |x_i| \xi_i(\omega)| < \eta\} = P\{\omega: \sum_{i=n+1}^{n+k} |x_i| \xi_i < \eta\} \ge$$

$$\ge P\{\omega: \xi_{n+i}(\omega) = 0,\ i=1,2,\dots,k\} = P\{\omega: \xi_{n+i}(\omega) < 1,\ i=1,2,\dots,k\} =$$

$$= \min_{1\le i\le k} \{F_{n+i}(1)\} = \sum_{j=1}^n a_j \nearrow 1 \text{ if } n \to \infty \text{ for every } k \in N.$$

A random variable $\xi_{|x|}(\omega) = P - \lim_{n\to\infty} \sum_{i=1}^n |x_i| \xi_i(\omega)$ must exist. As for every $n \in N$

$$\sum_{i=1}^n |x_i| \xi_i(\omega) \le \sum_{i=1}^{n+1} |x_i| \xi_i(\omega) \quad \text{a.s.}$$

then $\xi_{|x|}(\omega) = \lim_{n\to\infty} \sum_{i=1}^n |x_i| \xi_i(\omega)$ a. s. and we can write

$\xi_{|x|}(\omega) = \sum_{n=1}^{\infty} |x_i|\xi_i(\omega)$, too.

We have proved that for every $x \in S$ there exists a sum $\xi_x(\omega) = \sum_1^{\infty} x_i \xi_i(\omega)$ satisfying

$$|\xi_x(\omega)| \le \xi_{|x|}(\omega) \quad \text{a. s.}.$$

The common probability distribution function $F_{1,2,3,\dots,n}(u_1,u_2,\dots,u_n)$ gives the probability distribution function G_n for the random variable

$$\xi_{|x|_n}(\omega) = \sum_{i=1}^{n} |x_i|\xi_i(\omega).$$

It can be easily shown that

$$P\{\omega\colon \xi_i(\omega) = \rho_i,\ i = 1,2,\dots,n\} = 0 \quad (\rho_i = 0 \text{ or } 1)$$

for every n-tuple $(\rho_1,\rho_2,\dots,\rho_n)$ if $\rho_i < \rho_{i+k}$ at least for one pair. In other words,

$P\{\omega\colon \xi_i(\omega) = \rho_i,\ i=1,2,\dots,n\} > 0$ only for the combinations:

$$\begin{array}{ll} \rho_1 = 1,\ \rho_i = 0 & i = 2,3,\dots,n \\ \rho_1 = 1,\ \rho_2 = 1,\ \rho_i = 0 & i = 3,4,\dots,n \\ \vdots & \\ \rho_1 = 1,\ \rho_2 = 1,\dots,\rho_{n-1} = 1,\ \rho_n = 0 & \\ \rho_1 = 1,\ \rho_2 = 1,\dots,\rho_{n-1} = 1,\ \rho_n = 1. & \end{array}$$

From this fact we can easily derive that (under assumption $|x_j| > 0$ for simplicity)

$$P\{\omega\colon \sum_1^n |x_j|\xi_j(\omega) = \sum_{j=1}^{i} |x_j|\} =$$

$= P\{\omega\colon \xi_1(\omega) = \xi_2(\omega) = \dots = \xi_i(\omega) = 1,\ \xi_{i+1}(\omega) = \dots = \xi_n(\omega) = 0\} = a_i$.

So $G_n(u) = \sum_{j=1}^{i} a_j$ for $\sum_{j=1}^{i} |x_j| \le u < \sum_{j=1}^{i+1} |x_j|$.

As $\sum_{i=1}^{n} |x_i|\xi_i(\omega) \nearrow \sum_{i=1}^{\infty} |x_i|\xi_i(\omega)$ a. s., $G_n(u) \underset{n\to\infty}{\to} G(u)$ weakly, where G is the probability distribution function belonging to $\sum_1^{\infty} |x_i|\xi_i$.

We obtain that

$$P\{\omega\colon \sum_{i=1}^{\infty} |x_i|\xi_i(\omega) < u\} = \mathcal{Y}_x(\omega). \quad \text{Q.E.D.}$$

We constructed a mapping $x \overset{T}{\to} \xi_x$ which to every $x \in (S,\mathcal{Y},\min)$ assigns a random variable $\xi_x = \sum_1^{\infty} x_i \xi_i$. At the first sight we see the mapping T is a linear and one-to-one mapping because

$$\xi_{\lambda x+\mu y} = \sum_1^{\infty}(\lambda x_i + \mu y_i)\xi_i = \lambda\sum_1^{\infty} x_i\xi_i + \mu\sum_1^{\infty} y_i\xi_i = \lambda\xi_x + \mu\xi_y$$

and the equality $\sum_1^{\infty} x_i\xi_i(\omega) = \sum_1^{\infty} y_i\xi_i(\omega)$ a. s. implies that $x_i = y_i$ for every $i \in N$ thanks to that fact that

$$P\{\omega\colon \xi_i(\omega) = 0,\ \xi_{i+k}(\omega) = 1\} = 0$$

for every $i \in N$ and every $k \in N$.
All random variables ξ_x for $x \in (S, \mathcal{Y}, \min)$ form a linear set which is closed with respect to convergence a. s. That follows from the decomposition $P(\bigcup_{i=1}^{\infty} A_i) = 1$ if we define

$$A_i = \{\omega: \xi_1(\omega) = \ldots = \xi_i(\omega) = 1, \ \xi_{i+1}(\omega) = 0 \ \forall k \in N\};$$

then $P(A_i) = a_i$. Now, we can easily prove that the mapping T is topological mapping because convergence $x_n \to 0$ in the ε-η-topology implies convergence $\xi_{x_n} \to 0$ a. s. and vice versa. Further, the convergence in probability of a sequence $\{\xi_{x_n}\}$ implies convergence in the sense a. s.

We proved in Theorem 3 the absolute convergence of $\sum_{i=1}^{\infty} x_i \xi_i$ a. s., i. e. for every $x \in (S, \mathcal{Y}, \min)$

$$\xi_{|x|}(\omega) = \sum_{i=1}^{\infty} |x_i| \xi_i(\omega) \text{ exists a. s..}$$

This random variable will be called a random seminorm defined on S. We are motivated by the following properties of $\xi_{|x|}$:

1. $\xi_{|x|}(\omega) \geq 0$ a. s., $\xi_{|x|}(\omega) = 0$ a. s. if and only if $x = 0$ in S
2. $\xi_{|\lambda x|}(\omega) = |\lambda| \xi_{|x|}(\omega)$ a. s.
3. $\xi_{|x+y|}(\omega) \leq \xi_{|x|}(\omega) + \xi_{|y|}(\omega)$ a. s.

Let Φ denote all real functions defined on S, i. e.

$\Phi = \{f;\ f: S \to R_1\}$. To every measurable cylinder set

$C = \{f \in \Phi;\ [f(x_1), f(x_2), \ldots, f(x_n)] \in B_n,\ x_i \in S,\ B_n \in \mathcal{B}_n\}$

we can assign a nonnegative number

$$\mu(C) = P\{\omega: [\xi_{|x_1|}(\omega), \xi_{|x_2|}(\omega), \ldots, \xi_{|x_n|}(\omega)] \in B_n\}.$$

In this way we define a set function μ on the algebra of all measurable cylinder sets in Φ that can be under certain conditions enlarged on Kolmogorov's σ-algebra in Φ into a probability measure. It is clear that in the construction of the set function μ we are not limited by a special case of the linear space S. Let L be any real linear set and let Φ_L be the function space defined on L, i. e. $\Phi_L = \{f;\ f: L \to R_1\}$. Let K_Φ be the smallest σ-algebra of subsets in Φ_L with respect to which every x f $\to$ f(x) becomes measurable. If μ is a probability measure defined on K_Φ then the triple (Φ_L, K_Φ, μ) forms the underlying probability space.

<u>Definition 2.</u> Let L be any real linear set, let (Φ_L, K_Φ, μ) be the probability space derived from L, let N be the subset of all seminorms on L. The triple (Φ_L, K_Φ, μ) is said to be a random semi-

normed space if there exists a probability measure ν_μ on the σ-algebra $\{N \cap A: A \in K_\Phi\}$ such that

$$\nu_\mu(N \cap A) = \mu(A) \text{ for every } A \in K_\Phi.$$

It is clear that (Φ_L, K_Φ, μ) is a random seminormed space if and only if $\mu^*(N) = 1$ where μ^* is the outer measure derived from μ. Similarly, as it is done in [4] in case of a random metric space we can give very simple necessary and sufficient conditions for the existence of a random seminormed space.

<u>Theorem 4.</u> Necessary and sufficient conditions for (Φ_L, K_Φ, μ) to be a random seminormed space in a linear set L are

(1) $\mu\{f \in \Phi_L: f(x) \geq 0\} = 1$ for every $x \in L$

(2) $\mu\{f \in \Phi_L: f(\lambda x) = |\lambda| f(x)\} = 1$ for every $x \in L$ and every $\lambda \in R_1$

(3) $\mu\{f \in \Phi_L: f(x+y) \leq f(x) + f(y)\} = 1$ for every $x, y \in L$.

Proof. The proof of Theorem 4 can be omitted because that is a simple application of Theorem 1 in [5] by aid of an obvious fact that the property of real valued functions on L to be a seminorm is extensible, hereditary and measurable with respect to the σ-directed covering class of all finite or countable-dimensional linear subspaces in L. Q.E.D.

At this situation, it is necessary to verify that a special case of a random seminormed space considered in $(S, \mathcal{V}, \min)$ is in accordance with Definition 2. It is sufficient to verify demands (1), (2), (3) in Theorem 4. Every random variable $\xi_{|x|}$ derived from $(S, \mathcal{V}, \min)$ in S takes a. s. at most countably many values forming the series $|x_1|$, $|x_1|+|x_2|, \ldots, \sum_{j=1}^{n} |x_j|, \ldots$ (if $x = \{x_i\}_{i=1}^{\infty}$). All members of this series are seminorms on S and this fact implies validity of (1), (2), (3) in this special case.

As it was remembered before, the topological dual space to $(S, \mathcal{V}, \min)$ is the subset of all vectors in S with a finite length. Now, our aim is a construction of a random seminorm in this dual space S*. On the basis of an analogy with Banach spaces we shall consider for $f \in S^*$, $f(x) = \sum_{i=1}^{M} f_i x_i$,

$$\sup_{\{x \in S: \xi_{|x|}(\omega)=1\}} \left\{\frac{|f(x)|}{\xi_{|x|}(\omega)}\right\} = \sup_{\{x \in S: \xi_{|x|}(\omega)=1\}} \left\{\left|\sum_{i=1}^{M} f_i x_i\right|\right\}.$$

Theorem 5. For every $f \in S^*$ there exists a random variable $\eta_{|f|}$ defined on the same probability space as all random variables $\{\xi_i\}_{i=1}^{\infty}$ such that

$$P\{\omega:\eta_{|f|}(\omega) = \sup_{\{x\in S:\xi_{|x|}(\omega)=1\}} \{|f(x)|\}\} = 1$$

and

$$P\{\omega:\eta_{|f|}(\omega) = \max\{|f_1|,|f_2|,\dots,|f_M|\}\} = \sum_{j=M}^{\infty} a_j$$

$$P\{\omega:\eta_{|f|}(\omega) = \infty\} = \sum_{j=1}^{M-1} a_j$$

if $f(x) = \sum_{i=1}^{M} f_i x_i$, $f_M \neq 0$.

Proof. Let (Ω,σ,P) be a probability space where all random variables $\{\xi_i\}_{i=1}^{\infty}$ are defined. We can easily derive from the properties of common probability distribution function of $\{\xi_i\}_{i=1}^{\infty}$ that Ω can be decomposed into

$$\Omega = \bigcup_{i=1}^{\infty} A_i \cup \Omega_0 \cup \{0\} \cup A$$

where $A_i = \{\omega \in \Omega:\xi_1(\omega)=\dots=\xi_i(\omega)=1,\ \xi_{i+k}(\omega) = 0 \text{ for every } k \leq 1\}$

$\Omega_0 = \{\omega \in \Omega: \exists j \in N \text{ such that } \xi_j(\omega)=0, \xi_{j+\ell}(\omega)=1 \text{ for some } \ell \in N\}$

$\{0\} = \{\omega \in \Omega:\ \xi_i(\omega) = 0 \text{ for every } i \in N\}$

$A = \{\omega \in \Omega:\ \xi_i(\omega) = 1, \forall i\}.$

All these sets are σ-measurable and $P\{A_i\} = a_i$, $P\{\Omega_0\} = P\{\{0\}\} = P\{A\} = 0$. Now, let $\omega \in A_i$, then for every $x \in S$, $x = \{x_j\}_{j=1}^{\infty}$,

$$\xi_{|x|}(\omega) = \sum_{j=1}^{i}|x_j| \quad \text{and hence}$$

$$\sup_{\{x\in S:\xi_{|x|}(\omega)=1} \{|f(x)|\} = \sup_{\{x:\sum_{j=1}^{i}|x_j|=1\}} \{|f(x)|\}.$$

At this situation we must consider two possibilities: a) $i<M$, b) $i\geq M$. In case a) it is easy to see that

$$\sup_{\{x:\sum_{j=1}^{i}|x_j|= 1\}} \{|f(x)|\} = +\infty.$$

In case b) we obtain

$$\sup_{\{x:\sum_{j=1}^{i}|x_j|=1\}} \{|f(x)|\} = \max_{1\leq k\leq M} \{|f_k|\}.$$

We can consider a random variable $\eta_{|f|}$ defined on Ω by the following relation

$$\eta_{|f|}(\omega) = \max_{1\leq k\leq M} \{|f_k|\} \quad \text{for } \omega \in \bigcup_{j=M}^{\infty} A_j$$

$$\eta_{|f|}(\omega) = \infty \quad \text{for } \omega \in \bigcup_{j=1}^{M-1} A_j$$

$$\eta_{|f|}(\omega) = 0 \quad \text{for } \omega \in \Omega_0 \cup \{0\} \cup A.$$

This construction yields immediately that the corresponding probability distribution function F_f is equal to

$$F_f(u) = 0 \qquad \text{for } u \le \max_{1\le k\le M} \{|f_k|\}$$
$$F_f(u) = \sum_{i=M}^{\infty} a_j \quad \text{for } u > \max_{1\le k\le M} \{|f_k|\}$$

and, further, we have

$$P\{\omega:\ \eta_{|f|}(\omega) = \sup_{\{x:\ \xi_{|x|}(\omega)=1\}} \{|f(x)|\}\} = 1,$$

which completes the proof. Q.E.D.

Theorem 5 enables to define a mapping $\mathcal{V}^*: S^* \to \mathcal{F}^*$ $\mathcal{V}^*_f(u) = F_f(u)$ where $\mathcal{F}^*$ denotes all nondecreasing left continuous real functions defined on reals with variation less or equal to 1. We shall study properties of the mapping $\mathcal{V}^*$. If f is the null functional on S, i. e. $f(x) = 0$ for every $x \in S$ then for every $\omega \in \bigcup_{i=1}^{\infty} A_i$

$$\eta_{|0|}(\omega) = \sup_{\{x:\ \sum_{j=1}^{i} |x_j| = 1\}} \{0\} = 0,$$

which means $\eta_{|0|}(\omega) = 0$ for every $\omega \in \Omega$. The corresponding probability distribution is $\mathcal{V}^*_0 = F_0 = H$ where $H(u) = 0$ for $u \le 0$, $H(u) = 1$ otherwise. If $\lambda \neq 0$ is any real number then

$$\lambda f(x) = \lambda \sum_1^M f_i x_i = \sum_{i=1}^M (\lambda f_i) x_i = (\lambda f)(x) \quad \text{and hence}$$

$\eta_{|\lambda f|}(\omega) = |\lambda| \eta_{|f|}(\omega)$ for every $\omega \in \Omega$. It follows that for every $u \in R_1$ and every $\lambda \neq 0$

$$\mathcal{V}^*_{\lambda f}(u) = \mathcal{V}^*_f\left(\frac{u}{\lambda}\right).$$

When $\lambda = 0$ it is reasonable to put $\mathcal{V}^*_f(\frac{u}{0}) = 1$ for every $u > 0$ and every $f \in S^*$. Further, let $f, g \in S^*$, $f(x) = \sum_{i=1}^M f_i x_i$, $g(x) = \sum_{j=1}^N g_j x_j$; then

$$(f+g)(x) = \sum_{i=1}^{\max(M,N)} (f_i + g_i) x_i$$

and $\mathcal{V}^*_{f+g}(u+v) = 0$ for $u + v \le \max_{1\le k\le \max(M,N)} \{|f_k + g_k|\}$. As for every k $|f_k + g_k| \le |f_k| + |g_k|$ we have $u + v \le \max_{1\le k\le M}\{|f_k|\} + \max_{1\le k\le N}\{|g_k|\}$ and hence at least one member of u, v must satisfy $u \le \max_{1\le k\le M}\{|f_k|\}$ or $v \le \max_{1\le k\le N}\{|g_k|\}$ and therefore $\mathcal{V}^*_f(u) = 0$ or $\mathcal{V}^*_g(v) = 0$. If $\mathcal{V}^*_{f+g}(u+v) = \sum_{k=\max(M,N)}^{\infty} a_k$, i. e. $u + v > \max_{1\le k\le \max(M,N)} \{|f_k + g_k|\}$, then $\sum_{k=\max(M,N)}^{\infty} a_k \le \min(\sum_{k=M}^{\infty} a_k, \sum_{k=N}^{\infty} a_k)$ and this inequality proves the generalized triangular inequality $\mathcal{V}^*_{f+g}(u+v) \ge \min(\mathcal{V}^*_f(u), \mathcal{V}^*_g(v))$. This result leads us to the following Definition 3.

Definition 3. Let F^* be the set of all real-valued left continuous nondecreasing nonnegative functions defined on reals with values less or equal to 1. Let L be a linear space and $\mathcal{V}^*$ be a mapping $\mathcal{V}^*: L \to F^*$ satisfying the following conditions:

1. $\mathcal{V}^*_x(u) = H(u)$ if and only if $x = 0$ in L
2. $\mathcal{V}^*_{\lambda x}(u) = \mathcal{V}^*_x(\frac{u}{|\lambda|})$ for every $u \in R_1$, $\lambda \neq 0$
3. $\mathcal{V}^*_{x+y}(u+v) \geq T(\mathcal{V}^*_x(u), \mathcal{V}^*_y(v))$ for every x, $y \in L$ and every u, $v \in R_1$ where T is a t-norm.

Then the triple $(L, \mathcal{V}^*, T)$ is called a generalized statistical linear space in the sense of Menger (GSLM-space).

We shall introduce a topology into $(L, \mathcal{V}^*, T)$ under assumption that the t-norm T is continuous. Let $\mathcal{U} = \{O(\varepsilon, \eta) = \{y \in L: \mathcal{V}^*_y(\eta) > 1 - \varepsilon\}, 0 < \varepsilon \leq 1, \eta > 0\}$. As it is proved in [1] this system $\mathcal{U}$ of neighbourhoods forms a base for topology in $(L, \mathcal{V}^*, T)$. We shall call this topology the ε-η-topology, too.

Theorem 6. The ε-η-topology in GSLM-space $(S^*, \mathcal{V}^*, \min)$ is stronger then the β-topology in S^*.

Proof. Let $\{f_n\}$ be a sequence in S^* convergent to 0 in the ε-η-topology, i. e.

$(\forall \varepsilon > 0\ \forall u > 0\ \exists n_0\ \forall n \geq n_0) \Rightarrow (\mathcal{V}^*_{f_n}(u) > 1 - \varepsilon) \Leftrightarrow (\sum_{j\ M_n}^{\infty} a_j > 1-\varepsilon)$,

if $f_n(x) = \sum_{i=1}^{M_n} f_i^n x_i$. Since $\{a_j\}$ are positive numbers then for every $\varepsilon \in (0,1\rangle$ a natural number M_ε must exist such that $\sum_{j \geq M_\varepsilon}^{\infty} a_j \geq 1 - \varepsilon \geq$ $\geq \sum_{j \geq M_\varepsilon + 1}^{\infty} a_j$ and hence for every $n \geq n_0$ $M_n \leq M$. It means that all functionals $f_n \in O(\varepsilon, \eta)$ have a uniformly bounded length. In other words, using the mapping $\mathcal{V}^*$ we can state that for every $n \geq n_0$

$$\max(|f_1^n|, \ldots, |f_{M_n}^n|) < u.$$

We have proved that $\lim_{n \to \infty} f_i^n = 0$ for every $i \in N$ and $M_n \leq M < \infty$ for every $n \in N$, too. This convergence in the dual space S^* is the so-called β-topology. Now, let us consider a sequence $\{f_n\}$ of functionals in S defined by

$$f_n(x) = \sum_{i=1}^{M} \frac{x_i}{n}$$

where M is fixed and $M > 1$. It is clear that $f_n \to 0$ in the β-topology but $f_n \not\to 0$ in the ε-η-topology. Surely,

$\mathcal{Y}^*_{f_n}(\frac{1}{n} + \delta) = \sum_{j=M}^{\infty} a_j < 1$ for every $n \in N$ and every $\delta > 0$.
If we choose ε^* in such a way that $\sum_{i=M}^{\infty} a_i \leq 1 - \varepsilon^*$ then

$$f_n \notin O(\varepsilon^*, \frac{1}{n} + \delta_n) \quad \text{where } \delta_n \downarrow 0.$$

This example proves that the ε-η-topology is stronger than the β-topology in the dual space $\mathcal{S}^*$. Q.E.D.

Theorem 7. A sequence $\{f_n\} \subset (S^*, \mathcal{Y}^*, \min)$ tends to the null element in the ε-η-topology if and only if

1. $\lim_{n\to\infty} f_i^n = 0$ for every $i \in N$,
2. $\lim_{n\to\infty} M_n = 1$ where M_n denotes the length of f_n, $f_n(x) = \sum_{i=1}^{M_n} f_i^n x_i$.

Proof. The proof can be omitted because it is an easy conclusion of Theorem 6 and the construction of $\mathcal{Y}^*$.

It is remarkable that the ε-η-topology in the dual space $(S^*, \mathcal{Y}^*, \min)$ is not a linear topology because for some $f \in S^*$, $f \neq 0$ and $\{\lambda_n\}_{n=1}^{\infty}$, $\lambda_n \to 0$ $\lambda_n f \not\to 0$ in the ε-η-topology. One can simply prove that the ε-η-topology in a GSLM-space $(S^*, \mathcal{Y}^*, T)$ is linear if and only if for every $x \neq 0$ $\lim_{u\to\infty} \mathcal{Y}^*_x(u) = 1$.

Now, we shall return to the SLM-space $(S, \mathcal{Y}, \min)$ again. We proved that for every $x = \{x_i\}_{i=1}^{\infty}$ the sum $\sum_1^{\infty} x_i \xi_i(\omega)$ is a. s. absolutely convergent and hence for every pair $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$ from S the sum

$$\sum_{i=1}^{\infty} x_i\, y_i\, \xi_i(\omega)$$

is absolutely convergent a. s., too. As $P\{\omega: \xi_i(\omega) = 0\} + P\{\omega: \xi_i(\omega) = 1\} = 1$ for every $i \in N$ then $\xi_i(\omega) = \xi_i^2(\omega)$ a. s. and we can write

$$\sum_{i=1}^{\infty} x_i\, y_i\, \xi_i(\omega) = \sum_{i=1}^{\infty} x_i\, y_i\, \xi_i^2(\omega) \text{ a. s.}$$

In this way we constructed a mapping which is defined on $S \times S$ and takes its values among random variables defined on the underlying probability space (Ω, σ, P) where all $\xi_i(.)$ are defined. This mapping satisfies the following properties. If we shall denote by $\xi_{<x,y>}(\omega) = \sum_1^{\infty} x_i\, y_i\, \xi_i^2(\omega)$ then

1. $\xi_{<x,y>}(\omega) + \xi_{<z,y>}(\omega) = \xi_{<x+z,y>}(\omega)$ a. s.

2. $\xi_{<\alpha x,y>}(\omega) = \alpha\xi_{<x,y>}(\omega)$ a. s., $\alpha \in R_1$

3. $\xi_{<x,x>}(\omega) \geq 0$ a. s.

4. $\xi_{<x,y>}(\omega) = \xi_{<y,x>}(\omega)$ a. s.

These properties lead us to call this mapping a random scalar product defined on S. This random scalar product defines a random seminorm on S by the relation

$$(\xi_{<x,x>}(\omega))^{\frac{1}{2}} = \xi_{|x|2}(\omega).$$

The inequality $|\sum_1^n x_i y_i| \leq (\sum_1^n x_i^2)^{\frac{1}{2}} (\sum_1^n y_i^2)^{\frac{1}{2}}$ holding for every $n \in N$ yields the inequality

$$|\xi_{<x,y>}(\omega)| \leq \xi_{|x|2}(\omega)\ \xi_{|y|2}(\omega) \text{ a. s.}$$

We can introduce a notion of orthogonality by aid of the random scalar product in S. We shall say that $x, y \in S$ are orthogonal if $\xi_{<x,y>}(\omega) = 0$ a. s. We immediately see that two vectors $x, y \in S$ are orthogonal if and only if

$$x_i \cdot y_i = 0 \text{ for every } i \in N.$$

Under orthogonality the Pythagorean theorem holds in the usual form

$$\xi^2_{|x+y|2}(\omega) = \xi^2_{|x|2}(\omega) + \xi^2_{|y|2}(\omega) \text{ a. s.}$$

Now, we need the probability distribution function of a random variable $\xi_{|x|2}(\omega)$, $x \in S$. According to the definition of the random scalar product on S we see that ($\lambda \geq 0$)

$$P\{\omega: \xi_{|x|2}(\omega) < \lambda\} = P\{\omega: (\sum_1^\infty x_i^2 \xi_i^2(\omega))^{\frac{1}{2}} < \lambda\} =$$

$$= P\{\omega: \sum_1^\infty x_i^2 \xi_i^2(\omega) < \lambda^2\} = \sum_{j=1}^{M(\lambda)} a_j \text{ where } M(\lambda) = \max\{n \in N: \sum_{i=1}^n x_i^2 < \lambda^2\} =$$

$$= \max\{n \in N: (\sum_{i=1}^n x_i^2)^{\frac{1}{2}} < \lambda\}.$$

We obtain a mapping $\mathcal{Y}_2 : S \to \mathcal{F}$

$$\mathcal{Y}_2(x)(\lambda) = P\{\omega: \xi_{|x|2}(\omega) < \lambda\}.$$

<u>Theorem 8.</u> The triple $(S, \mathcal{Y}_2, \min)$ is a SLM-space and the corresponding ε-η-topology in S is equivalent to the coordinatewise convergence in S.

Proof. When $x = 0$ in S then $\xi_{|x|2}(\omega) = 0$ a. s. and hence $\mathcal{V}_2(x)(u) = H(u)$ for every $u \in R_1$. On the contrary, if $\xi_{|x|2}(\omega) = 0$ a. s. then $P\{\omega: \sum_1^\infty x_i^2 \xi_i(\omega) < u\} = 1$ for every $u > 0$ and it implies $x = 0$ in S. Let $\lambda \neq 0$ then $\mathcal{V}_2(\lambda x)(u) =$

$$= P\{\omega: (\sum_1^\infty (\lambda x_i)^2 \xi_i^2(\omega))^{\frac{1}{2}} < u\} = P\{\omega: |\lambda|(\sum_1^\infty x_i^2 \xi_i^2(\omega))^{\frac{1}{2}} < u\} =$$

$$= P\{\omega: (\sum_1^\infty x_i^2 \xi_i^2)^{\frac{1}{2}} < \frac{u}{|\lambda|}\} = \mathcal{V}_2(x)\ (\frac{u}{|\lambda|}).$$

It lasts to prove the generalized triangular inequality with the t-norm min. Let $x, y \in S$; $u, v \in R_1$ (we can consider $u > 0$, $v > 0$ only because other cases are quite trivial). We know that $\mathcal{V}_2(x+y)(u+v) = P\{\omega: (\sum_1^\infty (x_i+y_i)^2 \xi_i(\omega))^{\frac{1}{2}} < u+v\} = \sum_{j=1}^{M(u+v)} a_j$, where $M(u+v) = \max\{n \in N: (\sum_{i=1}^n (x_i+y_i)^2)^{\frac{1}{2}} < u+v\}$. Using $(\sum_1^n (x_i+y_i)^2)^{\frac{1}{2}} \leq$ $\leq (\sum_{i=1}^n x_i^2)^{\frac{1}{2}} + (\sum_{i=1}^n y_i^2)^{\frac{1}{2}}$ we see that $u+v \leq (\sum_{i=1}^{M(u+v)+1} (x_i+y_i)^2)^{\frac{1}{2}} \leq$ $\leq (\sum_{i=1}^{M(u+v)+1} x_i^2) + (\sum_{i=1}^{M(u+v)+1} y_i^2)^{\frac{1}{2}}$ and hence either $(\sum_1^{M(u+v)+1} x_i^2) \geq u$ or $(\sum_1^{M(u+v)+1} y_i^2)^{\frac{1}{2}} \geq v$; in every case either $P\{\omega: (\sum_1^\infty x_i^2 \xi_i(\omega))^{\frac{1}{2}} < u\} \leq$ $\leq \sum_{j=1}^{M(u+v)} a_j$ or $P\{\omega: (\sum_1^\infty y_i^2 \xi_i(\omega))^{\frac{1}{2}} < v\} \leq \sum_{j=1}^{M(u+v)} a_j$. Summarizing these facts we obtain

$$\mathcal{V}_2(x+y)(u+v) \geq \min[\mathcal{V}_2(x)(u), \mathcal{V}_2(y)(v)].$$

We have, further, thanks to the inequality $\sum_1^n |x_i| \geq (\sum_1^n x_i^2)^{\frac{1}{2}}$

$$\xi_{|x|}(\omega) \leq \xi_{|x|2}(\omega) \text{ a. s.}$$

for every $x \in S$. That implies that the ε-η-topology induced by the random seminorm $\xi_{|x|}$ is stronger than the ε-η-topology derived from $\xi_{|x|2}$. But if $x_n \to 0$ in S in the ε-η-topology induced by $\xi_{|x|2}$ then $(\forall \varepsilon>0\ \forall \eta>0\ \exists n_0\ \forall n \geq n_0) \Rightarrow (x_n \in O(\varepsilon,\eta)) \Leftrightarrow (\mathcal{V}_2(x_n)(\eta) > 1-\varepsilon) \Leftrightarrow$ $\Leftrightarrow (\sum_{j=1}^{m_n} a_j > 1-\varepsilon)$. As $\{\sum_1^n a_j\}$ is increasing then for every $n \geq n_0$ $m_n \geq m_\varepsilon + 1$ where $\sum_{j=1}^{m_\varepsilon} a_j \geq 1 - \varepsilon$; at the same moment $(\sum_{j=1}^{m_\varepsilon} (x_j^n)^2)^{\frac{1}{2}} < \eta$ must hold for every $n \geq n_0$. With respect to arbitrariness of ε, η we can state that $\lim_{n\to\infty} x_j^n = 0$ for every $j \in N$. We proved the equivalence between the topology generated by the coordinatewise convergence and the ε-η-topology induced by the mapping $\mathcal{V}_2$. Q.E.D.

The system $\{\xi_{\langle x,y \rangle}, x,y \in S\}$ of random variables enables to introduce a probability measure into the measurable space (Φ, K) where Φ is the set of all real-valued functions defined on $S \times S$ and

K is the smallest σ-algebra generated by all measurable cylinder sets

$$\{f \in \Phi : [f(x_1,y_1), f(x_2,y_2), \dots, f(x_n,y_n)] \in B_n\}, \quad x_i, y_i \in S,$$

B_n is a Borel subset in n-dimensional Euclidean space. In the σ-algebra K can be defined a probability measure μ by

$$\mu(C) = P\{\omega : [\xi_{\langle x_1,y_1\rangle}(\omega), \xi_{\langle x_2,y_2\rangle}(\omega), \dots, \xi_{\langle x_n,y_n\rangle}(\omega)] \in B_n\}.$$

If we denote by $\pi \subset \Phi$ the subset of all semiscalar products defined on S then one can affirm that

$$\mu^*(\pi) = 1$$

(μ^* is the outer measure corresponding to the measure μ). In this way we constructed a probability space (π, K_π, ν) where π is the set of all semiscalar products on S, $K_\pi = K \cap \pi$ and

$$\nu(A \cap \pi) = \mu(A), \quad A \in K.$$

This example enables a generalization. Let L be any vector space, let $\Phi_{L\times L}$ be the set of all real-valued functions defined on L × L and let K be the smallest σ-algebra generated by all cylinder sets of the form

$$\{f \in \Phi_{L\times L} : [f(x_1,y_1), \dots, f(x_n,y_n)] \in B_n\},$$

where $x_i, y_i \in L$ and B_n is an n-dimensional Borel subset.

<u>Definition 4.</u> A triple $(\Phi_{L\times L}, K, \mu)$ will be called a random semiunitary (unitary resp.) space if there exists a probability measure ν on the σ-algebra $\pi_L \cap K$ such that $\nu(A \cap \pi_L) = \mu(A)$ for every $A \in K$ where π_L is the subset of all semiscalar (scalar resp.) products on L.

Without any proof we assert that the property "to be a semiscalar (scalar resp.) product on L" is extensible, hereditary and K-measurable with respect to the measurable space $(\Phi_{L\times L}, K)$. Using Theorem 1 in [5] again we can formulate Theorem 9.

<u>Theorem 9.</u> Necessary and sufficient conditions for the existence of a random semiunitary (unitary resp.) space $(\Phi_{L\times L}, K, \mu)$ in a vector space L are

1. $\mu\{f \in \Phi_{L\times L} : f(x,x) \geq 0\} = 1$ for every $x \in L$
2. $\mu\{f \in \Phi_{L\times L} : f(\alpha x + \beta y, z) = \alpha f(x,z) + \beta f(y,z)\} = 1$ for every $x,y,z \in L$ and every $\alpha, \beta \in R_1$

3. $\mu\{f \in \Phi_{L\times L}: f(x,y) = f(y,x)\} = 1$ for every $x, y \in L$

(1. $\mu\{f \in \Phi_{L\times L}: f(x,x) > 0\} = 1$ for every $x \neq 0$ in L

2. $\mu\{f \in \Phi_{L\times L}: f(\alpha x+\beta y,z) = \alpha f(x,z)+\beta f(x,z)\} = 1$ for every $x,y,z \in L$ and every $\alpha,\beta \in R_1$

3. $\mu\{f \in \Phi_{L\times L}: f(x,y) = f(y,x)\} = 1$ for every $x,y \in L$ resp.).

Proof. The proof of Theorem 9 can be omitted.

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