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ON EXTENDED SHANNON ENTROPIES AND THE EPSILON ENTROPY

Miroslav KATĚTOV

Abatract: On the class of all metrized probability spaces, a certain modification of one of the extended Shannon entropies introduced by the author coincides (up to a multiplicative constant) with the epsilon entropy as introduced by Posner, Rodemich, and Rumsey.

Key words: Extended Shannon entropies, epsilon entropy.

Classification: 94A17

When examining the extended Shannon entropies in [1] and [2], the author aimed, among other things, at introducing a concept from which various kinds of entropies (such as e.g. the ε-entropy of totally bounded metric spaces and the differential entropy) could be obtained in a natural way. In the present note, the epsilon entropy in the sense of Posner, Rodemich, and Humphrey (which is closely related to the ε-entropy of metric spaces) is shown to coincide with a fairly natural modification of the entropy C_E (see [1]).

1

1.1. The letters R and N have their usual meaning. We put $\overline{R} = \{-\infty\} \cup R \cup \{\infty\}$, $R_+ = \{x \in R : x \geq 0\}$, $\overline{R}_+ = \{x \in \overline{R} : x \geq 0\}$, $R_+^* = \{x \in R : x > 0\}$, $N_1 = \{n \in N : n \geq 1\}$, $[m,n] = \{k \in N : m \leq k \leq n\}$ for $m,n \in N$. - Instead of $\log_2$ we write log. We put $\hat{L}(0) = L(0) = 0$,

$\hat{L}(x) = -x\ln x$, $L(x) = -x \log x$ for $x \in R_+^*$. Instead of $\hat{L}(x)$ and $L(x)$ we often write, respectively, $\hat{L}x$ and Lx. - If $K \neq \emptyset$ is a set, then $\ell_1^+(K)$ denotes the set of all $x = (x_k : k \in K)$ such that $x_k \in R_+$ and $\Sigma x_k < \infty$. If $x = (x_k : k \in K) \in \ell_1^+(K)$, then we put $H(x) = H(x_k : k \in K) = \Sigma(Lx_k : k \in K) - L\Sigma(x_k : k \in K)$, $\hat{H}(x) = \hat{H}(x_k : k \in K) = \Sigma(\hat{L}x_k : k \in K) - \hat{L}\Sigma(x_k : k \in K)$. - A function (or a functional) is a mapping $f : X \to \overline{R}$.

1.2. <u>Facts</u>. A) If $x \in \ell_1^+(K)$, $a \in R_+$, then $\hat{H}(x) = H(x) \cdot \ln 2$, $H(ax) = aH(x)$. - B) If $x = (x_1, \dots, x_n) \in \ell_1^+(n)$, then $H(x_1, \dots, x_n) \leq \Sigma x_i \cdot \log n$. - C) If $x \in \ell_1^+(J \times K)$, then $H(x) = \Sigma(H(x_{jk} : k \in K) : j \in J) + H(\Sigma(x_{jk} : k \in K) : j \in J)$.

1.3. A measure is always a finite measure on a set $Q \neq \emptyset$, i.e. a σ-additive $\mu : \mathcal{A} \to R_+$, where $\mathcal{A}$ (denoted by dom μ) is a σ-algebra of subsets of Q. If $f : Q \to \overline{R}$ is $\overline{\mu}$-measurable, $\overline{\mu}\{x \in Q : f(x) < 0\} = 0$ and $\int f d\mu < \infty$; then $X \mapsto \int_X f d\mu$, defined on dom μ , is a measure, which will be denoted by $f \cdot \mu$. If $Y \in \text{dom}\, \overline{\mu}$, then we put $Y \cdot \mu = i_Y \cdot \mu$, where i_Y is the indicator of Y.

1.4. If $\rho : Q \times Q \to R_+$ satisfies $\rho(x,x) = 0$, $\rho(x,y) = \rho(y,x)$, then ρ is called a semimetric on Q and $\langle Q, \rho \rangle$ is called a semimetric space. If $\langle Q, \rho \rangle$ is a metric space, then $\mathfrak{B} = \mathfrak{B}\langle Q, \rho \rangle$ denotes the collection of all Borel sets $X \subset Q$. - For any set Q and any $a \in R_+^*$, a_Q or a denotes the metric ρ on Q satisfying $\rho(x,y) = a$ for $x \neq y$.

1.5. <u>Definition</u>. Let μ and ρ be, respectively, a measure and a $[\mu \times \mu]$-measurable semimetric on Q. Then $P = \langle Q, \rho, \mu \rangle$ is called a semimetrized measure space or a W-space. For any W-space $P = \langle Q, \rho, \mu \rangle$, we put $wP = \mu Q$. - The class of all W-spaces is denoted by $\mathfrak{W}$. A W-space $\langle Q, \rho, \mu \rangle$ will be called (1) an FWspace, (2) a graph W-space or a GW-space, (3) a metric W-space if, respectively, (1) Q is finite, dom μ = exp Q, (2) $[\mu \times \mu]$ $\{(x,y) \in Q \times Q : 0 \neq \rho(x,y) \neq 1\} = 0$, (3) ρ is a metric. The corresponding classes (i.e. that of all FW-spaces, etc.) will be denoted by (1) $\mathfrak{M}_F$, (2) $\mathfrak{M}_G$, (3) $\mathfrak{M}_M$.

1.6. Let $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$. If ν is a measure, dom ν = = dom μ, $\nu \leq \mu$, then we call $S = \langle Q, \rho, \nu \rangle$ a subspace of P and write $S \leq P$; if $\nu = Y \cdot \mu$ for some $Y \in \operatorname{dom} \bar{\mu}$, then S is called pure. If $K \neq \emptyset$ is a countable set, $P_k = \langle Q, \rho, \mu_k \rangle \in \mathfrak{M}$, $k \in K$, $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$ and $\mu = \Sigma(\mu_k : k \in K)$, then we put P = = $\Sigma(P_k : k \in K)$ and call $(P_k : k \in K)$ an ω-partition of P. An ω-partition $(P_k : k \in K)$ of P is called a partition if K is finite, pure if all P_k are pure. If $\mathcal{U} = (U_k : k \in K)$ and $\mathcal{V} = (V_m : m \in M)$ are ω-partitions of P and there is a partition $(M_k : k \in K)$ of the set M such that, for each $k \in K$, either $\Sigma(V_m : m \in M_k) = U_k$ or $U_k = \emptyset \cdot P$, $M_k = \emptyset$, then $\mathcal{V}$ is said to refine $\mathcal{U}$.

1.7. Let $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$. If f is a function such that $f \cdot \mu$ is defined (see 1.3), then we put $f \cdot P = \langle Q, \rho, f \cdot \mu \rangle$. If $X \in \operatorname{dom} \bar{\mu}$, we put $X \cdot P = \langle Q, \rho, X \cdot \mu \rangle$. - For any $S \leq P$, there exists a function f such that $S = f \cdot P$.

1.8. Fact. If $\langle Q, \rho \rangle$ is a separable metric space, μ is a measure on $\langle Q, \rho \rangle$ and $\mathcal{B} \subset \operatorname{dom} \bar{\mu}$, then $\langle Q, \rho, \mu \rangle \in \mathfrak{M}$.

Proof. Let $a \in R_+$. The set $G = \{(x,y) : \rho(x,y) < a\}$ is open in $Q \times Q$, and therefore, $Q \times Q$ being separable, it is of the form $\cup(X_n \times Y_n : n \in N)$, where X_n, Y_n are open in Q. Since X_n, Y_n are in dom $\bar{\mu}$, we get $G \in \operatorname{dom}[\mu \times \mu]$.

1.9. Notation. The class of all (P_1, P_2) such that $P_1 \leq P$, $P_2 \leq P$ for some $P \in \mathfrak{M}$ will be denoted by $\mathfrak{A}$. If $P_i = \langle Q, \rho, \mu_i \rangle$, i = 1,2, and $(P_1, P_2) \in \mathfrak{A}$, then we put (1) $r(P_1, P_2)$ = $= \int \rho \, d(\mu_1 \times \mu_2)/wP_1 \cdot wP_2$ if $wP_1 \cdot wP_2 > 0$, $r(P_1, P_2) = 0$ if $wP_1 \cdot wP_2 = 0$,

(2) $d(P_1,P_2) = \inf\{a \in \overline{R}_+ : [\mu \times \mu]\{(x,y): \rho(x,y) > a\} = 0\}$;

(3) $E(P_1,P_2) = d(P_1+P_1, P_1+P_2)$. For any $P \in \mathfrak{M}$, we put $d(P) = d(P,P)$. The functionals $(P_1,P_2) \mapsto r(P_1,P_2)$ and $(P_1,P_2) \mapsto E(P_1,P_2)$, defined on $\mathfrak{A}$, will be denoted by r and E, respectively.

1.10. In [1], 3.4 and 3.7, normal gauge functionals (NGF) have been defined (they are functionals on $\mathfrak{A}$ satisfying certain conditions) and, for any NGF τ, the functionals C_τ and C^*_τ have been introduced. We do not state again the definition of an NGF as only two NGF's, r and E, defined in 1.9, will be considered here (for the fact that r, denoted r_1 in [1], 3.2, 3.5, and E are NGF's see [1], 3.5). The definition of C_τ and C^*_τ will be given below in a form different from, but equivalent to (for any NGF τ) that in [1].

1.11. The concatenation of finite sequences x and y is denoted by $x \cdot y$ or xy (or also by xb if $y = (b)$ and by ay if $x = (a)$). The letter Δ denotes the collection of all finite non-void $D \subset \bigcup(\{0,1\}^n : n \in N)$ such that if $x = (x_i : i < k) \in D$, then (1) $(x_i : i < j) \in D$ for all $j < k$, (2) $x0 \in D$ iff $x1 \in D$. If $D \in \Delta$, then we put $D' = \{x \in D : x0 \in D\}$, $D'' = D \setminus D'$. - We call $\mathcal{P} = (P_x : x \in D)$ a dyadic expansion of $P \in \mathfrak{M}$ if $D \in \Delta$, $P_\emptyset = P$, $P_{x0} + P_{y1} = P_x$ for each $x \in D'$. If all $P_x \leq P$ are pure, then $\mathcal{P}$ is called pure. If $\mathcal{P} = (P_x : x \in D)$ is a dyadic expansion, then $\mathcal{P}''$ denotes the indexed set $(P_x : x \in D'')$. - See [1], 4.1-4.4.

1.12. Let τ be an NGF, $P \in \mathfrak{M}$. If $U \leq P$, $V \leq P$, then we put $\Gamma_\tau(U,V) = H(wU,wV)\tau(U,V)$. If $\mathcal{P} = (P_x : x \in D)$ is a dyadic expansion of P, then we put $\Gamma_\tau(\mathcal{P}) = \Sigma(\Gamma_\tau(P_{x0},P_{x1}) : x \in D')$. - See [1], 4.10.

1.13. Definition (see [1], 4.29, 4.11). Let τ be an NGF and let $P \in \mathfrak{M}$. Then $C_\tau(P)$ **(respectivelv,** $C^*_\tau(P)$) denotes the infimum of all $a \in \overline{R}_+$ such that, for any partition (pure partition) $\mathcal{U}$ of P, there is a dyadic expansion (pure dyadic expansion) $\mathcal{P}$ such that $\mathcal{P}^*$ refines $\mathcal{U}$ and $\Gamma_\tau(\mathcal{P}) \leq a$. The functionals $P \mapsto C_\tau(P)$ and $P \mapsto C^*_\tau(P)$ are denoted by C_τ and C^*_τ, respectively. - Instead of C_E and C^*_E, we will often write E and E^*.

1.14. If τ is an NGF, $\mathcal{U} = (U_k : k \in K)$ is a partition of $P \in \mathcal{M}$ and $\tau(U_i, U_j) < \infty$ for $i \neq j$, then $[\mathcal{U}]_\tau$ denotes the W-space $\langle K, \sigma, \nu \rangle$, where $\sigma(i,j) = \tau(U_i, U_j)$ for $i \neq j$, $\nu X = w(\Sigma(U_i : i \in X))$ for all $X \subset K$. - See [1], 3.6.

1.15. Theorem (see [1], 3.14-3.19). Let τ be an NGF and let $\varphi = C_\tau$ (respectively, $\varphi = C^*_\tau$). Let $P \in \mathcal{M}$. Then $\varphi(P)$ is equal to the infimum of all $b \in \overline{R}_+$ such that, for any partition (pure partition) $\mathcal{U}$ of P there is a finer partition (pure partition) $\mathcal{V}$ with $C^*_\tau[\mathcal{V}]_\tau \leq b$.

1.16. Facts (see [1]). Let τ be an NGF and let $P \in \mathcal{M}$. Then (1) $\tau \leq E$, (2) if $\varphi = C_\tau$ (respectively, $\varphi = C^*_\tau$) and $U+V = P$ (respectively, $U+V = P$ and U, V are pure), then $\varphi(P) \leq \varphi(U) + \varphi(V) + \Gamma_\tau(U,V)$, (3) if $\tau \geq r$ and $P = \langle Q, 1, \mu \rangle \in \mathcal{M}_F$, then $C_\tau(P) = C^*_\tau(P) = H(\mu\{q\} : q \in Q)$, (4) if ψ is an NGF, $\psi \geq \tau$, then $C_\psi(P) \geq C_\tau(P)$, $C^*_\psi(P) \geq C^*_\tau(P)$.

1.17. Definition. If $a, b \in \overline{R}$, we put $a * b = 0$ if $a \geq b$, $a * b = 1$ if $a < b$. If $f : X \to \overline{R}$ and $\varepsilon \in R$, then $\varepsilon * f$ denotes the function $x \mapsto \varepsilon * f(x)$. - If $\varepsilon \in R^*_+$ and $P = \langle Q, \rho, \mu \rangle \in \mathcal{M}$, then $\langle Q, \varepsilon * \rho, \mu \rangle$ is a W-space, which will be denoted by $\varepsilon * P$. For any $P \in \mathcal{M}$, the mapping $\varepsilon \mapsto \varepsilon * P$, defined on R^*_+, will be called the graded representation of P. For any $\varphi : \mathcal{M} \to \overline{R}$, the function $\varepsilon \mapsto \varphi(\varepsilon * P)$, defined on R^*_+, will be denoted by $G\varphi(P)$; the mapping $P \mapsto G\varphi(P)$ will be called the graded modification of φ and will be denoted by $G\varphi$.

1.18. In [3], Posner, Rodemich and Rumsey have defined the epsilon entropy for spaces X of the form $X = \langle X, d, \mu \rangle$, where $\langle X, d \rangle$ is a complete separable metric space and μ is a measure of the form $\mu = \overline{\nu}$, dom $\nu = \mathcal{B}$. By 1.8, these spaces are W-spaces, and it is easy to see that the definition of the epsilon entropy presented in [3] can be extended to all W-spaces. We are going to present the extended definition in a form which coincides with that given in [3] for spaces mentioned above.

1.19. Definition. Let $P = \langle Q, \varphi, \mu \rangle \in \mathcal{M}$, $\varepsilon \in R_+^*$. Then $(X_k : k \in K)$, where $K \neq \emptyset$ is countable, is called an ε-partition of P if $X_k \in \operatorname{dom} \overline{\mu}$, diam $X_k \leq \varepsilon$, $X_i \cap X_j = \emptyset$ for $i \neq j$, $\overline{\mu}(\cup(X_k : k \in K)) = \mu Q$, and the infimum of all $\hat{H}(\overline{\mu} X_k : k \in K)$, where $(X_k : k \in K)$ is an ε-partition of P, is denoted by $\hat{H}_\varepsilon(P)$. The function $\varepsilon \longmapsto \hat{H}_\varepsilon(P)$, defined on R_+^*, will be called the epsilon entropy of P and will be denoted by $\hat{H}(P)$.

1.20. Notation. For any $P = \langle Q, \varphi, \mu \rangle \in \mathcal{M}$, $\eta(P)$, $\eta^*(P)$, $\eta_f(P)$ and $\eta_f^*(P)$ denote, respectively, the infimum of all $H(wU_k : k \in K)$, where $(U_k : k \in K)$ is an ω-partition (pure ω-partition, partition, pure partition) of P such that $d(U_k) = 0$ for all $k \in K$, and $\overline{\eta}(P)$ denotes the infimum of all $H(\overline{\mu} X_k : k \in K)$, where $(X_k \cdot P : k \in K)$ is a pure ω-partition of P and diam $X_k = 0$ for all $k \in K$ (thus, $\overline{\eta}(P) = \infty$ if there is no such partition, and similarly for $\eta(P)$, etc.).

1.21. Evidently, $\hat{H}_\varepsilon(P) = \overline{\eta}(\varepsilon * P) \cdot \ell n\, 2$ for all $\varepsilon \in R_+^*$. - It will be proved below that, for any $P \in \mathcal{M}_M$ and any $\varepsilon \in R_+^*$, $E(\varepsilon * P)$, $E^*(\varepsilon * P)$, $\eta(\varepsilon * P)$ and $\eta^*(\varepsilon * P)$ coincide and are equal, at least for small $\varepsilon > 0$, to $\overline{\eta}(\varepsilon * P)$.

2

2.1. Proposition. If $P \in \mathcal{M}_G$ and $\eta_f(P) < \infty$ (i.e., there is a partition $(U_k : k \in K)$ with $d(P_k) = 0$ for all $k \in K$), then

$E(P) = E^*(P) = \eta_f(P) = \eta_f^*(P)$. - See [2], 10.6.

2.2. Lemma. Let τ be an NGF, $P \in \mathcal{M}$, $d(P) < \infty$, $P_n \leqq P$, $n \in N$, and let $w(P-P_n) \longrightarrow 0$ for $n \longrightarrow \infty$. Then $\varphi(P) \leqq \underline{\lim}\ \varphi(P_n)$, where $\varphi = C_\tau$ or $\varphi = C_\tau^*$.

Proof. We consider the case $\varphi = C_\tau$; the other case is analogoua. Put $a = \underline{\lim}\ \varphi(P_n)$; we can assume that $a < \infty$ and $d(P) = 1$. It is enough to prove that, for any $b > a$ and any partition $\mathcal{U} = (f_i \cdot P : i \in [1,m])$ of P, there is a dyadic expansion $\mathcal{P}$ such that $\mathcal{P}''$ refines $\mathcal{U}$ and $\Gamma_\tau(\mathcal{P}) < b$. - Choose $\varepsilon > 0$ such that $a < b - 2\varepsilon$. Choose $n \in N$ such that $w(P-P_n) \cdot \log m < \varepsilon$, $H(wP_n, w(P-P_n)) < \varepsilon$, $\varphi(P_n) < b - 2\varepsilon$. Put $S = P_n$, $T = P-S$. Choose functions s, t such that $S = s \cdot P$, $T = t \cdot P$, and put $s_i = f_i s$, $t_i = f_i t$ for $i \in [1,m]$. Put $\mathcal{U}_S = (s_i \cdot P : i \in [1,m])$, $\mathcal{U}_T = (t_i \cdot P : i \in [1,m])$. Clearly, $\mathcal{U}_S$ and $\mathcal{U}_T$ are partitions of S and T, respectively. Since $\varphi(S) < b - 2\varepsilon$, there is a dyadic expansion $\mathcal{S} = (S_x : x \in D_S)$ of S such that $\mathcal{S}''$ refines $\mathcal{U}_S$ and $\Gamma_\tau(\mathcal{S}) < b - 2\varepsilon$. It is easy to see that there is a dyadic expansion $\mathcal{T} = (T_y : y \in D_T)$ of T such that $\mathcal{T}''$ refines $\mathcal{U}_T$ and $\Gamma_\tau(\mathcal{T}) \leqq H(w(t_i \cdot P) : i \in [1,m])$, hence, by 1.2 B, $\Gamma_\tau(\mathcal{T}) \leqq wT \cdot \log m$. Let D consist of $\emptyset$, all $(0) \cdot x$, $x \in D_S$, and all $(1) \cdot y$, $y \in D_T$. Then $D \in \Delta$ and there is a dyadic expansion $\mathcal{P} = (P_z : z \in D)$ of P such that $P_{(0) \cdot x} = S_x$ for $x \in D_S$, $P_{(1) \cdot y} = T_y$ for $y \in D_T$. Clearly, $\mathcal{P}''$ refines $\mathcal{U}$, and $\Gamma_\tau(\mathcal{P}) = \Gamma_\tau(\mathcal{S}) + \Gamma_\tau(\mathcal{T}) + \Gamma_\tau(S,T) \leqq b - 2\varepsilon + wT \cdot \log m + H(wS, wT) < b$.

2.3. Proposition. Let $P = \langle Q, \wp, \mu \rangle \in \mathcal{M}$, $S \leqq P$. Then $E(S) \leqq E(P)$, and if S is pure, then also $E^*(S) \leqq E^*(P)$.

Proof. We prove $E(S) \leqq E(P)$; the proof of $E^*(S) \leqq E^*(P)$ is analogous. We can assume that $E(P) < \infty$. It is enough to prove that, for any $b > E(P)$ and any partition $\mathcal{U} = (U_k : k \in K)$ of S, there is a dyadic expansion $\mathcal{S}$ of S such that $\mathcal{S}''$ refines $\mathcal{U}$ and $\Gamma_E(\mathcal{S}) < b$. - Let z non$\in K$, put $K' = K \cup (z)$, and put $U_z = P-S$, $\mathcal{V} =$

$= (U_k : k \in K')$. Since $E(P) < b$, there exists a dyadic expansion $\mathcal{P} = (P_x : x \in D)$ of P such that $\mathcal{P}''$ refines $\mathcal{V}$ and $\Gamma_E(\mathcal{P}) < b$. Since $\mathcal{P}''$ refines $\mathcal{V}$, there is a partition $(M(k) : k \in K')$ of D" such that $\Sigma(P_x : x \in M(k)) = U_k$ for each $k \in K'$. Clearly, there is a dyadic expansion $\mathcal{S} = (S_x : x \in D)$ of S such that $S_x = P_x$ if $x \in \bigcup(M(k) : k \in K)$ and $S_x = \langle Q, \varphi, 0)$ if $x \in M(2)$. Then we have $S_x \leq P_x$ for each $x \in D$, and therefore $\Gamma_E(S_{x0}, S_{x1}) \leq \Gamma_E(P_{x0}, P_{x1})$ for each $x \in D'$ Hence $\Gamma_E(\mathcal{S}) \leq \Gamma_E(\mathcal{P}) < b$. Clearly, $\mathcal{S}''$ refines $\mathcal{U}$.

2.4. <u>Proposition</u>. Let $P \in \mathfrak{M}$, $d(P) < \infty$, $P_n \leq P$, $n \in N$, and let $w(P-P_n) \longrightarrow 0$ for $n \longrightarrow \infty$. Then $E(P_n) \longrightarrow E(P)$, and if P_n are pure, then also $E^*(P_n) \longrightarrow E^*(P)$.

This follows at once from 2.2 and 2.3.

2.5. <u>Fact</u>. If (S,T) is a partition of $P \in \mathfrak{M}$, then $\max(\eta(S), \eta(T)) \leq \eta(P) \leq \eta(S) + \eta(T) + H(wS,wT) \leq \eta(P) + wP$.

Proof. We prove the third inequality; the proof of the first two is easy and can be omitted. We can assume that $\eta(P) < \infty$. Let $\varepsilon > 0$. Then there is an ω-partition $(U_k : k \in N) = (f_k \cdot P : k \in N)$ of P such that $H(wU_k : k \in N) < \eta(P) + \varepsilon$ and $d(U_k) = 0$ for all $k \in N$. Let $S = g_1 \cdot P$, $T = g_2 \cdot P$. For $k \in N$, $i = 1,2$, put $V_{ik} = f_k g_i \cdot P$. By 1.2 C, we have $H(wV_{1k} : k \in N) + H(wV_{2k} : k \in N) + H(wS,wT) = H(wV_{ik} : i = 1,2; k \in N) = H(wU_k : k \in N) + \Sigma(H(wV_{1k}, wV_{2k}) : k \in N)$. Since $H(wV_{1k}, wV_{2k}) \leq wU_k$, we get $\Sigma(H(wV_{1k}, wV_{2k}) : k \in N) \leq wP$ and therefore $\eta(S) + \eta(T) + H(wS,wT) \leq \eta(P) + \varepsilon + wP$.

2.6. <u>Lemma</u>. Let $P \in \mathfrak{M}$, $P_n \leq P_{n+1} \leq P$ for $n \in N$, $w(P-P_n) \longrightarrow 0$ for $n \longrightarrow \infty$. If $\{\eta(P_n) : n \in N\}$ is bounded, then $\eta(P_m - P_n) \longrightarrow 0$ for $m \longrightarrow \infty$, $n \longrightarrow \infty$, $m > n$.

Proof. Put $a = \sup\{\eta(P_n) : n \in N\}$. Let $\varepsilon > 0$. Choose $k \in N$ such that $w(P-P_k) < \varepsilon/2$. Put $b = \sup\{\eta(P_n - P_k) : n > k\}$. Clearly, $b \leq a < \infty$. Choose $t > k$ such that $b - \eta(P_t - P_k) < \varepsilon/2$; then, by 2.5 (first inequality), $b - \eta(P_n - P_k) < \varepsilon/2$ for each $n \geq t$. If

$m,n \in N$, $m > n \geq t$, then, by 2.5, $\eta(P_m - P_n) + \eta(P_n - P_k) + H(w(P_m - P_n), w(P_n - P_k)) \leq \eta(P_m - P_k) + w(P_n - P_k)$, hence $\eta(P_m - P_n) < \eta(P_m - P_k) - \eta(P_n - P_k) + \varepsilon/2 \leq b - \eta(P_n - P_k) + \varepsilon/2 < \varepsilon$.

2.7. Lemma. Let $P \in \mathcal{M}$. Let $P_n \leq P_{n+1} \leq P$ for $n \in N$ and let $w(P - P_n) \to 0$. Then $\eta(P_n) \to \eta(P)$, $\eta^*(P_n) \to \eta^*(P)$.

Proof. We prove $\eta(P_n) \to \eta(P)$; the proof of $\eta^*(P_n) \to \eta^*(P)$ is analogous. Put $a = \sup\{\eta(P_n) : n \in N\}$. Since, by 2.5, $\eta(P_n) \leq \eta(P)$ for all $n \in N$, it is enough to show that $\eta(P) \leq a$. We can assume that $a < \infty$ and $wP = 1$. - Let $\varepsilon > 0$. Choose $\delta > 0$ such that $3\delta + H(\delta, 1-\delta) < \varepsilon$. By 2.6, there are $s(k) \in N$ such that, for each $k \in N$, (1) $s(k) < s(k+1)$, (2) $w(P - P_{s(k)}) < \delta/2^{k+1}$, (3) $m > n \geq s(k)$ implies $\eta(P_m - P_n) < \delta/2^k$. Put $S_0 = P_{s(o)}$, $S_k = P_{s(k)} - P_{s(k-1)}$ for $k \in N_1$. Then $\eta(S_0) \leq a$, $w(P - S_0) < \delta/2$, $\eta(S_k) < \delta/2^k$. $wS_k < \delta/2^k$ for $k \in N_1$. For each $k \in N_1$, there is an ω-partition $(U_{kj} : j \in N)$ of S_k such that $d(U_{kj}) = 0$, $H(wU_{kj} : j \in N) < \delta/2^k$. Clearly, $(U_{kj} : k \in N_1, j \in N)$ is an ω-partition of $P - S_0$, and, by 1.26, $H(wU_{kj} : k \in N_1, j \in N) = H(wS_k : k \in N_1) + \Sigma(H(w(U_{kj} : j \in N) : k \in N_1) < H(\delta/2^k : k \in N_1) + \delta$. It is easy to see that $H(2^{-k} : k \in N_1) = 2$. Hence we get $\eta(P - S_0) < 3\delta$. By 2.5, $\eta(P) \leq \eta(S_0) + \eta(P - S_0) + H(wS_0, w(P - S_0)) < a + 3\delta + H(1-\delta, \delta) < a + \varepsilon$.

2.8. Lemma. Let $P \in \mathcal{M}_G$. Assume that there exists a partition $(U_k : k \in K)$ of P such that $d(U_k) = 0$ for all $k \in K$. Then $E(P) = E^*(P) = \eta(P) = \eta^*(P) = \eta_f(P) = \eta_f^*(P)$.

Proof. By 2.1, it is enough to show that $\eta_f(P) \leq \eta(P)$, $\eta_f^*(P) \leq \eta^*(P)$, for the inequalities $\eta(P) \leq \eta_f(P)$, $\eta^*(P) \leq \eta_f^*(P)$ are evident. We prove only $\eta_f(P) \leq \eta(P)$, as the proof of $\eta_f^*(P) \leq \eta^*(P)$ is completely analogous. - Put $a = \eta(P)$; we can assume that $a < \infty$. Let $(U_1, \dots, U_m)$ be a partition of P such that $d(U_i) = 0$. Let $\varepsilon > 0$ and let $(V_k : k \in N)$ be an ω-partition such that $d(V_k) = 0$ for all $k \in N$ and (1) $H(wV_k : k \in N) < a + \varepsilon/2$.

Let $U_i = g_i \cdot P$, $V_k = f_k \cdot P$. Choose n such that (2) $w(\Sigma(V_k : k > n) \cdot \log m < \varepsilon/4$, (3) $H(\Sigma(wV_k : k \leqq n), \Sigma(wV_k : k > n)) < \varepsilon/4$. Put $f = \Sigma(f_k : k > n)$, $T_k = V_k$ for $k \in [0,n]$, $T_k = g_{k-n} f \cdot P$ for $k \in [n+1, n+m]$, and put $\mathcal{T} = (T_0, \ldots, T_{n+m})$. Clearly, $\mathcal{T}$ is a partition of P and $d(T_k) = 0$ for $k \in [0, n+m]$. By (1), we have $H(wT_k : k \in [0,n]) < a + \varepsilon/2$. By (2) and 1.2 B, we get $H(wT_k : k \in [n+1, n+m]) < \varepsilon/4$. Clearly, $H(wT_k : k \in [0, n+m]) = H(wT_k : k \in [0,n]) + H(wT_k : k \in [n+1, n+m]) + H(\Sigma(wT_k : k \in [0,n]), \Sigma(wT_k : k \in [n+1, n+m]))$. Using (3), we obtain $H(wT_k : k = 0, \ldots, n+m) < a + \varepsilon$.

2.9. Proposition. Let P be a GW-space and assume that there exists an ω-partition $(U_k : k \in K)$ of P such that $d(U_k) = 0$ for all $k \in K$. Then $E(P) = E^*(P) = \eta(P) = \eta^*(P)$.

Proof. For each $n \in N$ put $P_n = \Sigma(U_k : k \leqq n)$. By 2.8, $E(P_n) = E^*(P_n) = \eta(P_n) = \eta^*(P_n)$ for each $n \in N$. By 2.4 and 2.7, this proves the proposition.

2.10. Definition. A Darboux measure is a measure μ such that, for any $X \in \operatorname{dom} \mu$ and any positive $b < \mu X$, there is a set $Y \in \operatorname{dom} \mu$ satisfying $Y \subset X$, $\mu Y = b$. A Darboux W-space is a $P \in \mathfrak{M}$ such that $U \leqq P$, $d(U) = 0$ implies $wU = 0$.

2.11. Fact. If $P \in \mathfrak{M}$, $d(P) > 0$, then there is a pure $S \leqq P$ such that $0 < wS < wP$. - See [2], 7.14.

2.12. Proposition. If $P = \langle Q, \varphi, \mu \rangle \in \mathfrak{M}$ is Darboux, then so is μ.

Proof. We show that if $X \in \operatorname{dom} \mu$, $\mu X > 0$, then there is a set $Z \in \operatorname{dom} \mu$ such that $Z \subset X$, $0 < \mu Z < \mu X$; by well-known theorems, this will imply that μ is Darboux. Since $w(X \cdot P) > 0$, we have $d(X \cdot P) > 0$, hence, by 2.11, there is a pure subspace $V \leqq X \cdot P$ such that $0 < wV < w(X \cdot P) = \mu X$. There is a set $Y \in \operatorname{dom} \bar{\mu}$ such that $V = Y.(X \cdot P)$. Choose a set $Z \in \operatorname{dom} \mu$ such that $Z \supset Y \cap X$, $\mu Z = \bar{\mu}(Y \cap X)$.

2.13. Proposition. Let P be a Darboux GW-space. If $wP > 0$, then $E(P) = E^*(P) = \eta(P) = \eta^*(P) = \infty$.

Proof. Let $n \in N_1$. By 2.12, there is a pure partition $\mathcal{U} = (U_1,\dots,U_n)$ of P such that $wU_k = wP/n$ for $k \in [1,n]$. Let $\mathcal{P} = (P_x : x \in D)$ be a dyadic expansion of P such that $\mathcal{P}''$ refines $\mathcal{U}$. Clearly, we can assume that $wP_x > 0$ for all $x \in D''$. Then, for each $x \in D''$, $d(P_x) > 0$ since P is Darboux, and therefore $d(P_x) = 1$ since $P \in \mathcal{M}_G$. It is now easy to see that $\Gamma_E(\mathcal{P}) = H(wP_x : x \in D'')$. Since $\mathcal{P}''$ refines $\mathcal{U}$, we obtain $\Gamma_E(\mathcal{P}) \geq H(wU_k : k \in [1,n]) = wP \cdot \log n$. This proves $E(P) = E^*(P) = \infty$. If $(U_k : k \in K)$ is an ω-partition of P, then, for some k, $wU_k > 0$ and therefore $d(U_k) > 0$. This implies $\eta(P) = \eta^*(P) = \infty$.

2.14. Proposition. Every W-space P has a pure ω-partition $(U_k : k \in N)$ such that U_0 is Darboux and $d(U_k) = 0$ for $k \in N_1$.

Proof. For every pure $S \leq P$ we can choose a pure $S' = \Phi(S) \leq P$ such that $d(S') = 0$ and $wS' \geq wT/2$ whenever $T \leq S$ is pure and $d(T) = 0$. Put $U_1 = \Phi(P)$ and $U_{k+1} = \Phi(P - \Sigma(U_i : 1 \leq i \leq k))$; put $U_0 = P - \Sigma(U_i : i \in N_1)$. Clearly, $d(U_k) = 0$ for all $k \in N_1$. - Suppose there is a pure $T \leq U_0$ such that $d(T) = 0$, $wT > 0$. Clearly, $wU_m \leq wT/2$ for some $m \in N_1$. Put $V = P - \Sigma(U_i : 1 \leq i < m)$. Then $U_m = \Phi(V)$, $T \leq V$, and we get a contradiction.

2.15. Proposition. If P is a graph W-space, then $E(P) = E^*(P) = \eta(P) = \eta^*(P)$.

Proof. Let $(U_k : k \in N)$ be a pure ω-partition with properties described in 2.14. If $wU_0 > 0$, then the equalities hold by 2.13 and 2.3. If $wU_0 = 0$, they hold by 2.9.

2.16. Proposition. For any W-space P and any positive number ε, $E(\varepsilon * P) = E^*(\varepsilon * P) = \eta(\varepsilon * P) = \eta^*(\varepsilon * P)$. - This follows from 2.15, since $\varepsilon * P \in \mathcal{M}_G$.

2.17. Lemma. If $P = \langle Q, \rho, \mu \rangle \in \mathcal{M}_M$, then there is a set $T \in \operatorname{dom} \mu$ such that $\mu T = \mu Q$, diam $T \leq 2\, d(P)$. If, in addition, there is a set $S \in \operatorname{dom} \bar{\mu}$ such that $\bar{\mu}(Q \setminus S) \neq 0$ and S is separable (as a subspace of $\langle Q, \rho \rangle$), then there exists a set $T \subset S$ closed in S and such that $T \in \operatorname{dom} \bar{\mu}$, $\bar{\mu} T = \mu Q$, diam T = d(P).

Proof. I. For $x \in Q$ put $V_x = \{y \in Q : \rho(y,x) > d(P)\}$. Then $[\mu \times \mu](\bigcup(\{x\} \times V_x : x \in Q)) = 0$, hence, by well-known theorems, there is a point $b \in Q$ such that $\bar{\mu} V_b = 0$. Choose a set $U \in \operatorname{dom} \mu$ such that $U \supset V_b$ and $\mu U = 0$. Put $T = Q \setminus U$. Clearly, diam $T \leqq 2d(P)$. - II. Let S be as described in the proposition. By [2], 7.24, we have $\mathfrak{B} \subset \operatorname{dom} \bar{\mu}$. Let G be the union of all open $V \subset Q$ satisfying $\bar{\mu}(S \cap V) = 0$. Since S is separable, it is easy to see that $\bar{\mu}(S \cap G) = 0$. Put $T = S \setminus G$. Then T is closed in S, $T \in \operatorname{dom} \bar{\mu}$ (due to $\mathfrak{B} \subset \operatorname{dom} \bar{\mu}$) and $\bar{\mu} T = \mu Q$. Clearly, if $X \subset T$ is open in T and $X \neq \emptyset$, then $\bar{\mu} X > 0$. Put $U = \{(x,y) \in T \times T : \rho(x,y) > d(P)\}$. Suppose $U \neq \emptyset$. Then, U being open, there are non-void A, B open in T such that $A \times B \subset U$, and we get $\bar{\mu} A > 0$, $\bar{\mu} B > 0$, hence $[\mu \times \mu](U) > 0$, which is a contradiction. We have shown that $U = \emptyset$, hence diam $T \leqq$ $\leqq d(P)$. Clearly, $d(P) \leqq$ diam T, since $\bar{\mu}(Q \setminus T) = 0$.

2.18. <u>Theorem</u>. Let $P = \langle Q, \rho, \mu \rangle$ be a metrized measure space. Then either the epsilon entropy $\hat{H}(P)$ and the graded E-entropy GE(P) coincide (up to the factor $\ln 2$) or both $\hat{H}_\varepsilon(P)$ and $E(\varepsilon * P)$ are infinite for all sufficiently small $\varepsilon > 0$.

Proof. I. If $E(\delta * P) = \infty$ for some $\delta > 0$, then, for all positive $\varepsilon \leqq \delta$, we have $E(\varepsilon * P) = \infty$, hence, by 2.16, $\eta(\varepsilon * P) = \infty$ and therefore $\bar{\eta}(\varepsilon * P) = \infty$, $\hat{H}_\varepsilon(P) = \infty$. - II. If $E(\delta * P) < \infty$ for all $\delta > 0$, then, by 2.17, there exist $T_{mn} \in \operatorname{dom} \bar{\mu}$, $m, n \in N$, such that, for all m and n, $\bar{\mu}(\bigcup(T_{mn} : n \in$ $\in N)) = \mu Q$ and diam $T_{mn} \leqq 2/m$. Let S be the closure of $\bigcap(\bigcup(T_{mn} :$ $: n \in N) : m \in N)$. Then S is closed separable and $\bar{\mu}(Q \setminus S) = 0$. - Let $X \in \operatorname{dom} \bar{\mu}$. Then the assumption in 2.17 (second part) are satisfied (for the space $X \cdot P$ and the set $X \cap S$). Therefore, there is a set $Y \subset X \cap S$ closed in $X \cap S$, hence in X, and satisfying $Y \in \operatorname{dom} \bar{\mu}$, $\bar{\mu} Y = \bar{\mu} X$ and diam $Y = d(X \cdot P)$. - Let $\varepsilon > 0$. By 2.16, $\eta^*(\varepsilon * P) = E(\varepsilon * P)$. We are going to prove that $\bar{\eta}(\varepsilon * P) = \eta^*(\varepsilon * P)$; by 1.24, this will complete the proof. Let $\vartheta > 0$. Let $(X_n \cdot (\varepsilon * P) : n \in N)$ be an ω-partition of $\varepsilon * P$ such that $H(\bar{\mu} X_n : n \in N) < \eta^*(\varepsilon * P) + \vartheta$ and, for each $n \in N$, $d(X_n \cdot (\varepsilon * P)) = 0$, hence $d(X_n \cdot P) \leqq \varepsilon$. Then there are sets Y_n such that, for each $n \in N$, $Y_n \subset X_n$, Y_n is closed in X_n, $\bar{\mu} Y_n = \bar{\mu} X_n$, diam $Y_n = d(X_n \cdot P) \leqq \varepsilon$, hence diam $Y_n = 0$ in $\langle Q, \varepsilon * \rho \rangle$. This proves that $\bar{\eta}(\varepsilon * P) < \eta^*(\varepsilon * P) + \vartheta$, and therefore, $\vartheta > 0$ being arbitrary, $\bar{\eta}(\varepsilon * P) \leqq \eta^*(\varepsilon * P)$, hence $\bar{\eta}(\varepsilon * P) = \eta^*(\varepsilon * P)$.

3

Let τ be a "standard" NGF, i.e. one of the NGF's introduced in [1], 3.2, and let $\tau \neq E$. Then the graded modifications GC^*_τ and GE^* (see 1.17) do not coincide, since $C^*_\tau \neq E^*$ on $\mathfrak{M}_F \cap \mathfrak{M}_G$ (see [2], 10.3, 10.7). We also have $GC_r \neq GE$ (cf. [2], 10.8). Thus, we cannot expect GE to coincide with some GC_τ or GC^*_τ on a not too narrow class of W-spaces. On the other hand, if τ is an NGF, $\tau \geqq r$, $\varphi = C_\tau$ or $\varphi = C^*_\tau$, $P = \langle Q, \rho, \mu \rangle$ and $\langle Q, \rho \rangle \subset R^n$ is bounded, then the limit behavior of $G\varphi(P)$ and $GE(P)$, or rather of $\varphi(\varepsilon * P)/|\log \varepsilon|$ and $E(\varepsilon * P)/|\log \varepsilon|$, is similar in the sense described below in 3.7. The motivation for considering $\varphi(\varepsilon * P)/|\log \varepsilon|$ lies in the fact that $P \mapsto \lim (E(\varepsilon * P)/|\log \varepsilon|)$ can be conceived as a dimension function (for W-spaces) closely connected with that introduced by A. Rényi (see e.g. [4]) for R^n-valued random variables.

3.1. In 3.2-3.6, we put $\xi(x) = 9 \log x + 16$ for $x \in R^*_+$.

3.2. Lemma. Let τ be an NGF and let $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}_F$, diam $\langle Q, \rho \rangle \leqq 1$, card $Q = n$. If $S \leqq P$, then $|C_\tau(P) - C_\tau(S)| \leqq$

$\leq \zeta(n)(wP)^{2/3}(w(P-S))^{1/3}$. - This is a special case of [2], 9.40.

3.3. Fact. Let τ be an NGF. Let $P = \langle Q, \rho, \mu \rangle$, $S = \langle T, \sigma, \nu \rangle$ be FW-spaces. Assume that there is an $f: Q \to T$ such that $\mu(f^{-1}Y) = \nu Y$ for each $Y \subset T$ and $\rho(x,y) = \sigma(fx, fy)$ for all $x, y \in Q$. Then $C^*_\tau(P) \geq C_\tau(S)$. - This is a special case of [1], 3.23.

3.4. Lemma. Let $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}_F$. Let $(V_0, \ldots, V_m)$ be a partition of Q and assume that $\rho(x,y) = 1$ if (x,y) is in $\bigcup(V_i \times V_j : i \neq j, i \neq 0 \neq j)$, $\rho(x,y) = 0$ if not. Then $C^*_r(P) \geq H(\mu V_i : i \in [1,m]) - \zeta(m+1)(wP)^{2/3}(\mu V_0)^{1/3}$.

Proof. For each $q \in Q$ put $f(q) = j$ if $q \in V_j$. Put $P' = \langle [0,m], \rho', \mu' \rangle$, where $\rho'(i,j) = 1$ if $i \neq j$, $i \neq 0 \neq j$, $\rho'(i,j) = 0$ if $i = j$ or $0 \in \{i,j\}$, $\mu' Y = \mu(f^{-1}Y)$ for each $Y \subset Q'$. By 3.2 and 1.16(3), $C_r(P') \geq H(\mu V_i : i \in [1,m]) - \zeta(m+1)(wP)^{2/3}(\mu V_0)^{1/3}$. By 3.3, $C^*_r(P) \geq C_r(P')$.

3.5. Lemma. Let $\varphi = C_r$ or $\varphi = C^*_r$. Let $P = \langle Q, \rho, \mu \rangle \in \mathfrak{M}$ and let $X_i \in \operatorname{dom} \bar{\mu}$, $i = 1, \ldots, m$. Let $\delta > 0$ and assume that $\rho(x,y) > \delta$ whenever $x \in X_i$, $y \in X_j$, $i, j \in [1,m]$, $i \neq j$. Put $X_0 = Q \setminus \bigcup(X_i : i \in [1,m])$. Then $\varphi(\delta * P) \geq H(\bar{\mu} X_i : i \in [1,m]) - \zeta(m+1)(wP)^{2/3} (\bar{\mu} X_0)^{1/3}$

Proof. By 1.15, it suffices to show that if a partition $\mathcal{U}$ of $\delta * P$ refines $\mathcal{X} = (X_i \cdot (\delta * P) : i \in [0,m])$, then the inequality holds with $\varphi(\delta * P)$ replaced by $C^*_r[\mathcal{U}]_{1}$. Let $\mathcal{U} = (U_k : k \in K)$. Since $\mathcal{U}$ refines $\mathcal{X}$, there is a partition $(A_j : j \in [0,m])$ of K such that $\Sigma(U_k : k \in A_j) = X_j \cdot (\delta * P)$ for all j. Put $[\mathcal{U}]_r = \langle K, \sigma, \nu \rangle$. For $k, k' \in K$ let $\hat{\sigma}(k,k') = 1$ if (k,k') is in $\bigcup(A_i \times A_j : i \neq j, i \neq 0 \neq j)$, $\hat{\sigma}(k,k') = 0$ if not. Put $S = \langle K, \hat{\sigma}, \nu \rangle$. Clearly, $\sigma \geq \hat{\sigma}$, hence $C^*_r[\mathcal{U}]_r \geq C^*_r(S)$. By 3.4, we have $C^*_r(S) \geq H(\nu A_i : i \in [1,m]) - \zeta(m+1)(wS)^{2/3}(\nu A_0)^{1/3}$. This proves the assertion, since $\nu A_i = \bar{\mu} X_i$.

3.6. Proposition. Let $P = \langle Q, \rho, \mu \rangle$ be a W-space, let $t = 1,2,\ldots$, and let $\langle Q, \rho \rangle$ be a bounded subspace of R^t (endowed with the metric $\rho((x_i),(y_i)) = \max |x_i - y_i|$). Then there exist positive numbers a and b such that if τ is an NGF, $\tau \geqq r$, $\varphi = C_\tau$ or $\varphi = C^*_\tau$, $\varepsilon > 0$, $p \in N$, $p > 2^t$, and $\delta = \varepsilon/5p$, then $\varphi(\delta * P) \geqq E(\varepsilon * P) - a(2^t/p)^{1/3}|\log \varepsilon| - b$.

Proof. Let Z be the set of all integers. For each $z \in Z^t$ put $G_z = \{x \in Q: \rho(x,(\varepsilon/2)z) < \varepsilon/2$. Put $K = \{z \in Z^t: G_z \neq \emptyset\}$, $n = \text{card } K$. Clearly, (1) $n \leqq (2 \text{ diam } P/\varepsilon + 2)^t$. For $k \in K$, $j \in [0,p]$ put $U(k,j) = \{x \in G_k: \rho(x, Q \setminus G_k) \geqq (p-j)\delta\}$, $X(k,j) = U(k,j) \setminus U(k,j-1)$ for $j > 0$, $X(k,0) = U(k,0)$. Clearly, $U(k,j) \subset U(k,j+1)$, and it is easy to see that $\bigcup(U(k,0): k \in K) = Q$. For each $k \in K$ choose $f(k) \in [1,p]$ such that (2) $\bar{\mu}(X(k,f(k)) \leqq \bar{\mu}(X(k,i)$ for $i \in [1,p]$, and put $V_k = X(k,f(k))$. Put $V = \bigcup(V_k: k \in K)$. – Since no $q \in Q$ is in more than 2^t sets G_k, we have $\Sigma(\bar{\mu}(G_k \setminus U(k,0)): k \in K) \leqq 2^t wP$. Hence, by (2), we get (3) $\bar{\mu} V \leqq \Sigma(\bar{\mu} V_k: k \in K) \leqq 2^t wP/p$. Choose a bijection $g: K \to [1,n]$ and put, for each $k \in K$, $T_k = U(k,f(k) - 1)$, $S_k = (T_k \setminus V) \setminus \bigcup(T_i: g(i) < g(k))$. It is easy to see that $\bigcup(S_k: k \in K) = Q \setminus V$ and $\rho(x,y) > \delta$ whenever $x \in S_i$, $y \in S_j$, $i \neq j$. By 3.5, 1.16 and (3), we get $\varphi(\delta * P) \geqq H(\bar{\mu} S_k: k \in K) - \xi(n+1)wP.(2^t/p)^{1/3}$. Clearly, $H(\bar{\mu} S_k: k \in K) \geqq E(\varepsilon * (Q \setminus V).P)$, since diam $S_k \leqq \varepsilon$. By 1.2B, we get $E(\varepsilon * (V.P)) \leqq \bar{\mu} V.\log n$. Hence, by 1.15, $\varphi(\delta * P) \geqq E(\varepsilon * P) - \xi(n+1)w\, P(2^t/p)^{1/3} - (2^t/p) \log n$, From this inequality, the assertion follows at once, since, by (1), $\log n \leqq a'|\log \varepsilon| + b'$ for appropriate numbers a', b' and all $\varepsilon > 0$.

3.7. Theorem. Let $P = \langle Q, \rho, \mu \rangle$ be a W-space such that $\langle Q, \rho \rangle$ is a bounded subspace of some R^t, $t = 1,2,\ldots$. Let τ be an NGF, $\tau \geqq r$, and let $\varphi = C_\tau$ or $\varphi = C^*_\tau$. Then the upper (respectively, lower) limit (for $\varepsilon \to 0$) of $\varphi(\varepsilon * P)/|\log \varepsilon|$ is equal to that of $E(\varepsilon * P)/|\log \varepsilon|$.

Proof. Put $a = \overline{\lim} (E(\varepsilon * P)/|\log \varepsilon|)$. Choose $\varepsilon_n > 0$ such that $\varepsilon_n \to 0$, $E(\varepsilon_n * P)/|\log \varepsilon_n| \to a$. For any $\varepsilon > 0$, put $g(\varepsilon) = |\log \varepsilon|^{1/2}$, $f(\varepsilon) = 2^{g(\varepsilon)}$. Put $p_n = f(\varepsilon_n)$, $\sigma_n = \varepsilon_n/5p_n$. Since $\sigma_n/\varepsilon_n \to 0$, $\log \sigma_n/\log \varepsilon_n \to 1$, we get, by 3.6, $\underline{\lim} (\varphi(\sigma_n * P)/|\log \sigma_n| - E(\varepsilon_n * P)/|\log \varepsilon_n|) \geqq 0$, which implies $\overline{\lim} (\varphi(\sigma_n * P)/|\log \sigma_n|) \geqq a$. Since, by 1.17, $\varphi(\varepsilon * P) \leqq E(\varepsilon * P)$ for all $\varepsilon > 0$, we obtain $\overline{\lim} (\varphi(\sigma * P)/|\log \sigma|) = a$. - For the lower limit, the proof is similar and can be omitted.

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