Věra Trnková

Simultaneous representations in discrete structures

Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 4, 633--649

Persistent URL: http://dml.cz/dmlcz/106482

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

27,4 (1986)

SIMULTANEOUS REPRESENTATIONS IN DISCRETE STRUCTURES

Věra TRNKOVÁ

Abstract: For every sequence of monoids

$$M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$$

there exists a directed graph (X,R) such that, for every $i=0,1,2,\dots$, M_i is isomorphic to the endomorphism monoid End $t^i(X,R)$, where

$$t^0(X,R)=(X,R),\ t(X,R)=(X,R\cup R\circ R),\ t^{i+1}(X,R)=t(t^i(X,R)).$$

A more general setting of simultaneous representation is introduced and stronger and more general results are presented.

Key words: Directed graph, representation of monoids, full embeddings of categories.

Classification: 18A22, 18B10, 20M30, 20M50, 05C20.

I. Introduction. Every group is isomorphic to the group of all automorphisms of a graph ([F,S]), every monoid (= semigroup with a unity) is isomorphic to the monoid of all endomorphisms of a graph ([HP,PH]). The next generalization leads to the investigation of full embeddings of small (or concrete) categories into the category of graphs. Let us recall that a functor

$$\Phi : \mathcal{K} \longrightarrow \mathcal{H}$$

of a category $\mathcal{K}$ into a category $\mathcal{H}$ is called a full embedding iff it is faithful (i.e. one-to-one on each set of morphisms with the same domain and codomain) and full (i.e. for every $a,b \in \operatorname{obj} \mathcal{K}$, Φ maps the set $\mathcal{K}(a,b)$ of all $\mathcal{K}$-morphisms with the domain a and codomain b onto the set $\mathcal{H}(\Phi(a), \Phi(b))$. The results of [F,S] and [HP,PH], mentioned above, fit into this field of problems by putting $\mathcal{K}$ to be a one-object category. The full embeddings of categories and related field of problems are investigated in the monograph [PT].

In the present paper, we investigate how some "standard constructions" influence the selecting of morphisms. For example, let R be a binary relation on a set X, $\widetilde{R}$ its transitive hull.

Then the monoid M of all endomorphisms of the directed graph (X,R) is a submonoid of the monoid $\widetilde{M}$ of all endomorphisms of $(X,\widetilde{R})$. Are there some other relations between M and $\widetilde{M}$? Our answer is no: for arbitrary two monoids M, $\widetilde{M}$ such that $M \subseteq \widetilde{M}$ there exists (X,R) with $M \simeq End(X,R)$ and $\widetilde{M} \simeq End(X,\widetilde{R})$.

We adopt the categorical language and categorical description of this idea. Let us formulate the main definition of the simultaneous representation. Denote by $\mathbb{C}at$ the category of all small categories and all functors. Let $\mathcal{C}: D \longrightarrow \mathbb{C}at$ be a diagram (a functor), let $\mathfrak{D}$ be a diagram over the same scheme D such that, for every $\sigma \in \operatorname{obj} D$, $\mathfrak{D}(\sigma)$ is also a category (but not necessarily small) and, for each morphism $m \in D(\sigma, \sigma')$, $\mathfrak{D}(m)$ is a functor of the category $\mathfrak{D}(\sigma)$ into the category $\mathfrak{D}(\sigma')$. A simultaneous representation of the diagram $\mathcal{C}$ in the diagram $\mathfrak{D}$ is a natural transformation $\underline{\Phi} = \{\Phi_\sigma \mid \sigma \in \operatorname{obj} D\}$ such that each Φ_σ is a full embedding of the category $\mathcal{C}(\sigma)$ into the category $\mathfrak{D}(\sigma)$.

If you admit "the category of all categories", the formulation of this general definition could be simplified. However, in the investigated problems, the diagram $\mathfrak{D}$ is given and its cheme is usually rather small. It is composed from some current categories (usually not small) and some current functors (describing some natural construction) and we ask which diagrams $\mathcal{C}$ in $\mathbb{C}at$ (over the same scheme as the given $\mathfrak{D}$) have simultaneous representations in $\mathfrak{D}$. In the above example, $\mathfrak{D}$ is the diagram

$$\text{Graph} \xrightarrow{h} \text{Graph},$$

where Graph is the category of all directed graphs (X,R) and all their compatible maps (see [PT]), and h is the functor which sends each (X,R) to its transitive hull $(X,\widetilde{R})$ and is identical on morphisms (clearly, h is really a functor). The investigated question: for which small categories k_1, k_2 and functors $K: k_1 \longrightarrow k_2$ does exist a simultaneous representation in $\mathfrak{D}$, i.e. there exist full embeddings $\Phi_1 : k_1 \longrightarrow \text{Graph}$, $\Phi_2 : k_2 \longrightarrow \text{Graph}$ such that

$$h \circ \Phi_1 = \Phi_2 \circ K \ ?$$

Since h is a faithful functor and Φ_1 , Φ_2 are required to be faithful, K must be also faithful, obviously. Is this condition sufficient? The positive answer is contained in the Theorem 1 of the present paper. This gives not only the above result about the monoids $M \subseteq \widetilde{M}$ (if k_1 and k_2 are both chosen to be one-object categories) but we obtain from it e.g. the following: there exists an arbitrarily large rigid set $\mathcal{R}$ of binary relations on the same set X (rigid in the sense that if there exists a compatible map $f:(X,R) \longrightarrow (X,R')$ for some $R,R' \in \mathcal{R}$, then necessarily $R=R'$ and f is the identity) such that the transitive hulls of all $R \in \mathcal{R}$ coincide and the unique obtained $(X,\widetilde{R})$ has its endomorphism monoid isomorphic to a prescribed monoid M. (This is obtained by the choice k_2 to be a one-object category with the morphism part formed by M and k_1 to be a discrete category, i.e. it has only unit morphisms, the functor K sends them all to the unique unit morphism of k_2.)

The main results of the present paper are concentrated in the last part III. We investigate there not only the transitive hull functor h, but also single steps in its construction sending (X,R) to $(X,R \cup R \circ R)$ (this leads to the result mentioned in the Abstract), the functor s, sending (X,R) to $(X,R \cup R^{-1})$ (similar problems are investigated in [N] and [HN]), and also the category Bi Graph of all bigraphs (X,R_1,R_2) (both $R_1,R_2 \subseteq X \times X$) simultaneously with the forgetful functors

$$\text{Bi Graph} \underset{F_2}{\overset{F_1}{\rightrightarrows}} \text{Graph}$$

sending (X,R_1,R_2) to (X,R_1) and to (X,R_2). The part II forms a technical basis for the constructions of simultaneous representations. The Lemma proved there is used not only in the part III of the present paper, but it is useful also in the construction of simultaneous representations in continuous structures (which will appear elsewhere).

Finally, let us mention that the definition of simultaneous representations can be further generalized - the categories $\mathcal{C}(\sigma)$ also need not be small. This further generalization is used in Part III of the present paper.

II. Technical basis of simultaneous representations

II.1. Let us recall that $\mathfrak{Cat}$ denotes the category of all small categories and all functors. A commutative square in $\mathfrak{Cat}$

$$\begin{tikzcd} & k_1 \arrow[dr, "K_1^3"] & \\ k_0 \arrow[ur, "K_0^1"] \arrow[dr, "K_0^2"'] & & k_3 \\ & k_2 \arrow[ur, "K_2^3"'] & \end{tikzcd}$$

is called a <u>subpullback</u> if the factorizing morphism $K: k_0 \longrightarrow h$

(where $\begin{tikzcd} & k_1 \arrow[dr, "K_1^3"] & \\ h \arrow[ur] \arrow[dr] & & k_3 \\ & k_2 \arrow[ur, "K_2^3"'] & \end{tikzcd}$ is a pullback in $\mathfrak{Cat}$)

is a full embedding.

II.2. Let us denote by $\mathfrak{G}$ the category of all directed graphs (X,R) (i.e. X is an arbitrary set, $R \subseteq X \times X$) such that

(i) (X,R) has no loops (i.e. never $(x,x) \in R$)

(ii) (X,R) is connected (i.e. for all $x,y \in X$ [not necessarily distinct]

there exist $x_0=x, x_1, \ldots, x_n=y$ in X such that, for every $i=1,\ldots,n$,

$$(x_{i-1}, x_i) \in R \cup R^{-1})$$

and all their compatible maps (i.e. $f:(X,R) \longrightarrow (X',R')$ is a morphism of $\mathfrak{G}$ iff $(x,y) \in R \Longrightarrow (f(x),f(y)) \in R'$). By [PT, 8.5 and 8.6 on p.53 and Theorem on p.104] every small category k can be fully embedded into $\mathfrak{G}$.

II.3. Let $D=(D,\leq)$ be a poset (=partially ordered set). We consider it also a thin category, i.e. if $d_1 \leq d_2$, then $D(d_1,d_2)$ contains precisely one morphism, let us denote it by $\binom{d_2}{d_1}$; if $d_1 \not\leq d_2$, then $D(d_1,d_2)=\emptyset$.

For every poset $(D,\leq)$ with a last element t and arbitrary $d_0 \in D$, let us denote by $\mathfrak{G}_{D,d_0}$ the following category: the objects are all pairs $(X,\{R_d \mid d \in D, d \geq d_0\})$ such that

a) (X,R_t) is an object of $\mathfrak{G}$;

b) if $d_1 \leq d_2$, then $R_{d_1} \subseteq R_{d_2}$;

morphisms are all

$$f:(X,\{R_d\mid d\in D, d\geq d_o\})\longrightarrow(X',\{R'_d\mid d\in D, d\geq d_o\})$$

which are $R_dR'_d$-compatible for all $d\in D$, $d\geq d_o$.

For every $(D,\leq)$ with a last element t, the diagram

$$\mathcal{G}_D$$

is defined in a natural way as follows:

for every $d\in D$, $\mathcal{G}_D(d) = \mathbb{G}_{D,d}$ (hence $\mathcal{G}_D(t) = \mathbb{G}$);

if $d_1\leq d_2$, then $\mathcal{G}_D\binom{d_2}{d_1}: \mathbb{G}_{D,d_1}\longrightarrow\mathbb{G}_{D,d_2}$ is the natural forgetful functor, i.e. it sends any $(X,\{R_d\mid d\geq d_1\})$ to $(X,\{R_d\mid d\geq d_2\})$ and every morphism f to f again. Let us denote $\mathcal{G}_D\binom{d_2}{d_1}$ by $\Sigma^{d_2}_{d_1}$.

II.4. Lemma. Let $D=(D,\leq)$ be a poset with a last element t. Let $\mathcal{C}:D\longrightarrow\mathbf{Cat}$ be a diagram such that, for every $d_1,d_2\in D$, $d_1\leq d_2$, the functor $\mathcal{C}\binom{d_2}{d_1}$ is faithful. Then $\mathcal{C}$ has a simultaneous representation in $\mathcal{G}_D$.

Moreover, if $d_o=d_1\wedge d_2$ in D and the square

$$\begin{tikzcd} & \mathcal{C}(d_1) \arrow[dr, "\mathcal{C}\binom{t}{d_1}"] & \\ \mathcal{C}(d_o) \arrow[ur, "\mathcal{C}\binom{d_1}{d_o}"] \arrow[dr, "\mathcal{C}\binom{d_2}{d_o}"'] & & \mathcal{C}(t) \\ & \mathcal{C}(d_2) \arrow[ur, "\mathcal{C}\binom{t}{d_2}"'] & \end{tikzcd}$$

is a subpullback in $\mathbf{Cat}$, then every $(X,\{R_d\mid d\in D, d\geq d_o\})$ representing an object of $\mathcal{C}(d_o)$ (i.e. being an image of an object of $\mathcal{C}(d_o)$ in the component $\Phi_{d_o}:\mathcal{C}(d_o)\longrightarrow\mathcal{G}_D(d_o)$ of the constructed simultaneous representation $\Phi = \{\Phi_d\mid d\in D\}$) fulfils

$$R_{d_o} = R_{d_1}\cap R_{d_2}.$$

Remark. Let $\Phi:\mathcal{C}\longrightarrow\mathcal{G}_D$ be a simultaneous representation of a diagram $\mathcal{C}:D\longrightarrow\mathbf{Cat}$. Since all the forgetful functors $\mathcal{G}_D\binom{d_2}{d_1}$ (for all $d_1\leq d_2$) are faithful, and Φ_{d_1}, Φ_{d_2} are also faithful, the equation

$$\Phi_{d_2} \circ \mathcal{C}\binom{d_2}{d_1} = \Sigma_{d_1}^{d_2} \circ \Phi_{d_1}$$

implies that $\mathcal{C}\binom{d_2}{d_1}$ must be faithful as well. Hence the lemma gives a necessary and sufficient condition for a representability of $\mathcal{C}:D \longrightarrow \mathcal{C}at$ in $\mathcal{G}_D$.

<u>Proof of the lemma</u>

II.5. Let k be a small category, U be a faithful functor of k into the category Set of all sets and all maps. Then there exists a full embedding

$$\Phi : k \longrightarrow \mathbb{G}$$

such that

for every $a \in \operatorname{obj} k$, $\Phi(a)=(X,R)$ contains U(a) (i.e. $U(a) \subseteq X$)

for every $m \in k(a,a')$, $\Phi(m)$ maps U(a) into U(a′) as U(m).

This follows from 7.3 on page 52, 1.5 on page 59 and Theorem in 1.11 on page 104 of [PT]. Following [PT], we say that Φ is an <u>extension</u> of U.

II.6. Let a poset $D=(D,\leq)$ with a last element t be given, let $\mathcal{C}:D \longrightarrow \mathcal{C}at$ be a diagram such that the functors $\mathcal{C}\binom{d_2}{d_1}$ are faithful for all $d_1 \leq d_2$. Let

$$M:k_t \longrightarrow \mathrm{Set}$$

be the Cayley-Mac Lane representation of the category $k_t = \mathcal{C}(t)$ (i.e. $M(a) = \bigcup_{b \in \operatorname{obj} k_t} k_t(b,a)$, M(m) sends every $p \in k(b,a)$ to $m \circ p$).

Let $\Psi : k_t \longrightarrow \mathbb{G}$ be a full embedding, which is an extension of M. We may suppose that no $\Psi(a)$, $a \in \operatorname{obj} k_t$, contains a cycle (see 3.5 on p.108 of [PT]).

Denote $\Psi(a)=(\widetilde{X}_a,\widetilde{R}_a)$. Thus $M(a) \subseteq \widetilde{X}_a$.

II.7. We may suppose that for every pair d_1, d_2 of distinct elements of D, the sets obj $\mathcal{C}(d_1)$ and obj $\mathcal{C}(d_2)$ are disjoint. Put

$$\Gamma = \bigcup_{d \in D} \operatorname{obj} \mathcal{C}(d).$$

Let $\{(Y_\gamma, T_\gamma) \mid \gamma \in \Gamma\}$ be a collection of objects of $\mathbb{G}$, rigid

in the following sense: if there is a compatible map $f:(Y_\gamma,T_\gamma) \to$
$\to (Y_{\gamma'},T_{\gamma'})$, then necessarily $\gamma=\gamma'$ and f is the identity (such a collection is the result of a full embedding of a discrete category h with obj h = Γ into $\mathbb{G}$). Moreover, we may suppose that no (Y_γ,T_γ) contains a cycle (see 3.5 on p.108 of [PT] again). For every $\gamma \in \Gamma$, we form a graph $H_\gamma = (A_\gamma,S_\gamma)$ as follows:

$$A_\gamma = Y_\gamma \cup \{y,p,q\} \cup \{(p,i) \mid i=1,\dots,7\} \cup \{(q,i) \mid i=1,\dots,9\},$$

$$\begin{aligned}S_\gamma = T_\gamma &\cup \{(p,q)\} \cup \{(q,\bar{y}),(\bar{y},y) \mid \bar{y} \in Y_\gamma\} \cup \\ &\cup \{((p,i).(p,i+1)) \mid i=1,\dots,5\} \cup \{(p,(p,1))\} \cup \\ &\cup \{(p,(p,7)),((p,7),(p,6)),((p,6),p)\} \cup \\ &\cup \{((q,i),(q,i+1)) \mid i=1,\dots,7\} \cup \{(q,(q,1))\} \cup \\ &\cup \{(q,(q,9)),((q,9),(q,8)),((q,8),q)\}.\end{aligned}$$

(Informally: denote by $G_{i,j}$ an i-cycle and j-cycle glued together along one arrow, denote its end by $\ell_{i,j}$; we form H_γ from (Y_γ,T_γ) by the adding of new three vertices y, p, q and joining p with q, q with every vertex in Y_γ and every vertex in Y_γ with y; finally we glue a copy of $G_{3,7}$ on p identifying p with $\ell_{3,7}$ and a copy of $G_{3,9}$ on q identifying q with $\ell_{3,9}$.)
Finally, denote by (B,Q) the graph $G_{3,5}$ (with $\ell = \ell_{3,5}$), i.e.

$$B = \{\ell,(\ell,i) \mid i=1,\dots,5\}$$

$$\begin{aligned}Q = &\{(\ell,(\ell,1))\} \cup \{((\ell,i),(\ell,i+1)) \mid i=1,2,3\} \cup \\ &\cup \{(\ell,(\ell,5)),((\ell,5),(\ell,4)),((\ell,4),\ell)\}.\end{aligned}$$

II.8. Now, we define $\Phi_t : k_t \longrightarrow \mathcal{Y}_D(t) = \mathbb{G}$. For every $a \in \text{obj } k_t$, put $\Phi_t(a) = (X_a,\bar{R}_a)$, where

$$X_a = \tilde{X}_a \cup \tilde{X}_a \times (B \setminus \{\ell\}) \cup M(a) \times \bigcup_{\gamma\in\Gamma} \{\gamma\} \times (A_\gamma \setminus \{y\})$$

$$\bar{R}_a = \tilde{R}_a \cup T_a^1 \cup T_a^2, \text{ with}$$

$$\begin{aligned}T_a^1 = &\{((x,r_1),(x,r_2)) \mid x \in \tilde{X}_a,\ (r_1,r_2) \in Q,\ r_1 \neq \ell \neq r_2\} \cup \\ &\cup \{((x,r_1),x) \mid x \in \tilde{X}_a, (r_1,\ell) \in Q\} \cup \{(x,(x,r_2)) \mid x \in \tilde{X}_a, (\ell,r_2) \in Q\},\end{aligned}$$

$$\begin{aligned}T_a^2 = &\{((x,(\gamma,r_1)),(x,(\gamma,r_2))) \mid x \in M(a),\ \gamma \in \Gamma, (r_1,r_2) \in S_\gamma, r_1 \neq y \neq r_2\} \cup \\ &\cup \{((x,(\gamma,r_1)),x) \mid x \in M(a), \gamma \in \Gamma, (r_1,y) \in S_\gamma\}.\end{aligned}$$

(Informally: we start from $\Psi(a) = (\tilde{X}_a,\tilde{R}_a)$ and glue a copy of $G_{3,5}$ on each element x of $\tilde{X}_a$ by identifying x with the vertex ℓ of this copy [this is described by T_a^1] ; then we glue a copy of each H_γ on each element x of M(a) by identifying x with the vertex y of this copy [this is described by T_a^2].)

The definition of $\Phi_t(m): \Phi_t(a) \longrightarrow \Phi_t(a')$, for $m \in k_t(a,a')$, is obvious: it extends $\Psi(m)$ and it maps the copy of $G_{3,5}$ (or H_γ) glued on x onto the copy of $G_{3,5}$ (or H_γ) glued on $\bar{x} = [\Psi(m)](x)$ such that it sends (x,r) (or (x,γ,r)) to $(\bar{x},r)$ (or $(\bar{x},\gamma,r)$, respectively).

Clearly, Φ_t is a faithful functor of k_t into $\mathbb{G}$. We show that it is full: a compatible map $f:(X_a,\bar{R}_a) \longrightarrow (X_{a'},\bar{R}_{a'})$ has to map a copy of $G_{3,5}$ onto a copy of $G_{3,5}$ again; hence it maps $\tilde{X}_a$ into $\tilde{X}_{a'}$ because $\Psi(a')$ and H_γ, $\gamma \in \Gamma$, contain no $G_{3,5}$; since Ψ is full, there exists $m \in k_t(a,a')$ such that $\Psi(m): X_a \longrightarrow \tilde{X}_{a'}$ is a domain-range-restriction of f; since f sends every copy of $G_{3,7}$ and $G_{3,9}$ on $G_{3,7}$ and $G_{3,9}$ again and since the collection $\{H_\gamma | \gamma \in \Gamma\}$ is rigid, necessarily f is equal to $\Phi_t(m)$.

II.9. Let $d_1 \in D$, $d_1 < t$, be given. We construct Φ_{d_1}:

$: \mathcal{C}(d_1) \longrightarrow \mathbb{G}_{D,d_1}$. For every $a \in \operatorname{obj} \mathcal{C}(d_1)$, denote $a(t) = [\mathcal{C}(^{t}_{d_1})](a)$. For every $\gamma \in \Gamma$, $x \in M(a(t))$, denote $v_{x,\gamma} = ((x,\gamma,p),(x,\gamma,q)) \in \bar{R}_{a(t)}$. We put

$$\Phi_{d_1}(a) = (X, \{R_d | d \in D, d \geq d_1\}),$$

where $(X,R_t) = \Phi_t(a(t))$ (i.e. $(X_{a(t)},\bar{R}_{a(t)})$ in the notation of II.8) and, for every $d_2 \in D$, $d_1 \leq d_2 < t$,

$$R_{d_2} = \{v_{x,\gamma} | \gamma \in \operatorname{obj} \mathcal{C}(d) \text{ with } d \leq d_2 \text{ and } x = [\mathcal{C}(^{t}_{d_2})] (\alpha) \text{ for some morphism } \alpha \in (\mathcal{C}(d_2))(\mathcal{C}(^{d_2}_{d})(\gamma), \mathcal{C}(^{d_2}_{d_1})(a))\}.$$

(In the definition of R_{d_2}, we use that $M:k_t \longrightarrow \mathrm{Set}$ is the Cayley-Mac Lane representation, hence the morphism α in the category $\mathcal{C}(d_2)$ [from the object $\mathcal{C}(^{d_2}_{d})(\gamma)$ into the object $\mathcal{C}(^{d_2}_{d_1})(a)$] is mapped by $\mathcal{C}(^{t}_{d_2})$ to an element of $M(a(t))$ so that the definition makes sense.)

II.10. Clearly

$$\Sigma^{d_2}_{d_1}(\Phi_{d_1}(a)) = \Phi_{d_2}(\mathcal{C}(^{d_2}_{d_1})(a)) \text{ for all } d_1 \leq d_2 \text{ and } a \in \operatorname{obj} \mathcal{C}(d_1),$$

because the definitions of the relations R_d, $d \geq d_2$, are the same for $\Phi_{d_1}(a)$ and for $\Phi_{d_2}(\mathcal{C}(^{d_2}_{d_1})(a))$. Define $\Phi_{d_1}(m)$ such that $\Sigma^{d_2}_{d_1} \circ \Phi_{d_1} = \Phi_{d_2} \circ \mathcal{C}(^{d_2}_{d_1})$ for all $d_1 \leq d_2 \leq t$.

II.11. We prove that $\Phi_{d_1}: \mathcal{C}(d_1) \longrightarrow \mathfrak{G}_{D,d_1}$ is really a functor: For every $a \in \mathrm{obj}\ \mathcal{C}(d_1)$, $\Phi_{d_1}(a)$ is really an object of $\mathfrak{G}_{D,d_1}$. Let m be a morphism in $\mathcal{C}(d_1)$ from an object a into a´. Denote $\Phi_{d_1}(a)=(X,\{R_d | d \geq d_1\})$, $\Phi_{d_1}(a')=(X',\{R'_d | d \geq d_1\})$. We have to verify that the map $f=\Phi'_{d_1}(m)$ is $R_dR'_d$-compatible, $d \geq d_1$. Since f maps elements of X into X´ as $\Phi_t(\mathcal{C}(^{t}_{d_1})(m))$, it is $R_tR'_t$-compatible . Let $d_2 \in D$, $d_1 \leq d_2 < t$ be given. We have to show that f is $R_{d_2}R'_{d_2}$-compatible. Denote $m_t = [\mathcal{C}(^{t}_{d_1})]\ (m)$. Let $v_{x,\gamma}$ be in R_{d_2} (i.e. $\gamma \in \mathrm{obj}\ \mathcal{C}(d)$ with $d \leq d_2$ and $x = \mathcal{C}(^{t}_{d_2})(\alpha)$ for some $\alpha \in \mathcal{C}(d_2)(\mathcal{C}(^{d_2}_{d})(\gamma), \mathcal{C}(^{d_2}_{d_1})(a)))$. Since M is the Cayley-Mac Lane representation of k_t, $M(m_t)$ sends x to $x' = m_t \circ x \in \Phi_t(\mathcal{C}(^{t}_{d_1})(a'))$, hence $\Phi_t(m_t)$ sends the arrow $v_{x,\gamma}$ on the arrow $v_{x',\gamma}$. But $x' = m_t \circ x = (\mathcal{C}(^{t}_{d_1})(m)) \circ (\mathcal{C}(^{t}_{d_2})(\alpha)) = [\mathcal{C}(^{t}_{d_2})]\ (\mathcal{C}(^{d_2}_{d_1})(m) \circ \alpha)$. Thus, by the definition of R'_{d_2}, $v_{x',\gamma}$ is in R'_{d_2} so that f is $R_{d_2}R'_{d_2}$-compatible.

II.12. We prove that Φ_{d_1} with $d_1 < t$ is full: let $a, a' \in \mathrm{obj}\ \mathcal{C}(d_1)$, denote $\Phi_{d_1}(a)=(X,\{R_d | d \geq d_1\})$, $\Phi_{d_1}(a') = (X',\{R'_d | d \geq d_1\})$; let $f: \Phi_{d_1}(a) \longrightarrow \Phi_{d_1}(a')$ be a morphism in $\mathcal{G}_D(d_1)$, i.e. $R_dR'_d$-compatible for all $d \geq d_1$. Then $\Sigma^{t}_{d_1}(f): \Sigma^{t}_{d_1}(\Phi_{d_1}(a)) \longrightarrow \Sigma^{t}_{d_1}(\Phi_{d_1}(a'))$ is a morphism in $\mathcal{G}_D(t)$. Since $\Sigma^{t}_{d_1} \circ \Phi_{d_1} = \Phi_t \circ \mathcal{C}(^{t}_{d_1})$ and Φ_t is full, by II.8, there exists a (unique) morphism $\beta \in \mathcal{C}(t)(\mathcal{C}(^{t}_{d_1})(a), \mathcal{C}(^{t}_{d_1})(a'))$ such that $\Phi_t(\beta) = \Sigma^{t}_{d_1}(f)$. The arrow $v_{x,a}$ with $x = \mathcal{C}(^{t}_{d_1})(1_a) = 1_{a(t)}$ is in R_{d_1}, hence its f-image v´ is in R'_{d_1}. But f sends it to the same arrow as $\Sigma^t_{d_1}(f)$, i.e. $v'=v_{x',a}$ with $x'=[\Sigma^t_{d_1}(f)]\,(1_{a(t)}) =$ $= [\Phi_t(\beta)]\,(1_{a(t)}) = [M(\beta)]\,(1_{a(t)}) = \beta$. Since $v'=v_{\beta,a}$ is in R'_{d_1}, there exists (by the definition of R'_{d_1}) $\alpha \in (\mathscr{C}(d_1)(a,a')$ such that $(\mathscr{C}(^{t}_{d_1}))(\alpha) = \beta$. Then $f = \Phi_{d_1}(\alpha)$ because the functors Φ_{d_1}, $\Sigma^t_{d_1}$, $\mathscr{C}(^{t}_{d_1})$, Φ_t are faithful and $\Sigma^t_{d_1} \circ \Phi_{d_1} =$ $= \Phi_t \circ \mathscr{C}(^{t}_{d_1})$.

II.13. Finally, let $d_0=d_1 \wedge d_2$ in D and

$$\begin{tikzcd}
 & \mathscr{C}(d_1) \arrow[dr, "\mathscr{C}(^{t}_{d_1})"] & \\
\mathscr{C}(d_0) \arrow[ur, "\mathscr{C}(^{d_1}_{d_0})"] \arrow[dr, "\mathscr{C}(^{d_2}_{d_0})"'] & & \mathscr{C}(t) \\
 & \mathscr{C}(d_2) \arrow[ur, "\mathscr{C}(^{t}_{d_2})"'] &
\end{tikzcd}$$

be a subpullback in $\mathfrak{Cat}$; a simple computation gives $R_{d_0}=R_{d_1} \cap R_{d_2}$ for every $a \in \operatorname{obj} \mathscr{C}(d_0)$, $\Phi_{d_0}(a)=(X,\{R_d \mid d \geq d_0\})$. In fact, if $v_{x,\gamma} \in$ $\in R_{d_1} \cap R_{d_2}$, then $\gamma \in \operatorname{obj} \mathscr{C}(d)$ for some $d \leq d_i$, $i=1,2$, hence $d \leq d_0$; and $x= \mathscr{C}(^{t}_{d_i})(\alpha_i)$ for some morphism $\alpha_i \in \mathscr{C}(d_i)(\mathscr{C}(^{d_i}_{d})(\gamma), \mathscr{C}(^{d_i}_{d_0})(a))$, $i=1,2$. Since $\mathscr{C}(^{t}_{d_1})(\alpha_1)=x= \mathscr{C}(^{t}_{d_2})(\alpha_2)$ and the square is a subpullback, there exists $\alpha_0 \in \mathscr{C}(d_0)(\mathscr{C}(^{d_0}_{d})(\gamma),a)$ such that $\mathscr{C}(^{d_i}_{d_0})(\alpha_0)= \alpha_i$, $i=1,2$, so that $v_{x,\gamma} \in R_{d_0}$.

II.14. Remarks. a) We constructed the simultaneous representation Φ such that, for every $d_0 \in D$ and $a \in \operatorname{obj} \mathscr{C}(d_0)$, no R_d in $\Phi_{d_0}(a)=(X,\{R_d \mid d \geq d_0\})$ contains a j-cycle with $j > 9$ and (X,R_t) admits no morphism into any $G_{3,j}$ with $j > 9$. This will be used in III.

b) Let us denote by $\mathfrak{G}_f$ (or $(\mathfrak{G}_{D,d})_f$) the full subcategory of $\mathfrak{G}$ (or $\mathfrak{G}_{D,d}$) generated by all the pairs (X,R) (or $(X,\{R_d \mid d\ldots\})$) with X finite. If k is a finite category, $M(a)$ is finite for all $a \in \operatorname{obj} k$. By [PT, 7.4 on p.52 and 5.1-3 on p.72], the Cayley-Mac Lane representation $M: k \longrightarrow \mathrm{Set}$ can be extended to a full embedding $\Psi: k \longrightarrow \mathbb{G}_f$. Inspecting the proof of the Lemma, we see that

> if D is finite and each $\mathcal{C}(d)$, $d \in D$, is finite then there is a simultaneous representation $\Phi : \mathcal{C} \longrightarrow \mathcal{G}_D$ such that, for every $d \in D$ and every $a \in \operatorname{obj} \mathcal{C}(d)$, $\Phi_d(a)$ is in $(\mathbb{G}_{D,d})_f$.

III. <u>The Main Theorems</u>. Let $R \subseteq X \times X$. Put $t(X,R) = (X, R \cup R \circ R)$ and

$$t^0(X,R) = (X,R), \qquad t^{i+1}(X,R) = t(t^i(X,R)) = (X, R_{i+1}).$$

By an infinite iteration we obtain the transitive hull of R

$$h(X,R) = (X, \bigcup_{i=1}^{\infty} R_i).$$

Let us denote by $h, t: \mathrm{Graph} \longrightarrow \mathrm{Graph}$ the corresponding functors, i.e. $t(X,R) = (X, R \cup R \circ R)$, $h(X,R) = (X, \bigcup_{i=1}^{\infty} R_i)$ on objects

$t(f) = h(f) = f$ on morphisms.

These functors naturally lead to the following commutative diagram $\mathcal{D}$ over the scheme $\omega + 1 = \{0,1,2,\dots,\omega\}$:

$$\begin{tikzcd}
\mathrm{Graph} \arrow[r, "t"'] \arrow[rrrr, bend left, "h"] & \mathrm{Graph} \arrow[r, "t"'] \arrow[rrr, bend left, "h"] & \mathrm{Graph} \arrow[r, "t"'] \arrow[rr, bend left, "h"] & \dots & \mathrm{Graph}
\end{tikzcd}$$

<u>Theorem 1</u>. A commutative diagram $\mathcal{C}$ over $\omega+1$ in $\mathbf{Cat}$

$$\begin{tikzcd}
k_0 \arrow[r, dashed, "K_0^1"'] \arrow[rrrr, bend left, "K_0^\omega"] & k_1 \arrow[r, dashed, "K_1^2"'] \arrow[rrr, bend left, "K_1^\omega"] & k_2 \arrow[r, "K_2^3"'] \arrow[rr, bend left, "K_2^\omega"] & \dots & k_\omega
\end{tikzcd}$$

has a simultaneous representation in $\mathcal{D}$ iff all the functors K_n^m are faithful.

<u>Proof</u>. 1) Since t, h and every $\Phi_n : k_n \longrightarrow \mathrm{Graph}$ in a simultaneous representation $\Phi = \{\Phi_n \mid n \in \omega+1\} : \mathcal{C} \longrightarrow \mathcal{D}$ are faithful, every K_n^m must be faithful.

2) To prove the opposite implication, let us suppose that all the K_n^m's are faithful. Then, by the Lemma in II, there is a simultaneous representation $\mathcal{C} \longrightarrow \mathcal{G}_{\omega+1}$, hence it is sufficient to find a simultaneous representation $\Psi = \{\Psi_n \mid n \in \omega+1\} : \mathcal{G}_{\omega+1} \to \mathcal{D}$. Let $G_{i,j}$, $\ell_{i,j}$, be as in II.7. For an object $(X, \{R_n \mid n \in \omega+1\})$ of

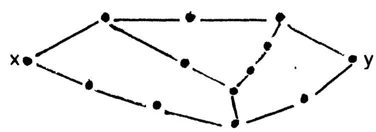
$\mathcal{G}_{\omega+1}(0)$ (by II.14, we may suppose that no R_n contains a j-cycle with $j > 9$ and (X,R_ω) admits no morphism into any $G_{3,j}$ with $j > 9$) put

$$(\tilde{X},\tilde{R}_\omega) = (X,R_\omega) \amalg \coprod_{i=0}^{\infty} G_{3,10+i},$$

where $\amalg$ denotes the coproduct in Graph and denote the vertex $\ell_{3,10+i}$ by ℓ_i.

3) In the graph $(\tilde{X},\tilde{R}_\omega)$, we replace each arrow $\overset{x}{\bullet} \longrightarrow \overset{y}{\bullet}$ by a copy of the undirected graph (Y,S) below .

(*) (Y,S):



(For the exact description of (Y,S), see p.68 in [PT]. The replacing of any arrow in a directed graph by a copy of (Y,S) [described with all details in IV.2 of [PT]] leads to a full embedding of Graph into the category of all undirected graphs, see 3.1 on p.107 in [PT].) Finally, we replace each edge $\overset{c}{\bullet} \text{——} \overset{d}{\bullet}$ in the obtained undirected graph by a copy of the directed graph $(\bar{Y},\bar{S})$ below.

$$(\bar{Y},\bar{S}): \quad \overset{c}{\bullet} \longleftarrow \overset{e}{\bullet} \longrightarrow \overset{d}{\bullet}$$

We obtain a directed graph again, say $(\bar{X},\bar{R}_\omega)$. This graph contains no consecutive arrows, hence $\bar{R}_\omega$ is transitive. We may suppose that during the replacing procedures we added some new vertices not changing the old ones so that $\tilde{X} \subseteq \bar{X}$. For every arrow $r=(a,b)$ in R_ω denote by E_r the set of the "middle vertices e" in all the copies of $(\bar{Y},\bar{S})$ replacing the edges of the copy of (Y,S) replacing r and put $E_n = \bigcup_{r \in R_n} E_r$. We define $\Psi_0 : \mathcal{G}_{\omega+1}(0) \longrightarrow$ Graph as follows: we construct $\Psi_0(a)$ such that we start from $(\bar{X},\bar{R}_\omega)$ and, for every $i=0,1,2,\dots,\ i < \omega$, we join every

$$\left.\begin{array}{ll} x \in E_\omega \setminus E_i & \text{by a copy of a directed path } p_i \\ & \quad \text{of the length } 2^{i+1} \\ \vdots & \\ x \in E_{i-k} \setminus E_{i-k-1} & \text{by a copy of } t^{k+1}p_i, k=0,\dots,i-1 \\ \vdots & \\ x \in E_0 & \text{by a copy of } t^{i+1}p_i \end{array}\right\} \text{ with } \ell_i .$$

If $a, a' \in \operatorname{obj} \mathcal{Y}_{\omega+1}(0)$, $f: a \longrightarrow a'$ is a morphism of $\mathcal{Y}_{\omega+1}(0)$, $\Psi_0(f)$ is equal to f on X, identical on the vertices of $\coprod_{i=0}^{\infty} G_{3,10+i}$ and every copy of (Y,S) (or $(\bar{Y},\bar{S})$) sends on the corresponding copy of it as "the identity". For $m \in \omega+1$, denote by $\langle a \rangle_m$ the object of $\mathcal{Y}_{\omega+1}(0)$ obtained from $a = (X, \{R_n \mid n \in \omega+1\})$ by replacing all the relations R_n with $n < m$ by the empty relation $\emptyset$. Since, in the graph $\Psi_0(a)$, consecutive arrows appear only in the copies of the paths p_i, we see that

$$t^m(\Psi_0(a)) = \Psi_0(\langle a \rangle_m) \text{ for } m < \omega \text{ , } h(\Psi_0(a)) = \Psi_0(\langle a \rangle_\omega).$$

This implies that $t \circ \Psi_n = \Psi_{n+1} \circ \mathcal{Y}_{\omega+1}\binom{n+1}{n}$ and $h \circ \Psi_n = \Psi_\omega \circ \mathcal{Y}_{\omega+1}\binom{\omega}{n}$ if we define $\Psi_n : \mathcal{Y}_{\omega+1}(n) \longrightarrow \text{Graph}$, $n \leq \omega$, by

$$\Psi_n(X, \{R_m \mid m \geq n\}) = \Psi_0(X, \{R_m \mid m \in \omega+1\}) \text{ with } R_m = \emptyset \text{ for } m < n.$$

This also reduces the proof that every $\Psi_n : \mathcal{Y}_{\omega+1}(n) \longrightarrow \text{Graph}$ is a full embedding to a proof that $\Psi_0 : \mathcal{Y}_{\omega+1}(0) \longrightarrow \text{Graph}$ is a full embedding. However, the steps in the construction of Ψ_0, namely

a) the adding of the graphs $G_{3,10+i}$, $i = 0,1,\ldots$

b) the replacing of each arrow (x,y) by a copy of (Y,S)

c) the replacing of each edge by a copy of $(\bar{Y},\bar{S})$

lead to a full embedding of (X, R_ω) into Graph. And the length of the paths p_i forces that a compatible map $f: \Psi_0(a) \to \Psi_0(a')$ "recognizes" R_n for all $n = 0,1,2,\ldots$ so that $f = \Psi_0(g)$ for a morphism $g: a \longrightarrow a'$ of $\mathcal{Y}_{\omega+1}(0)$.

Remark. For every $m \in \omega$, every diagram

$$k_0 \xrightarrow{K_0^1} k_1 \xrightarrow{K_1^2} k_2 \xrightarrow{K_2^3} \cdots \xrightarrow{K_{n-1}^m} k_m$$

of finite categories and faithful functors has a simultaneous representation $\Phi = \{\Phi_i \mid i = 0,\ldots,m\}$ in the diagram

$$\mathbb{G}_f \xrightarrow[t]{} \mathbb{G}_f \xrightarrow[t]{} \mathbb{G}_f \xrightarrow[t]{} \cdots \xrightarrow[t]{} \mathbb{G}_f$$

such that, for every $a \in \operatorname{obj} k_m$, $\Phi_m(a)$ is already transitive. In fact, use II.14.b) and modify the proof of Theorem 1 (consider $(\tilde{X},\tilde{R}_n) = (X,R_n) \sqcup \coprod_{i=0}^{m} G_{3,10+i}$ instead of the infinite coproduct).

III.2. Theorem 2. Let $s: \text{Graph} \longrightarrow \text{Graph}$ be the functor

$$s(X,R) = (X, R\cup R^{-1}), \quad s(f) = f.$$

Then a diagram $\mathcal{C}$ in $\mathbf{Cat}$

$$k_0 \xrightarrow{K} k_1$$

has a simultaneous representation in the diagram

$$\text{Graph} \xrightarrow{s} \text{Graph}$$

iff K is faithful.

<u>Proof</u>. 1) If there is a simultaneous representation $\Phi: \mathcal{C} \to \mathcal{D}$ then K must be faithful because s, Φ_0, Φ_1 are faithful.

2) If K is faithful, then there is a simultaneous representation $\mathcal{C} \to \mathcal{G}_2$ (where $2=\{0,1\}, 0\leq 1$), hence it is sufficient to construct a simultaneous representation $\Psi: \mathcal{G}_2 \to \mathcal{D}$. For an object $(X,\{R_0,R_1\})$ of $\mathcal{G}_2(0)$ (we may suppose that R_1 contains no j-cycle with $j>9$, see II.14) put

$$(\tilde{X},\tilde{R}_1) = (X,R_1) \amalg G_{3,10}.$$

In the graph $(\tilde{X},\tilde{R}_1)$, we replace each arrow $\overset{x}{\bullet} \longrightarrow \overset{y}{\bullet}$ by a copy of the graph (Y,S) (see ($\ast$)) and then replace each edge $\overset{c}{\bullet} \!-\!\!-\!\!-\! \overset{d}{\bullet}$ by the two arrows (c,d) and (d,c). Denote the obtained symmetric graph by $(\bar{X},\bar{R}_1)$. We may suppose $\tilde{X}\subseteq\bar{X}$. Choose a vertex z in Y, $x \neq z \neq y$, and denote by z_r the vertex z of the copy of (Y,S), replacing the arrow $r \in R_1$. We define $\Psi_0: \mathcal{G}_2(0) \to \text{Graph}$ such that, for $a=(X,R_0,R_1) \in \text{obj}\, \mathcal{G}_2(0)$, we have

$$\Phi_0(a) = (\bar{X}, \bar{R}_1 \cup T) \quad \text{where}$$

$T=\{(\ell_{3,10}, z_r) \mid r\in R_1\} \cup \{(z_r, \ell_{3,10}) \mid r \in R_0\}$. The rest of the proof is analogous to the previous proof.

<u>Remark</u>. If k_1, k_2 are finite (and K faithful) then $\mathcal{C}$ has a simultaneous representation in

$$\mathbb{G}_f \xrightarrow{s} \mathbb{G}_f.$$

III.3. Let us denote by Bi Graph the category of all bigraphs, i.e. all (X,R_1,R_2), where X is a set and $R_i \subseteq X\times X$, i=1,2, and all $f:(X,R_1,R_2) \to (X',R_1',R_2')$, which are R_iR_i'-compatible, i=1,2. There are two natural forgetful functors F_i: Bi Graph $\to$ Graph, i=1,2, namely

$$F_1(X,R_1,R_2)=(X,R_1), F_2(X,R_1,R_2)=(X,R_2), \quad F_1(f)=F_2(f)=f.$$

Thus, we obtain a diagram

$$\mathcal{D}:\ \begin{tikzcd}[row sep=small] & \mathrm{Graph} \\ \mathrm{BiGraph} \arrow[ur, "F_1"] \arrow[dr, "F_2"'] & \\ & \mathrm{Graph} \end{tikzcd}$$

Theorem 3. A diagram $\mathcal{C}$ in $\mathcal{C}at$

$$\begin{tikzcd}[row sep=small] & k_1 \\ k_0 \arrow[ur, "K_1"] \arrow[dr, "K_2"'] & \\ & k_2 \end{tikzcd}$$

has a simultaneous representation in $\mathcal{D}$ iff the pushout in $\mathcal{C}at$

$$(**)\qquad \begin{tikzcd}[row sep=small] & k_1 \arrow[dr, "H_1"] & \\ k_0 \arrow[ur, "K_1"] \arrow[dr, "K_2"'] & & h \\ & k_2 \arrow[ur, "H_2"'] & \end{tikzcd}$$

is a subpullback in $\mathcal{C}at$ and all the functors K_i, H_i, $i=1,2$, are faithful.

Proof. 1) Let there be a simultaneous representation $\Phi = \{\Phi_0, \Phi_1, \Phi_2\}: \mathcal{C} \longrightarrow \mathcal{D}$. Then K_1, K_2 are faithful because F_i, Φ_i, $i=1,2$, and Φ_0 are faithful. Let $(**)$ be the pushout of K_1 and K_2 in $\mathcal{C}at$. Denote by $F:\mathrm{Graph} \longrightarrow \mathrm{Set}$ the forgetful functor (i.e. $F(X,R)=X$). Then $F\circ F_1=F\circ F_2$, consequently $F\circ \Phi_1 \circ K_1= =F\circ F_1\circ \Phi_0=F\circ F_2\circ \Phi_0=F\circ \Phi_2\circ K_2$ hence there exists $\bar{F}:h \longrightarrow \mathrm{Set}$ such that $\bar{F}\circ H_i=F\circ \Phi_i$, $i=1,2$. Since $F\circ \Phi_i$ are faithful, H_i must be also faithful. To prove that the pushout $(**)$ is a subpullback, it is sufficient to prove the following: if $a,a' \in \mathrm{obj}\, k_0$ and $\alpha_i \in k_i\ (K_i(a),K_i(a'))$, $i=1,2$, are morphisms such that $H_1(\alpha_1)= =H_2(\alpha_2)$, then there exists $\alpha \in k_0(a,a')$ such that $K_i(\alpha)=\alpha_i$ for $i=1,2$. But $H_1(\alpha_1)=H_2(\alpha_2)$ implies $(F\circ \Phi_1)(\alpha_1)=(F\circ \Phi_2)(\alpha_2)$, i.e. the morphisms $\Phi_1(\alpha_1):(X,R_1) \longrightarrow (X',R_1')$ and $\Phi_2(\alpha_2):(X,R_2) \longrightarrow (X',R_2')$ are carried by the same set-mapping $f:X \longrightarrow X'$. Since f is both R_1R_1'-compatible and R_2R_2'-compatible, it is a morphism of BiGraph. Since Φ_0 is full, there exists $\alpha \in k_0(a,a')$ with $\Phi_0(\alpha)=f$. Then $K_i(\alpha)=\alpha_i$ for $i=1,2$.

2) Let us suppose that the pushout $(**)$ is a subpullback

in $\mathbf{Cat}$ and all the functors K_i, H_i, $i=1,2$, are faithful. Put $D=\{0,1,2,3\}$, where 1 and 2 are incomparable and both are less than 3, $0=1\wedge 2$ and apply the Lemma on the diagram (**). Hence it is sufficient to construct a full embeddings $\Psi_0: \mathbb{G}^*_{D,0} \longrightarrow$ $\longrightarrow$ BiGraph, $\Psi_i: \mathbb{G}^*_{D,i} \longrightarrow$ Graph (where $\mathbb{G}^*_{D,0}$ is the full subcategory of $\mathbb{G}_{D,0}$, generated by all the $a=(X,\{R_i \mid i=0,\dots,3\})$ with $R_0=R_1\cap R_2$, $R_j \subseteq R_3$ for $j=1,2$, R_3 contains no j-cycle with $j>9$, and (X,R_3) admit no morphism into any $G_{3,j}$ with $j>9$, see II.14; and $\mathbb{G}^*_{D,i}$, $i=1,2$, is a Σ^i_0-image of $\mathbb{G}^*_{D,0}$) such that $\Psi_i \circ \Sigma^i_0 = F_i \circ \Psi_0$, $i=1,2$. We construct the bigraph $\Psi_0(a)=(\bar{X},T_1,T_2)$ as follows: We start from

$$(\tilde{X},\tilde{R}_3) = (X,R_3) \amalg G_{3,10} \amalg G_{3,11}.$$

Choose one fixed orientation of the edges in the graph (Y,S) (see (*)) such that there is no directed path from x to y and no directed path from y to x in the obtained directed graph; denote it by $(Y,\tilde{S})$; choose $z\in Y$, $x\neq z\neq y$. In the graph $(\tilde{X},\tilde{R}_3)$, replace each arrow by a copy of $(Y,\tilde{S})$, denote the obtained directed graph by $(\bar{X},\bar{R}_3)$. If $r\in R_3$, denote by z_r the vertex z in the copy of $(Y,\tilde{S})$ which replaces r. Put $\Psi_0(a)=(\bar{X},T_1,T_2)$, where

$$T_1 = \bar{R}_3 \cup \{(\ell_{3,10},z_r) \mid r\in R_1\},\quad T_2 = \bar{R}_3 \cup \{(\ell_{3,11},z_r) \mid r\in R_2\}$$

If $f:a \longrightarrow a'$ is a morphism of $\mathbb{G}^*_{D,0}$, put $(\Psi_0(f))=f(x)$ on X, $(\Psi_0(f))(x)=x$ on $G_{3,10}\amalg G_{3,11}$ and $\Psi_0(f)$ maps each copy of $(Y,\tilde{S})$ on the corresponding copy of it as "the identity".

The proof that Ψ_0 is a full embedding and that $F_i \circ \Psi_0 = \Psi_i \circ \Sigma^i_0$, $i=1,2$ (where $\Psi_1((X,\{R_1,R_3\})$ is defined as $F_1(\Psi_0(X,\{R_0,R_1,R_2,R_3\}))$ with $R_0=R_2=\emptyset$ and analogously for $\Psi_2((X,\{R_2,R_3\}))$, is analogous to the previous proofs and it is left to the reader.

Remarks. a) If k_0, k_1, k_2 are finite, then the represented bigraphs and graphs can be constructed also finite, see II.14.

b) The Theorem 3 can be generalized such that we investigate not only bigraphs but n-graphs with larger n and more complex scheme for the diagram of the forgetful functors. Since this generalization is straightforward, we do not describe it here.

References

[F] R. FRUCHT: Herstellung von Graphen mit vorgegebener abstrakter Gruppe, Compos. Math. 6(1938), 239-250.

[HN] P. HELL, J. NEŠETŘIL: Homomorphisms of graphs and of their orientations, Monatsh. für Math. 85(1978), 39-48.

[HP] Z. HEDRLÍN, A. PULTR: Relations (graphs) with given finitely generated semigroups, Monats. Math. 68(1964), 213-217.

[N] J. NEŠETŘIL: On symmetric and antisymmetric relations, Monatsh. Math. 76(1972), 323-327.

[PH] A. PULTR, Z. HEDRLÍN: Relations (graphs) with given infinite semigroups, Monatsh. Math. 68(1964), 421-425.

[PT] A. PULTR, V. TRNKOVÁ: Combinatorial, Algebraic and Topological Representation of Groups, Semigroups and Categories, North Holland 1980.

[S] G. SABIDUSSI: Graphs with given infinite groups, Monatsh. Math. 64(1960), 64-67.

Mathematical Institute, Charles University, Sokolovská 83, 18600 Praha 8, Czechoslovakia

(Oblatum 11.7. 1986)