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ON THE MULTIPLICITY POINTS OF MONOTONE OPERATORS ON SEPARABLE BANACH SPACES

Libor VESELÝ

Abstract: It is proved that the set of multiplicity points of monotone operator T on a separable real Banach space is contained in a countable union of Lipschitz hypersurfaces with "linearly finite convexity on a subset". If T is a subdifferential of a proper convex function, the hypersurfaces are δ-convex. Analogous results are obtained for the sets of n-dimensional and n-codimensional multiplicities. Applications to singular points of convex sets are given. This paper improves and generalizes the results of L.Zajíček.

Key words: Multiplicity points of monotone operators, linearly finite convexity, Lipschitz surfaces in Banach spaces, convex analysis, subdifferentials of proper convex functions, singular points of convex sets, δ-convex functions.

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1. Introduction

Let T be a set-valued monotone operator on a separable real Banach space X (i.e. $T:X \to \exp X^*$ and $\langle x-y, x^*-y^* \rangle \geq 0$ whenever $x^* \in Tx$, $y^* \in Ty$) and let

$A_n = \{x \in X: \dim(\mathrm{co}Tx) \geq n\}$,

$A^n = \{x \in X: \mathrm{co}Tx$ contains a ball of codimension $n\}$,

where coTx denotes a convex hull of the set Tx.

The smallness of the sets A_n, A^n was investigated by E.H. Zarantonello [8], N.Aronszajn [1] and L.Zajíček [6],[7]. The theorems were applied to operators F_M, V_M ("vertex-" and "face--operator") being connected with singular points of a closed convex set M, in [8],[7].

In this paper, the results from [6] and [7] were improved and generalized.

L.Zajíček has proved (see [7]) that the set A_n can be covered by countably many Lipschitz surfaces of codimension n. If $T=\partial f$ for some continuous convex function on an open convex set $U \subset X$ then it is possible to write "δ-convex surfaces" instead of "Lipschitz surfaces" (see [6]). In case X is a Hilbert space or $n=1$ and X^* is separable, the set A^n of a general monotone operator T can be covered by a countable union of Lipschitz surfaces of dimension n (see [7]).

1.1 Problem: It is still an open problem whether the set A_n (or A^n, if X^* is separable, respectively) can be covered by countably many δ-convex surfaces of codimension n (or dimension n, respectively) if T is a general monotone operator.

Following main results of the present article suggest that the answer could be positive:

a/ The Lipschitz surfaces from [7] have an additional property - "linearly finite convexity on a subset". This result easily gives an existence of a Lipschitz surface of codimension n (dimension n, respectively) which cannot be a subset of A_n (A^n, respectively) for any monotone operator T.

b/ If X^* is separable then the set A^1 is contained in a countable union of curves with finite convexity. It gives a positive answer to 1.1 in the special case $X=\mathbb{R}^2$.

c/ The result from [6] is generalized to the case $T=\partial f$, where f is a proper convex function. It makes possible to improve the results from [7], [8] concerning singular points of convex sets.

d/ It is shown that the Lipschitz surfaces covering the set A^n are in a certain sense δ-convex on a subset if $T=\partial f$.

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2. Definitions and auxiliary propositions

All linear spaces of present paper will be real linear spaces. Let M be a subset of the real line $\mathbb{R}$. We shall denote by $\mathcal{P}(M)$ the system of all sets $A \subset M$, which contain at least three elements.

Let X be a Banach space and $f: M \to X$. For any $a, b \in M$, $a \neq b$, we define $Q_f(a,b) = \frac{f(b)-f(a)}{b-a}$. We shall write $Q(a,b)$ instead of $Q_f(a,b)$ when it is clear which mapping is concerned to.

<u>2.1 Definition</u> (cf. [2]): Let X be a Banach space, $M \subset \mathbb{R}$ and $f: M \to X$. For $P = \{x_0 < x_1 < \dots < x_n < x_{n+1}\} \in \mathcal{P}(M)$ we define

$$K(f,P) = \sum_{i=1}^{n} \| Q_f(x_{i-1}, x_i) - Q_f(x_i, x_{i+1}) \|$$

and put

$$\mathcal{K}(f,M) = \begin{cases} \sup \{K(f,P) : P \in \mathcal{P}(M)\} & \text{if } \mathcal{P}(M) \neq \emptyset , \\ 0 \quad \text{if} \quad \mathcal{P}(M) = \emptyset . \end{cases}$$

$\mathcal{K}(f,M)$ is called convexity of f on M.

<u>2.2 Lemma</u>: Let X be a Banach space, $M \subset \mathbb{R}$ and $f: M \to X$. Then $K(f,P) \leq K(f, P \cup \{m\})$ holds for any $P \in \mathcal{P}(M)$, $m \in M$.

<u>Proof</u>: Let $P = \{x_0 < x_1 < \dots < x_n < x_{n+1}\} \in \mathcal{P}(M)$. There are four possible positions of the point m.

a/ $m \in P$;

b/ $m < x_0$ or $x_{n+1} < m$;

c/ $x_0 < m < x_1$ or $x_n < m < x_{n+1}$;

d/ $x_j < m < x_{j+1}$ for some $1 \leq j \leq n-1$.

We shall perform the proof of the most complicated case d/ only, since the proof of c/ is similar and a/,b/ are obvious.

If we shortly denote $x = x_{j-1}$, $y = x_j$, $z = x_{j+1}$, $w = x_{j+2}$, we have following situation:

$$x < y < m < z < w.$$

Let $k \in X$ be such that

$$\frac{k-f(y)}{m-y} = Q(y,z) = \frac{f(z)-k}{z-m} .$$

Then $\|Q(x,y) - Q(y,z)\| + \|Q(y,z) - Q(z,w)\| = \|Q(x,y) - \frac{k-f(y)}{m-y}\| + \|\frac{f(z)-k}{z-m} - Q(z,w)\| \leq \|Q(x,y) - Q(y,m)\| + \frac{\|f(m)-k\|}{m-y} + \frac{\|f(m)-k\|}{z-m} + \|Q(m,z) - Q(z,w)\| = \|Q(x,y) - Q(y,m)\| + \|Q(y,m) - Q(m,z)\| + \|Q(m,z) - Q(z,w)\|$ and hence $K(f,P) \leq K(f, P \cup \{m\})$.

(We have used following equalities:

$\frac{\|f(m)-k\|}{m-y} + \frac{\|f(m)-k\|}{z-m} = \|\frac{f(m)-k}{m-y} - \frac{k-f(m)}{z-m}\| = \|Q(y,m) - Q(m,z)\|$.)

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2.3 Proposition: Let X be a Banach space, $M \subset \mathbf{R}$, $f: M \to X$. If $\mathcal{K}(f,M) < \infty$ then f is a Lipschitz mapping on M.

Proof: Suppose f is not Lipschitz. It is evident that there exist two points $a,b \in M$ such that $a < b$ and f is not Lipschitz on at least one of the sets $M_+ = M \cap (b,+\infty)$, $M_- = M \cap (-\infty,a)$. We can assume f to be not Lipschitz on M_+ without any loss of generality. There exist $u,v \in M_+$ such that $u < v$ and $\|Q(u,v)\| > \mathcal{K}(f,M) + \|Q(a,b)\|$. Then $\mathcal{K}(f,M) < \|Q(u,v)\| - \|Q(a,b)\| \leq \|Q(a,b) - Q(u,v)\| \leq \|Q(a,b) - Q(b,u)\| + \|Q(b,u) - Q(u,v)\| = K(f,\{a,b,u,v\}) \leq \mathcal{K}(f,M)$ and this is a contradiction. ///

2.4 Proposition: Let X be a Banach space, $M \subset \mathbf{R}$, $f: M \to X$ and $\mathcal{K}(f,M) < \infty$. If $x \in M$ is a limit point of M from the right (from the left, respectively), there exist

$$f'_+(x,M) = \lim_{\substack{y \to x+ \\ y \in M}} Q_f(x,y) \quad \left(f'_-(x,M) = \lim_{\substack{y \to x- \\ y \in M}} Q_f(x,y), \text{ resp.} \right).$$

Proof: Suppose $f'_+(x,M)$ doesn't exist. Then there must exist $\varepsilon > 0$ such that for any $\delta > 0$ there exist $y,z,w \in M$ satisfying $x < y < z < w < x+\delta$ and $\|Q(x,y) - Q(x,z)\| > \varepsilon$.
But $\|Q(x,y) - Q(x,z)\| \leq \|Q(x,y) - Q(y,z)\| + \|Q(y,z) - Q(w,z)\| + \|Q(x,z) - Q(w,z)\| = K(f,\{x,y,z,w\}) + K(f,\{x,z,w\}) \leq 2\mathcal{K}(f,M \cap [x,x+\delta)) = 2\mathcal{K}(f,M \cap (x,x+\delta))$. (The last equality is an easy consequence of 2.3.)
Hence $\mathcal{K}(f,M \cap (x,x+\delta)) > 2^{-1}\varepsilon$ for any $\delta > 0$. Let $N > \frac{2}{\varepsilon}\mathcal{K}(f,M)$ be positive integer. Since we have for any $\delta > 0$ an existence of P from $\mathcal{P}(M \cap (x,x+\delta))$ such that $K(f,P) > 2^{-1}\varepsilon$, it is possible to find $P_1, P_2, \dots, P_N \in \mathcal{P}(M)$ with following properties:

$$\max P_{k+1} < \min P_k \quad \text{for } k=1,2,\dots,N-1$$
$$K(f,P_j) > 2^{-1}\varepsilon \quad \text{for } j=1,2,\dots,N.$$

Then $\mathcal{K}(f,M) < N\frac{\varepsilon}{2} < \sum_{k=1}^{N} K(f,P_k) \leq K(f, \bigcup_{k=1}^{N} P_k) \leq \mathcal{K}(f,M)$ and this is a contradiction.
The proof of existence of $f'_-(x,M)$ is analogous. ///

2.5 Theorem: Let X be a Banach space, $M \subset \mathbf{R}$, $f: M \to X$. Then there exists a mapping $F: \mathbf{R} \to X$ such that

$$\forall x \in M \quad F(x) = f(x) \tag{1}$$
$$\mathcal{K}(F,\mathbf{R}) = \mathcal{K}(f,M). \tag{2}$$

Proof: If $\mathcal{K}(f,M)=+\infty$, F can be an arbitrary extension of f. If M has two or less elements then F can be defined as affine mapping satisfying (1). Suppose M has at least three elements and $\mathcal{K}(f,M)<\infty$. The needed extension will be constructed in several steps.

a/ Extension on $\overline{M}$ (closure of M).

X is complete and f is Lipschitz on M (by 2.3). Hence f has a unique continuous extension g on $\overline{M}$. Choose $\varepsilon>0$ and arbitrary $P=\{x_0<x_1<\dots<x_{n+1}\}\in\mathcal{P}(\overline{M})$. The continuity of the mapping $q(u,v)=Q_g(u,v)$ on the set $\{[u,v]\in\overline{M}\times\overline{M}: u\neq v\}$ gives existence of $P_1=\{y_0<y_1<\dots<y_{n+1}\}\in\mathcal{P}(M)$ such that

$$|Q_g(x_j,x_{j+1})-Q_g(y_j,y_{j+1})|<\frac{\varepsilon}{2n}\ ;\quad j=0,1,\dots,n\ .$$

Then

$$K(g,P)<K(g,P_1)+2n.\frac{\varepsilon}{2n}=K(f,P_1)+\varepsilon\leq\mathcal{K}(f,M)+\varepsilon\ .$$

Hence $\mathcal{K}(g,\overline{M})=\sup\{K(g,P): P\in\mathcal{P}(\overline{M})\}\leq\mathcal{K}(f,M)+\varepsilon$.
Since ε was arbitrary and the inequality $\mathcal{K}(f,M)\leq\mathcal{K}(g,\overline{M})$ is evident, we have $\mathcal{K}(g,\overline{M})=\mathcal{K}(f,M)$.

b/ Extension on $M_1=\{x\in\mathbb{R}: \sigma\leq x\leq s\}$, where $\sigma=\inf M$, $s=\sup M$.
The complement of $\overline{M}$ can be written as a finite or countable union of disjoint open intervals:

$$\mathbb{R}\setminus\overline{M}=J_-\cup\bigcup_{k\in A}J_k\cup J_+\ ,$$

where $A\subset\{1,2,3,\dots\}$, $J_-=(-\infty,\sigma)$, $J_+=(s,+\infty)$, $J_k=(a_k,b_k)$, $a_k<b_k$, $k\in A$. J_-,J_+ can be empty and, obviously, $a_k,b_k\in\overline{M}$ for any $k\in A$.

It is easy to see that $M_1=\overline{M}\cup\bigcup J_k$. Let us define

$$h(x)=\begin{cases} g(x) & \text{if } x\in\overline{M}\\ g(a_k)+Q_g(a_k,b_k)(x-a_k) & \text{if } x\in(a_k,b_k)\ .\end{cases}$$

Obviously $Q_h(a_k,x)=Q_h(x,b_k)=Q_h(a_k,b_k)=Q_g(a_k,b_k)$ for any $x\in(a_k,b_k)$ and $h=f$ on M. (3)

For arbitrary $P\in\mathcal{P}(M_1)$ we define

$$P_1=P\cup\bigcup_{\substack{k\in A\\ P\cap J_k\neq\emptyset}}\{a_k,b_k\}\ ,\quad P_2=P_1\setminus\bigcup_{k\in A}J_k\ .$$

Then P_2 contains at least two points and $P_2\subset\overline{M}$. If P_2 contains just two points then $P\subset J_k$ for convenient $k\in A$ and then

$K(h,P)=0\leq\mathcal{K}(f,M)$. Let P_2 contain more than two elements. Then $P_2\in\mathcal{P}(\bar{M})$ and by 2.2 and (3):

$$K(h,P)\leq K(h,P_1)=K(h,P_2)=K(g,P_2)\leq\mathcal{K}(g,\bar{M})=\mathcal{K}(f,M) .$$

Since $P\in\mathcal{P}(M_1)$ was arbitrary then $\mathcal{K}(h,M_1)=\mathcal{K}(f,M)$.

c/ Extension on $\mathbf{R}$.

Define

$$F(x)=\begin{cases} h(x) & \text{if } x\in M_1 \\ h(s)+h'_-(s,M_1)(x-s) & \text{if } x\in J_+ \\ h(\sigma)+h'_+(\sigma,M_1)(x-\sigma) & \text{if } x\in J_- \end{cases} .$$

Let us suppose $J_+\neq\emptyset, J_-\neq\emptyset$. The other cases are more simple. Let $P\in\mathcal{P}(\mathbf{R})$ and $\varepsilon>0$. Choose $P_1\in\mathcal{P}(\mathbf{R})$ such that $P\subset P_1$, $J_-\cap P_1\neq\emptyset$, $J_+\cap P_1\neq\emptyset$, $M_1\cap P_1\neq\emptyset$. Define

$$P_2=P_1\cup\{\sigma,s\}=\{x_0<x_1<\dots<x_i<\sigma<x_{i+1}<\dots<x_m<s<x_{m+1}<\\ <\dots<x_n\}$$

and $P_3=\{x_i<\sigma<x_{i+1}<\dots<x_m<s<x_{m+1}\}$.

There exist $y\in(\sigma,x_{i+1})$, $z\in(x_m,s)$ such that

$$\|Q_F(\sigma,x_{i+1})-Q_F(y,x_{i+1})\|<\tfrac{1}{6}\varepsilon ,$$

$$\|Q_F(x_i,\sigma)-Q_F(\sigma,y)\|=\|F'_+(\sigma)-Q_F(\sigma,y)\|<\tfrac{1}{6}\varepsilon ,$$

$$\|Q_F(x_m,s)-Q_F(x_m,z)\|<\tfrac{1}{6}\varepsilon ,$$

$$\|Q_F(z,x_{m+1})-Q_F(z,s)\|=\|F'_-(s)-Q_F(z,s)\|<\tfrac{1}{6}\varepsilon .$$

Then 2.2 , (3) and simple triangle inequalities imply

$$K(F,P)\leq K(F,P_1)\leq K(F,P_2)=K(F,P_3)<$$
$$<K(F,\{\sigma,y,x_{i+1},\dots,x_m,z,s\})+\varepsilon=K(h,\{\sigma,y,x_{i+1},\dots,x_m,z,s\})+\varepsilon$$
$$\leq\mathcal{K}(h,M_1)+\varepsilon=\mathcal{K}(f,M)+\varepsilon .$$

P and ε were arbitrary, hence $\mathcal{K}(F,\mathbf{R})=\mathcal{K}(f,M)$. ///

__2.6 Definition__: Let X,Y be Banach spaces, $M\subset X$, $\varphi:M\to Y$ and $x,h\in X$. Let $M_{x,h}=\{t\in\mathbf{R}: x+th\in M\}$ and let us define mapping $\varphi_{x,h}:M_{x,h}\to Y$ by the formula

$$\varphi_{x,h}(t)=\varphi(x+th) .$$

We shall say that φ has linearly finite convexity on M , if $\sup\{\mathcal{K}(\varphi_{x,h},M_{x,h}): x,h\in X, \|h\|=1\}$ is finite.

Thus φ has linearly finite convexity on M iff its restriction on any straight line p has finite convexity on $M \cap p$ and all these convexities have a common upper bound.

Let us note that a mapping φ, possessing a linearly finite convexity on a neighbourhood of a point $x \in X$, has all one-sided directional derivatives at x (by 2.4).

2.7 Definition: Let X,Y be Banach spaces, $M \subset X$ and $\varphi : M \rightarrow Y$. The mapping φ is said to be δ-convex on M iff there exists a convex Lipschitz function g on X with property:
for each $y^* \in Y^*, \|y^*\| = 1$, there exists a convex Lipschitz function h_{y^*} on X such that $y^* \circ \varphi = h_{y^*} - g$ on M .

2.8 Observation: A real function f on a subset M of a Banach space X is δ-convex on M iff f can be extended to a function on X representable as a difference of two convex Lipschitz functions.

2.9 Remark: Let $M \subset \mathbf{R}$, $f : M \rightarrow \mathbf{R}$. Then f is δ-convex on M iff $\mathcal{K}(f,M)$ is finite. This yields from well-known results (cf. [2]) and 2.5 .

2.10 Observation: Let M be a subset of a Banach space X and $\varphi : M \rightarrow \mathbf{R}^n$, $\varphi = [\varphi_1, \dots, \varphi_n]$. Then
/i/ φ is δ-convex on M iff φ_k is δ-convex on M for $k=1,\dots,n$.
/ii/ If φ is δ-convex on M, there exists a δ-convex extension of φ defined on the whole space X. Both propositions are easy consequences of the definition 2.7, /ii/ yields from /i/.

Let us note that if X,Y are metric spaces, $M \subset X$ and $f : M \rightarrow Y$ is a Lipschitz mapping, then there exists a Lipschitz extension $F : X \rightarrow Y$ of f in the following cases:

/i/ $Y = \mathbf{R}^n$

/ii/ X,Y are Hilbert spaces

/iii/ $X = \mathbf{R}$ and Y is a Banach space.

(For references see [7]).

It is not known to the author whether there exist extensions of mappings with linearly finite convexity keeping this property, if $\dim X > 1$ (even in case $X = \mathbf{R}^2, Y = \mathbf{R}$), and δ-convex extensions of δ-convex mappings if $\dim Y = \infty$.

2.11 Definition: Let E be a subset of a Banach space X and $n < \dim X$ be a positive integer. We shall say that E is a Lipschitz fragment of dimension n (of codimension n, respectively) and denote $E \in \mathcal{K}_n$ ($E \in \mathcal{K}^n$, respectively) if the following condition is satisfied: There exist a subspace Z of X of codimension n (of dimension n, resp.), a topological complement W of the space Z in X, a set $M \subset W$ and a Lipschitz mapping $\varphi : M \to Z$ such that $E = \{w + \varphi(w) : w \in M\}$.

If Z,M,W can be chosen in such way that in addition φ is δ-convex on M or φ has linearly finite convexity on M then we shall say that E is a δ-convex fragment or E is an LFC-fragment of given dimension or codimension. The notation will be following: $E \in DC_n$, $E \in DC^n$, $E \in LFC_n$, $E \in LFC^n$.

Fragments with M=W are called surfaces. Surfaces of dimension 1 (of codimension 1, resp.) are called curves (hypersurfaces, resp.).

2.12 Notation: Let $\mathcal{Y}$ be a given system of subsets of a Banach space X. By $\sigma\mathcal{Y}$ we denote the system of all sets representable as a union of countably many elements from $\mathcal{Y}$. (For example: $E \in \sigma DC^n$ means that E can be written as a countable union of δ-convex fragments of codimension n).

2.13 Observations: a/ Every $E \in \mathcal{K}_n$ has σ-finite n-dimensional Hausdorff measure. In particular, if $X = \mathbb{R}^m$, $m > n$, then E is of Lebesgue measure zero.

b/ Every surface from $\mathcal{K}_n$ has infinite but σ-finite n-dimensional Hausdorff measure and its Hausdorff dimension is n.

c/ As consequences of 2.5 , 2.10 and extensiontheorems for Lipschitz mappings we obtain the following propositions:

Every $E \in \mathcal{K}^n$ is a subset of a Lipschitz surface of codimension n.

Every $E \in \mathcal{K}_1$ is a subset of a Lipschitz curve.

Every $E \in DC^n$ is a subset of a δ-convex surface of codimension n.

Every $E \in LFC_1$ is a subset of an LFC-curve.

If X is a Hilbert space then every $E \in \mathcal{K}_n$ is a subset of a Lipschitz surface of dimension n.

3. Multiplicity points of monotone operators

By exp A we shall denote the system of all subsets of a set A and by coA the convex hull of A.

The dimension (codimension, resp.) of a convex set is meant as the dimension (codimension, resp.) of its affine hull.

Let X be a Banach space with dual space X^* and $T: X \to \exp X^*$ be a monotone operator. We shall use the following notation:

$$A_n = \{x \in X: \dim(\mathrm{co}Tx) \geq n\}$$

$$A^n = \{x \in X: \mathrm{co}Tx \text{ contains a ball of codimension } n\}$$

$$\mathrm{gph}\, T = \{[x,x^*] \in X \times X^*: x^* \in Tx\}.$$

3.1 Definition: Let $T, \tilde{T}$ be monotone operators on X. We shall write $T \subset \tilde{T}$ if $\mathrm{gph}\, T \subset \mathrm{gph}\, \tilde{T}$. T is called a maximal monotone operator if $T \subset \tilde{T}$ implies $T = \tilde{T}$.

3.2 Observation: a/ For every monotone operator T there exists a maximal monotone operator T_{max} such that $T \subset T_{max}$, by Zorn's lemma.

b/ It is easy to see that Tx is always convex if T is a maximal monotone operator.

By a proper convex function (cf. [3]) it is meant a mapping $f: X \to \mathbb{R} \cup \{+\infty\}$ satisfying following two conditions:

$$\forall x,y \in X \quad \forall \lambda \in (0,1) \quad f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \qquad (4)$$

$$\mathrm{dom}\, f = \{x \in X: f(x) < +\infty\} \neq \emptyset . \qquad (5)$$

3.3 Definition: Let f be a proper convex function on a Banach space X and $x \in X$. If $x \in \mathrm{dom}\, f$, we define

$$\partial f(x) = \{x^* \in X^*: \forall z \in X \quad f(z) \geq f(x) + \langle z-x, x^* \rangle\}.$$

We put $\partial f(x) = \emptyset$ in case $f(x) = +\infty$. The mapping $\partial f: x \mapsto \partial f(x)$ is called subdifferential of f.

It will fit to define $\partial f \equiv \emptyset$ for $f \equiv +\infty$.

3.4 Remark: Subdifferentials of proper convex functions are monotone but not conversely. There exist monotone operators which are not subdifferentials ([3]). The characterization of subdifferentials of proper convex functions using the notion of a cyclically monotone operator is due to R.T.Rockafellar (see [4]).

The main result of this paper is contained in the following two theorems.

3.5 Theorem: Let T be a monotone operator on a separable Banach space X and $n < \dim X$ be a positive integer. Then $A_n \in \sigma LFC^n$. If in addition $T \subset \partial f$ for some proper convex function f on X then $A_n \in \sigma DC^n$.

3.6 Theorem: Let T be a monotone operator on a Banach space X with separable dual space X^* and $n < \dim X$ be a positive integer. Then $A^n \in \sigma LFC_n$. If in addition $T \subset \partial f$ for some proper convex function f on X then $A^n \in \sigma DC_n$.

These theorems say that the sets A_n and A^n can be written as a countable union of images of special Lipschitz mappings (defined on a subset of a Banach space of codimension n or dimension n, respectively).

Both proofs are practically equal and we shall do it simultaneously.

At first we state the following simple lemmas without a proof (see [7], Lemma 1, Lemma 2). An open ball with centre c and radius $r > 0$ is denoted by $\Omega(c,r)$.

3.7 Lemma: Let X be a separable Banach space. Then there exist a countable system $\mathcal{V}$ of n-codimensional subspaces of X^* and a countable system $\mathcal{L}$ of n-codimensional affine subsets of X^* such that: /i/ Any n-dimensional subspace $P \subset X^*$ has a topological complement $V \in \mathcal{V}$.

/ii/ If P,V are as in /i/, $c^* \in X^*$, $\varepsilon > 0$ then there exists $t \in (c^*+P) \cap \Omega(c^*,\varepsilon)$ such that $L=t+V \in \mathcal{L}$.

3.8 Lemma: Let Y be a separable Banach space. Then there exist a countable system $\mathcal{V}$ of n-dimensional subspaces of Y and a countable system $\mathcal{L}$ of n-dimensional affine subsets of Y such that: /i/ Any subspace $P \subset Y$ of codimension n has a topological complement $V \in \mathcal{V}$.

/ii/ If P,V are as in /i/, $c^* \in Y$, $\varepsilon > 0$ then there exists $t \in (c^*+P) \cap \Omega(c^*,\varepsilon)$ such that $L=t+V \in \mathcal{L}$.

3.9 Proof of 3.5 and 3.6: Let X,T be as in 3.5 (in 3.6, resp.) and $A=A_n$ ($A=A^n$, resp.). Without any loss of generality we can suppose Tx to be convex for any x (see 3.2).

a/ Decomposition of A.

Let x be an arbitrary element of A. Then there exist a point $c_x \in Tx$, a positive rational number r_x and a subspace $P_x \subset X^*$ of dimension n (of codimension n, resp.) such that

$$(c_x+P_x) \cap \Omega(c_x,r_x) \subset Tx .$$

Let m_x be a rational number such that $\|c_x\| < m_x$. Lemma 3.7 (3.8, resp.) guarantees an existence of a topological complement $V_x \in \mathcal{V}$ of P_x and a point $t_x \in (c_x+P_x) \cap \Omega(c_x, \frac{1}{2}\cdot r_x)$ such that $L_x = t_x + V_x \in \mathcal{L}$.

Let us find a rational number q_x such that $\|\pi_x\| < q_x$, where $\pi_x : X^* \to P_x$ is a projection in the direction of V_x.

For any r,m,q positive rational, $V \in \mathcal{V}$, $L \in \mathcal{L}$ let us denote

$$B(r,m,V,q,L) = \{x \in A: r_x=r,\ m_x=m,\ V_x=V,\ q_x=q,\ L_x=L\} .$$

It is clear that $A=\bigcup B(r,m,V,q,L)$ and the union is countable.

Let r,m,V,q,L be fixed. We shall show that the set $B=B(r,m,V,q,L)$ is a Lipschitz fragment of codimension n (of dimension n, resp.).

b/ "Parametrization" of B.

Define $Z={}^{\perp}V$. Let W be an arbitrary topological complement of Z in X and $Y=W^{\perp}$. Then Y is a topological complement of V in X^*.

The following proposition is true: (6)

$z^* \in Z^*$ iff there exists $y^* \in Y$ such that $z^*=y^*$ on Z .

There exists a point $y_0 \in Y$ such that $L=y_0+V$. Let us denote

$$M = \{w \in W: \exists z \in Z \quad w+z \in B\} ,$$

i.e. M is a projection of B on the subspace W in the direction of Z .

c/ B is a Lipschitz fragment.

Let $B \neq \emptyset$. Let $w_1,w_2 \in M$, $z_1,z_2 \in Z$ such that $x_i=w_i+z_i \in B$ for i=1,2. Let us denote $t_i=t_{x_i}$, $\pi_i=\pi_{x_i}$

Let $y^* \in Y$ be an arbitrary functional from a unit sphere in Y. Define

$$t_1^+ = t_1 + \frac{r}{2q}\pi_1(y^*)$$

$$t_1^- = t_1 - \frac{r}{2q}\pi_1(y^*).$$

The fact $t_1^+, t_1^- \in Tx_1$ follows from inequalities

$$\|t_1^+ - c_{x_1}\| \leq \|t_1 - c_{x_1}\| + \|\tfrac{r}{2q}\pi_1(y^*)\| < r \ , \ \|t_1^- - c_{x_1}\| < r \ .$$

The monotonicity of T implies

$$0 \leq \langle x_1 - x_2, t_1 - t_2 \pm \tfrac{r}{2q}\pi_1(y^*)\rangle =$$

$$= \langle w_1 - w_2, t_1 - t_2 \pm \tfrac{r}{2q}\pi_1(y^*)\rangle \pm \langle z_1 - z_2, \tfrac{r}{2q} y^*\rangle.$$

(We have used the fact that the functionals $t_1 - t_2$, $y^* - \pi_1(y^*)$ are elements of V.) Now we obtain

$$\mp \langle z_1 - z_2, y^*\rangle \leq \tfrac{2q}{r}\langle w_1 - w_2, t_1 - t_2 \pm \tfrac{r}{2q}\pi_1(y^*)\rangle \leq$$

$$\leq \tfrac{2q}{r}\|w_1 - w_2\|\left(\|t_1 - c_{x_1}\| + \|c_{x_1}\| + \|t_2 - c_{x_2}\| + \|c_{x_2}\| + \tfrac{r}{2q}\|\pi_1\|\right)$$

$$< \tfrac{2q}{r}\|w_1 - w_2\|\left(\tfrac{r}{2} + m + \tfrac{r}{2} + m + \tfrac{r}{2}\right) = \frac{q(3r+4m)}{r}\|w_1 - w_2\| \ .$$

Then by (6)

$$\|z_1 - z_2\| = \sup\{|\langle z_1 - z_2, y^*\rangle| : y^* \in Y, \|y^*\| = 1\} \leq \frac{q(3r+4m)}{r}\|w_1 - w_2\| \ .$$

If we take $\varphi(w) \in Z$ (for $w \in M$) such that $w + \varphi(w) \in B$, we obtain a correctly defined mapping which is Lipschitz on M and satisfies

$$B = \{w + \varphi(w) : w \in M\} \ .$$

d/ φ has linearly finite convexity on M.

Let $w_o \in W$, $h \in W$, $\|h\| = 1$. Denote $D = M_{w_o,h}$, $F = \varphi_{w_o,h}$ (see 2.6).

If D contains less than three elements then $\mathcal{K}(F,D) = 0$ by the definition. Let D have at least three elements and

$$\{d_o < d_1 < \dots < d_s < d_{s+1}\} \in \mathcal{P}(D) \ .$$

For $0 \leq j \leq s+1$ let us introduce following simplifications:

$$x_j = w_o + d_j h + F(d_j)$$

$$t_j = t_{x_j}$$

$$\pi_j = \pi_{x_j}$$

x_j's are obviously points from B. The monotonicity of T implies

$$0 \leq i < j \leq s+1 \implies \langle h, t_j - t_i\rangle \geq 0 \ .$$

Let us choose an arbitrary number $i \in \{1, 2, \dots, s\}$ and a functional $y^* \in Y$ such that $\|y^*\| = 1$.

Denote $t_i^+ = t_i + \frac{r}{2q}\pi_i(y^*)$, $t_i^- = t_i - \frac{r}{2q}\pi_i(y^*)$. We have $t_i^+, t_i^- \in Tx_i$ similarly as in the part c/. Using the monotonicity of T we obtain

$$0 \le \langle x_i - x_{i-1}, t_i^- - t_{i-1}\rangle = (d_i - d_{i-1})\langle h, t_i - t_{i-1}\rangle -$$
$$- \frac{r}{2q}(d_i - d_{i-1})\langle h, \pi_i(y^*)\rangle - \frac{r}{2q}\langle F(d_i) - F(d_{i-1}), y^*\rangle \quad \text{and hence}$$
$$\langle Q_F(d_{i-1}, d_i), y^*\rangle \le \frac{2q}{r}\langle h, t_i - t_{i-1}\rangle - \langle h, \pi_i(y^*)\rangle .$$

Analogous calculations with

$$0 \le \langle x_i - x_{i-1}, t_i^+ - t_{i-1}\rangle$$
$$0 \le \langle x_{i+1} - x_i, t_{i+1} - t_i^-\rangle$$
$$0 \le \langle x_{i+1} - x_i, t_{i+1} - t_i^+\rangle$$

will afford the following inequalities:

$$-\langle Q_F(d_{i-1}, d_i), y^*\rangle \le \frac{2q}{r}\langle h, t_i - t_{i-1}\rangle + \langle h, \pi_i(y^*)\rangle$$
$$-\langle Q_F(d_i, d_{i+1}), y^*\rangle \le \frac{2q}{r}\langle h, t_{i+1} - t_i\rangle + \langle h, \pi_i(y^*)\rangle$$
$$\langle Q_F(d_i, d_{i+1}), y^*\rangle \le \frac{2q}{r}\langle h, t_{i+1} - t_i\rangle - \langle h, \pi_i(y^*)\rangle .$$

Then for any $y^* \in Y$, $\|y^*\| = 1$

$$|\langle Q_F(d_{i-1}, d_i) - Q_F(d_i, d_{i+1}), y^*\rangle| \le \frac{2q}{r}\langle h, t_{i+1} - t_{i-1}\rangle$$

and hence by (6)

$$\| Q_F(d_{i-1}, d_i) - Q_F(d_i, d_{i+1})\| \le \frac{2q}{r}\langle h, t_{i+1} - t_{i-1}\rangle .$$

Then

$$\sum_{i=1}^{s} \|Q_F(d_{i-1}, d_i) - Q_F(d_i, d_{i+1})\| \le \frac{2q}{r}\langle h, t_{s+1} + t_s - t_1 - t_0\rangle \le$$
$$\le \frac{2q}{r}(\|t_{s+1}\| + \|t_s\| + \|t_1\| + \|t_0\|) < \frac{4q(r+2m)}{r}$$

because $\quad \|t_j\| \le \|t_j - c_{x_j}\| + \|c_{x_j}\| < \frac{r}{2} + m$.

So we managed to estimate $\mathcal{K}(F,D)$ from above independently on the choice of w_o and h, and that is why φ has linearly finite convexity on M.

d/ φ is δ-convex on M if $T \subset \partial f$.

Let $T \subset \partial f$ for some proper convex function on X. Without any loss of generality we can suppose $T = \partial f$. Now Tx is always convex

Let $y^* \in Y$, $\|y^*\| = 1$. For any $x \in B$ we shall denote w_x the projection of x on W in the direction of Z. Then $w_x \in M$ and $x = w_x + \varphi(w_x)$. A functional $t_x^- = t_x - \frac{r}{2q}\pi_x(y^*)$ is an element of Tx because $\|\frac{r}{2q}\pi_x(y^*)\| < \frac{r}{2}$.

Let us denote

$$g_1(w_x) = f(x) - \langle \varphi(w_x), t_x\rangle$$
$$h_1(w_x) = f(x) - \langle \varphi(w_x), t_x^-\rangle .$$

g_1, h_1 are finite real functions on M.

Let $x_0 \in B$ be fixed. We shall define two continuous affine functions on W:
$$a_{x_0}(w) = f(x_0) + \langle w-x_0, t_{x_0}\rangle$$
$$b_{x_0}(w) = f(x_0) + \langle w-x_0, t^-_{x_0}\rangle \ .$$
For any $x \in B$ the functionals $t_x - t_{x_0}, t^-_x - t^-_{x_0}$ are in V and $t_{x_0}, t^-_{x_0}$ are in $\partial f(x_0)$, hence
$$a_{x_0}(w_x) = f(x_0) + \langle x-x_0, t_{x_0}\rangle - \langle \varphi(w_x), t_{x_0}\rangle \leq$$
$$\leq f(x) - \langle \varphi(w_x), t_{x_0}\rangle = f(x) - \langle \varphi(w_x), t_x\rangle = g_1(w_x) \ ,$$
$$a_{x_0}(w_{x_0}) = f(x_0) - \langle \varphi(w_{x_0}), t_{x_0}\rangle = g_1(w_{x_0}) \ .$$
Similarly $b_{x_0}(w_x) \leq h_1(w_x)$, $b_{x_0}(w_{x_0}) = h_1(w_{x_0})$.

The functions a_{x_0}, b_{x_0} are Lipschitz with the constant $m+r$ (since $\| t^-_{x_0}\| < m+r$, $\| t_{x_0}\| < m + \frac{r}{2} < m+r$).

The former properties enable us to say that the functions
$$g(w) = \sup\{a_{x_0}(w) : x_0 \in B\}$$
$$h(w) = \sup\{b_{x_0}(w) : x_0 \in B\}$$
are Lipschitz convex functions on W satisfying $g=g_1$, $h=h_1$ on M and the function g does not depend on the choise of y^*. For any $x \in B$
$$h_1(w_x) - g_1(w_x) = \frac{r}{2q} \langle \varphi(w_x), \pi_x(y^*)\rangle = \frac{r}{2q} \langle \varphi(w_x), y^*\rangle \ .$$
Put $G(w) = \frac{2q}{r} g(w)$, $H_{y^*}(w) = \frac{2q}{r} h(w)$.

We have proved that for any $y^* \in Y$: $y^* \circ \varphi = H_{y^*} - G$ on M where H_{y^*}, G are convex Lipschitz functions on W and G is independent on y^*. Hence φ is δ-convex on M regarding (6).

The theorems 3.5, 3.6 are proved. ///

The following proposition is a direct consequence of 3.5, 3.6 and 2.13.

<u>3.10 Corollary</u>: Let T be a monotone operator on a separable Banach space X and $n < \dim X$ be a positive integer. Then the **set** A_n can be covered by **countably many** Lipschitz surfaces of codi-

mension n. If $T \subset \partial f$ for some proper convex function f then the set A_n can be covered by countably many DC-surfaces of codimension n.
If X^* is separable then the set A^1 for a general monotone operator T can be covered by countably many LFC-curves.
If X is a separable Hilbert space then A^n can be covered by countably many Lipschitz surfaces of codimension n.

3.11 Observation: Let us observe that in case $X = \mathbb{R}^2$, 3.10 ensures a countable covering of the set A_1 of a general monotone operator T on $\mathbb{R}^2$ by LFC-curves which are simultaneously DC-hypersurfaces in this case. (Compare the problem 1.1.)

There are sometimes considered monotone operators on X^* with values in X, e.g. an operator T_{-1} "inverse" to a monotone operator T on X:

$$T_{-1}: X^* \to \exp X$$
$$T_{-1}(x^*) = \{x \in X:\ x^* \in Tx\}.$$

In this cases the following version of 3.6 is useful. (The proof is similar; instead of $\|x^*\| = \sup\{\langle x^*, x^{**}\rangle :\ \|x^{**}\| = 1\}$ use $\|x^*\| = \sup\{\langle x, x^*\rangle :\ \|x\| = 1\}$ and change the roles of X and X^*.)

3.12 Theorem: Let X be a separable Banach space, $T: X^* \to \exp X$ be a monotone operator and $n < \dim X$ be a positive integer. Then $A^n \in \sigma LFC_n$. If $T \subset \partial f$ for some proper convex function f on X^* then $A^n \in \sigma DC_n$.

4. Operators V_M, F_M

Let M be a nonvoid convex subset of a Banach space X. We shall state the definition of a vertex-operator $V_M: X \to \exp X^*$ and a face-operator $F_M: X^* \to \exp X$ which are in close connection with singular points of M (cf. [8]).

4.1 Definition: Let

$$\delta_M(x) = \begin{cases} 0 & \text{if } x \in M, \\ +\infty & \text{if } x \notin M; \end{cases}$$

$$s_M(x^*) = \sup\{\langle m, x^*\rangle :\ m \in M\},\quad x^* \in X^*.$$

δ_M is called indicator-function of M and is a proper convex function on X. The function s_M satisfies $s_M(tx^*)=t.s_M(x^*)$, $s_M(x^*+y^*)\leq \leq s_M(x^*)+s_M(y^*)$ for any $t>0$, $x^*,y^*\in X^*$. Hence if dom s_M is not empty then s_M is a proper convex function on X^*.

<u>4.2 Definition</u>:

$$V_M(x) = \begin{cases} \{y^*\in X^*: \langle x,y^*\rangle = s_M(y^*)\} & \text{if } x\in M, \\ \emptyset & \text{if } x\notin M; \end{cases}$$

$$F_M(x^*) = \{y\in M: \langle y,x^*\rangle = s_M(x^*)\}, \qquad x^*\in X^*.$$

<u>4.3 Note</u>: <u>a</u>/ If $X=\mathbb{R}^m$ then $V_M(x)$ is the set of all normals of M at x and is called vertex of M at x. The set $F_M(x^*)$ forms a face of M perpendicular to x^*.
<u>b</u>/ It is obvious that the operators V_M, F_M are monotone and their images $V_M(x), F_M(x^*)$ of each point are convex. Following simple lemma says a little more.

<u>4.4 Lemma</u>: $V_M = \partial\delta_M$, $F_M \subset \partial s_M$.

<u>Proof</u>: <u>a</u>/ If $x\notin M$ then $V_M(x)=\emptyset=\partial\delta_M(x)$. Let $x\in M$. Then the following equivalences hold:

$x^*\in V_M(x) \Leftrightarrow \forall m\in M \ \ 0\geq\langle m-x,x^*\rangle \Leftrightarrow \forall z\in X \ \ \delta_M(z)\geq\delta_M(x)+ +\langle z-x,x^*\rangle \Leftrightarrow x^*\in\partial\delta_M(x)$.

<u>b</u>/ If $F_M(x^*)=\emptyset$ then $F_M(x^*)\subset\partial s_M(x^*)$ is evident. Let $x\in F_M(x^*)$. Then any $z^*\in X^*$ satisfies $s_M(z^*)\geq\langle x,z\rangle = s_M(x^*)+\langle x,z^*-x^*\rangle$ and hence $x\in\partial s_M(x^*)$. ///

<u>4.5 Theorem</u>: If X is separable then $A_n(V_M)\in\sigma DC^n(X)$,
$A^n(F_M)\in\sigma DC_n(X^*)$.

If X^* is separable then $A^n(V_M)\in\sigma DC_n(X)$, $A_n(F_M)\in\sigma DC^n(X^*)$.

<u>Proof</u>:
The propositions of the theorem yield from 3.5, 3.6, 4.4 . ///

Using known extension theorems it is possible to obtain following new result.

<u>4.6 Theorem</u>: Let M be a nonempty convex subset of a separable Banach space X. Then:

/i/ The set of points $x\in M$ for which $V_M(x)$ is at least n-dimensional can be covered by countably many DC-surfaces of codimension n.

/ii/ If in addition X^* is separable then the set of all normals x^* to M at faces $F_M(x^*)$ being at least n-dimensional can be covered by countably many DC-surfaces of codimension n, and the set of all points $x \in M$ with a vertex $V_M(x)$ containing a ball of codimension 1 can be covered by countably many LFC-curves.

5. Existence of "bad" Lipschitz surfaces

We shall show that there exist Lipschitz surfaces of codimension n (dimension n, respectively) which cannot be a subset of A_n (A^n, resp.) for any monotone operator T satisfying assumptions of the theorems 3.5, 3.6 .

We shall use the local geometric term of a contingent of a set at a point (cf. [5]).

5.1 Definition: Let X be a Banach space, $x \in X$, $M \subset X$. Then we define cont(M,x) as the set of all nonzero vectors $v \in X$ which satisfy the following condition:

There exist sequences $\{x_n\} \subset X$, $\{\lambda_n\} \subset \mathbb{R}$ such that

/i/ $x_n \in M$,

/ii/ $\lambda_n > 0$,

/iii/ $\lambda_n \to 0$,

/iv/ $\left\| \frac{x_n - x}{\lambda_n} - v \right\| \to 0$.

5.2 Construction: Let X be a Banach space, W,Z closed subspaces of X such that $X = W \oplus Z$ (i.e. X is a topological sum of W,Z). Let $h \in W$, $z_o \in Z$ be nonzero vectors and U be a topological complement of $\mathrm{lin}\{h\}$ in the space W. We shall define a Lipschitz mapping $F: W \to Z$ by the formula

$$F(th+u) = f(t) z_o \qquad t \in \mathbb{R},\ u \in U ,$$

where f is a real Lipschitz function on $\mathbb{R}$ which has right derivative $f'_+(t)$ at no rational point t. (Existence of f is guaranteed by a standard category argument.)

Denote $E = \{w+F(w): w \in W\}$. Let $q \in \mathbb{R}$, $u_o \in U$, $x=qh+u_o+f(q)z_o \in E$. It is easy to prove that $\mathrm{cont}(E,x)$ contains the set $C = \{\alpha h+u+\beta z_o+y: \alpha > 0,\ u \in U,\ \alpha D_+f(q) \le \beta \le \alpha D^+f(q),\ y \in Y\}$, where D_+f, D^+f denote the lower and upper Dini derivatives of f and Y is a topological complement of $\mathrm{lin}\{z_o\}$ in Z.
Hence $\mathrm{int}(\mathrm{cont}(E,x)) \ne \emptyset$ if $x=qh+u_o+f(q)z_o$ with q rational.(7)

<u>5.3 Lemma</u>: Let X be a Banach space, W,Z be closed subspaces of X such that $X=W \oplus Z$. Let $w_o \in W$ and $G:W \to Z$ be a Lipschitz mapping having all one-sided directional derivatives at w_o. Denote

$M = \{w+G(w): w \in W\}$,

$x = w_o + G(w_o)$,

$\pi_W: X \to W$ a projection in the direction of Z .

Then, if $v_1,v_2 \in \mathrm{cont}(M,x)$, $\pi_W(v_1)=\pi_W(v_2)$ then $v_1=v_2$.

<u>Proof</u>: Let $v_1,v_2 \in \mathrm{cont}(M,x)$, $\pi_W(v_1)=\pi_W(v_2)=\nu$. The vector ν is nonzero because G is Lipschitz. Let $z_1,z_2 \in Z$ be such that $v_i=\nu+z_i$ $(i=1,2)$. Let U_ν be a topological complement of $\mathrm{lin}\{\nu\}$ in W, $\pi_\nu: W \to \mathrm{lin}\{\nu\}$ a projection in the direction of U_ν , $\pi: W \to U_\nu$ a projection in the direction of ν . By 5.1 we have

$$x_{n,i}=w_{n,i}+G(w_{n,i}),\quad \lambda_{n,i}>0,\quad \lambda_{n,i}\xrightarrow[n]{}0,$$

$$A_{n,i}=\left(\frac{x_{n,i}-x}{\lambda_{n,i}}-v_i\right)\xrightarrow[n]{}0 \qquad (i=1,2).$$

Let $a_{n,i} \in \mathbb{R}$ be such that $a_{n,i}\nu = \pi_\nu(w_{n,i}-w_o)$. Then

$$\lim_{n\to\infty}\frac{a_{n,i}}{\lambda_{n,i}} = 1 \tag{8}$$

because

$$\left\|\left(\frac{a_{n,i}}{\lambda_{n,i}}-1\right)\nu\right\| = \|\pi_\nu(\pi_W(A_{n,i}))\| \to 0 .$$

Without any loss of generality we can suppose $a_{n,i} > 0$ $(i=1,2$, $n=1,2,\dots)$. Then

$$\left\|\frac{G(w_o+a_{n,i}\nu)-G(w_o)}{\lambda_{n,i}} - z_i\right\| = \left\|\frac{w_{n,i}+G(w_{n,i})-w_o-G(w_o)}{\lambda_{n,i}} - v_i + \right.$$

$$\left. + \frac{\lambda_{n,i}\nu - a_{n,i}\nu}{\lambda_{n,i}} + \frac{a_{n,i}\nu-(w_{n,i}-w_o)}{\lambda_{n,i}} + \frac{G(w_o+a_{n,i}\nu)-G(w_{n,i})}{\lambda_{n,i}}\right\| \le$$

$$\leq |A_{n,i}| + |1-\frac{a_{n,i}}{\lambda_{n,i}}| . |\nu| + (1+L) |\pi(\pi_W(A_{n,i}))| \xrightarrow[n]{} 0 ,$$

where L is the constant from the Lipschitz property of G. Then (8) and the existence of a directional derivative $\delta_+ G(w_0,\nu)$ imply $z_1 = \delta_+ G(w_0,\nu) = z_2$. ///

5.4 Theorem: Let X be a separable Banach space (X has separable dual X^*, resp.), $n < \dim X$ be a positive integer. Then the set E from 5.2 with dim Z=n (codim Z=n, resp.) is a Lipschitz surface of codimension n (of dimension n, resp.) which cannot satisfy $E \subset A_n$ ($E \subset A^n$, resp.) for any monotone operator T on X.

Proof: Let us assume the existence of T such that $E \subset A_n$ ($E \subset A^n$, resp.). Then (in the notation of 3.9) $E \subset \bigcup B(r,m,V,q,L)$. There exist r_0, m_0, V_0, q_0, L_0, a positive number δ and a point $x_0 \in E$ such that the set $B_0 = B(r_0,m_0,V_0,q_0,L_0)$ is dense in $E \cap \Omega(x_0,\delta)$, by the Baire Category Theorem.
Let $Z_0 = {}^{\perp}V_0$, W_0 be a topological complement of Z_0 in X and π_0: $X \rightarrow W_0$ be a projection in the direction of Z_0. The set $M_0 = \pi_0(B_0)$ is dense in $S = \pi_0(E \cap \Omega(x_0,\delta))$, which is an open set containing the point $\pi_0(x_0)$. By the part d/ of 3.9, there exists a Lipschitz mapping $\varphi_0 : M_0 \rightarrow Z_0$ with a linearly finite convexity on M_0 such that $B_0 = \{w + \varphi_0(w) : w \in M_0\}$.
φ_0 has unique continous extension $\bar{\varphi}_0$ on $\bar{M}_0$. This extension is Lipschitz, has linearly finite convexity on $\bar{M}_0$ and has by 2.4 all one-sided directional derivatives at each point $\pi_0(x) \in S$.
 $\mathrm{int}(\mathrm{cont}(E,x)) = \emptyset$ for every $x \in E \cap \Omega(x_0,\delta)$ by 5.3 .
But the construction of E implies that there exists a point $\tilde{x} = qh + u + f(q) z_0 \in E \cap \Omega(x_0,\delta)$ with q rational. Then cont(E,x) has nonempty interior by (7) and this is the needed contradiction.

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R e f e r e n c e s

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