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Commentationes Mathematicae Universitatis Carolinae, Vol. 27 (1986), No. 3, 499--518

Persistent URL: http://dml.cz/dmlcz/106472

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
27,3 (1986)

A CONSTRUCTION OF COMMUTATIVE MOUFANG LOOPS AND QUASIMODULES

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Abstract: A construction based on trilinear mappings is found.

Key words: Commutative Moufang loop, quasimodule.

Classification: 20N05

In the last time, an attention is paid to the theory of commutative Moufang loops and related structures (quasimodules, trimedial quasigroups, distributive quasigroups, Manin quasigroups, Hall triple systems). However, started from the first examples (see [2]), only one general construction of commutative Moufang loops (up to subloops) is known. It is based on the following idea: Let G(+) be an abelian group and T a triadditive mapping of G^3 into G. Define a new binary operation $*$ on G by $x*y=x+y+$ $+T(x,y,x-y)$. Under certain conditions, $G(*)$ turns out to be a commutative Moufang loop. Clearly, the basic identity $(x*x)*(y*z)=$ $=(x*y)*(x*z)$ of a commutative Moufang loop can be rewritten into an identity for the ternary ring $G(+,T)$. But this one is rather complicated and it is difficult to use it for constructions of commutative Moufang loops. One possible way how to proceed is to cut the identity into smaller pieces. This was done in [4] and generalized in [3]. The construction works well enough up to the nilpotence class 3 but is not general for higher classes. Here, we present another set of identities yielding a construction which involves as a special case the construction used in [1] for solving the Bruck's problem (about the nilpotence class of free commutative Moufang loops).

1. Introduction. Any loop $Q(*)$ satisfying the identity $(x*x)*(y*z)=(x*y)*(x*z)$ is commutative and is known as a commutative Moufang loop. We denote by $C(Q(*))$ the centre of $Q(*)$, by $A(Q(*))$ the associator subloop of $Q(*)$ and, for all $a,b,c \in Q$, by $[a,b,c]$ the associator of a,b,c, i.e. $[a,b,c]=((a*b)*c)*(a*(b*c))^{-1}$.

Let R be an associative ring. By a (left R-) quasimodule Q we mean a commutative Moufang loop $Q(*)$ together with a scalar multiplication by elements from R such that the usual module identities are satisfied, i.e. $r(x*y)=rx*ry$, $(r+s)x=rx*sx$ and $r(sx)=(rs)x$ for all $r,s \in R$ and $x,y \in Q$. Further, if F is a (ring) homomorphism of R into the three-element field Z_3 then we shall say that the quasimodule Q is F-special if $rx \in C(Q(*))$ for all $r \in \operatorname{Ker}(F)$ and $x \in Q$.

By a ternary ring $G=G(+,T)$ we mean an abelian group $G(+)$ with a triadditive mapping T of G^3 into G, i.e. $T(x+y,u,v)=T(x,u,v)+T(y,u,v)$, $T(u,x+y,v)=T(u,x,v)+T(u,y,v)$ and $T(u,v,x+y)=T(u,v,x)+T(u,v,y)$ for all $x,y,u,v \in G$.

By a ternary (left R-)algebra $A=A(+,rx,T)$ we mean a (left R-) module $A(+,rx)$ with a trilinear mapping T of A^3 into A, i.e. T is triadditive and $T(rx,y,z)=T(x,ry,z)=T(x,y,rz)=rT(x,y,z)$ for all $x,y,z \in A$ and $r \in R$.

2. Some identities for ternary rings. In this section, let $G=G(+,T)$ be a ternary ring. We shall consider the following

identities:

(a1) $3T(x,y,y)=0$ for all $x,y \in G$.

(a2) $3T(x,y,z)+3T(x,z,y)=0$ for all $x,y,z \in G$.

(a3) $3T(x,x,y)=0$ for all $x,y \in G$.

(a4) $3T(x,y,z)=0$ for all $x,y,z \in G$.

(b1) $T(x,x,y)=0$ for all $y,y \in G$.

(b2) $T(x,y,z)+T(y,x,z)=0$ for all $x,y,z \in G$.

(c1) $T(T(x,y,y),x,z)=0$ for all $x,y,z \in G$.

(c2) $T(T(x,y,y),z,v)+T(T(z,y,y),x,v)=0$ for all $x,y,z,v \in G$.

(c3) $T(T(x,y,z),x,v)+T(T(x,z,y),x,v)=0$ for all $x,y,z,v \in G$.

(c4) $T(T(x,y,z),u,v)+T(T(x,z,y),u,v)+T(T(u,y,z),x,v)+$
$+T(T(u,z,y),x,v)=0$ for all $x,y,z,u,v \in G$.

(d1) $T(T(x,y,y),z,z)-T(x,y,T(y,z,z)=0$ for all $x,y,z \in G$.

(d2) $T(T(x,y,z),v,v)+T(T(x,z,y),v,v)-T(x,y,T(z,v,v))-$
$-T(x,z,T(y,v,v))=0$ for all $x,y,z,v \in G$.

(d3) $T(T(x,y,y),z,v)+T(T(x,y,y),v,z)-T(x,y,T(y,z,v))-T(x,y,$
$T(y,v,z))= 0$ for all $x,y,z,v \in G$.

(d4) $T(T(x,y,z),u,v) +T(T(x,z,y),u,v)+T(T(x,y,z),v,u)+T(T(x,z,y),$
$v,u)-T(x,y,T(z,u,v))-T(x,z,T(y,u,v))-T(x,y,T(z,v,u))-T(x,z,$
$T(y,v,u))=0$ for all $x,y,z,u,v \in G$.

2.1. Lemma. (i) (a1) implies (a2).

(ii) (a4) implies (a1) and (a3).

(iii) (b1) is equivalent to (b2) and (a3).

(iv) (c1) implies (c2),(c3) and (c4).

(v) (c2) implies (c4).

(vi) (c3) implies (c4).

(vii) (c4) and (a1) imply (c1).

(viii) (d1) implies (d2),(d3) and (d4).

(ix) (d2) implies (d4).

(x) (d3) implies (d4).

(xi) (d4) and (a1) imply (d1).

Proof. Easy.

2.2. Lemma. Suppose that the conditions (b1) and (c2) are satisfied. Then

$$T(T(x,y,y),y,z)=0$$

for all $x,y,z \in G$.

Proof. Easy.

2.3. Lemma. Suppose that (b1) and (c1) are satisfied. Then

$$T(T(x,y,y),T(y,z,z),v)=0 ,$$

$$T(T(x,y,y),T(z,y,y)),v)=0$$

for all $x,y,z,v \in G$.

Proof. We have $((x,y,y),(y,z,z),v)=-(((y,z,z),y,y),x,v)=0$ by (c2) and (c1) and $((x,y,y),(z,y,y),v)=-(((z,y,y),y,y),x,v)=0$ by 2.2.

2.4. Lemma. Suppose that (a1),(c3) and (d4) are satisfied. Then

$$T(x,y,T(z,x,x))+T(x,z,T(y,x,x))=0$$

for all $x,y,z \in G$.

Proof. By (d4), $((x,y,z),x,x)+((x,z,y),x,x)+((x,y,z),x,x)+$ $+((x,z,y),x,x)-(x,y,(z,x,x))-(x,z,(y,x,x))-(x,y,(z,x,x))-$ $-(x,z,(y,x,x))=0$. Using (a1) and (c3) we get the desired equality.

2.5. Lemma. Suppose that (a1),(b1),(c3) and (d4) are satisfied. Then

$$T(x,y,T(y,x,x))=0,$$

$$T(x,y,T(x,y,y))=0$$

for all $x,y \in G$.

Proof. This is an easy consequence of 2.4.

2.6. Lemma. Suppose that (a1),(b1),(c2),(c3), and (d4) are satisfied. Then

$$T(T(x,y,y),T(x,z,z),y)+T(y,T(x,z,z),(x,y,y))=0$$

for all $x,y, z\in G$.

Proof. By (d4), $((x,y,y),(x,z,z),y)+((x,y,y),(x,z,z),y)+$
$+((x,y,y),y,(x,z,z))+((x,y,y)y,(x,z,z))-(x,y,(y,(x,z,z),y))-$
$-(x,y(y,(x,z,z),y))-(x,y,(y,y,(x,z,z))-(x,y,(y,y,(x,z,z))=0$.
From this, by (a1),(b1),(b2) and 2.2, we have
$(y,x,((x, z,z),y,y))-((x,y,y),(x,z,z),y)=0$.
The result now follows from 2.4.

2.7. Lemma. Suppose that (a1),(b1),(c1) and (d4) are satisfied. Then

$$T(x,T(y,z,z),T(z,x,x))=0$$

for all $x,y,z\in G$.

Proof. By (d4), $((x,y,y),(z,x,x),y)+((x,y,y),(z,x,x),y)+$
$+((x,y,y),y,(z,x,x))+((x,y,y),y,(z,x,x))-(x,y,(y,(z,x,x),y))-$
$-(x,y,(y,(z,x,x),y))-(x,y,(y,y,(z,x,x))-(x,y,(y,y,(z,x,x))=0$.
However, $((x,y,y),(z,x,x),y)= -((z,x,x),(x,y,y),y)$ by (b2) and 2.3, $((x,y,y),y,(z,x,x))=0$ by (b1) and (c2) and so $(y,z,((z,x,x),y,y))=0$ by (b1). From this, by 2.4, $(y,(z,x,x),(x,y,y))=0$.

2.8. Lemma. Suppose that (a1),(b1),(c1) and (d4) are satisfied. Then

$$T(x,T(y,z,z),T(z,y,y))=0$$

for all $x,y,z\in G$.

Proof. We have $(x,(y,z,z),(z,y,y))= -((y,z,z),x,(z,y,y))=$
$=((x,z,z),y,(z,y,y))= -(y,(x,z,z), (x,y,y))=0$ by 2.7.

2.9. Lemma. Suppose that (b2),(c1) and (d1) are satisfied. Then

$$T(x\ T(v.z.z),T(y,z,z))=0$$

for all $x,y,z\in G$.

Proof. $(x,(y,z,z),(y,z,z))= -((y,z,z),x,(y,z,z))=((x,z,z),y,(y,z,z))=(((x,z,z),y,y),z,z)= -((z,y,y),(x,z,z),z)=((x,z,z),(z,y,y),z)=(((z,y,y),z,z),x,z)=0$.

2.10. Lemma. Suppose that (b2),(c1) and (d1) are satisfied. Then

$$T(x,T(y,z,z),T(x,z,z))=0$$

for all $x,y,z \in G$.

Proof. $(x,(y,z,z),(x,z,z))= -((y,z,z),x,(x,z,z))=((x,z,z),y,(x,z,z))= -(y,(x,z,z),(x,z,z))=0$ by 2.9.

2.11. Lemma. Suppose that (a1),(b1),(c1) and (d4) are satisfied. Then

$$T(x,y,T(T(z,y,y),x,z))=0$$

for all $x,y,z \in G$.

Proof. We have $(x,y,((z,y,y),x,z)=(y,x,((x,y,y),z,z))$ by (b2) and (c2). Further, by (a1) and (d4), $(y,x,((x,y,y),z,z)+$ $+(y,(x,y,y),(x,z,z))- ((y,x,(x,y,y)),z,z)-((y,(x,y,y),x),z,z)=0$. But $((y,x,(x,y,y)),z,z)=0$ by 2.5, $((y,(x,y,y),x),z,z)=$ $= -(((x,y,y),y,x),z,z)=(((y,y,y),x,x),z,z)=0$ by (b1) and (c2) and, finally, $(y,(x,y,y),(x,z,z))= -((x,y,y),y,(x,z,z))=0$ by (c1).

2.12. Lemma. Suppose that (a1),(b1),(c1) and (d1) are satisfied. Then

$$T(x,T(y,z,z),T(x,y,y))+T(z,T(y,x,x),T(z,y,y))=0$$

for all $x,y,z \in G$.

Proof. We have $(x,(y,z,z),(x,y,y))= -((y,z,z),x,(x,y,y))=$ $=((x,z,z),y,(x,y,y))= -(y,(x,z,z),(x,y,y))=(y,x,((x,z,z),y,y))=$ $=(y,x,(x,z,(z,y,y)))$ by 2.4. Further, $(y,x,(x,z,(z,y,y)))+$ $+(y,x,(x,(z,y,y),z))-((y,x,x),z,(z,y,y))-((y,x,x),(z,y,y),z)=0$. But $((y,x,x),(z,y,y),z)= -((z,y,y),(y,x,x),z)=(((y,x,x),y,y),z,z)=$ $=0$ and we have by 2.11 $(y,x,(x,z,(z,y,y)))+(z,(y,x,x),(z,y,y))=0$.

3 Auxiliary results. In this section, let $G=G(+,T)$ be a ternary ring satisfying (a1) and (b1). We define a new binary operation $*$ on G by $x*y=x+y+T(x,y,x-y)=x+y+T(x,y,x)+T(y,x,y)=$ $=x+y-T(y,x,x)-T(x,y,y)$ for all $x,y\in G$.

3.1. Lemma. $x*y=y*x$, $x*0=x$ and $x*(-x)=0$ for all $x,y\in G$.

Proof. Easy.

3.2. Lemma. Suppose that (c1) and (d1) are satisfied. Then $(-x)*(x*y)=y$ for all $x,y\in G$.

Proof. Let $a,b\in G$. Then $c=(-a)*(a*b)=(-a)*(a+b+(a,b,a-b))=b+(a,b,a-b)+(-a,a+b+(a,b,a-b)),-2a-2b-(a,b,a-b))=b+d+e+f+g$ where $d=-(b,a,a)-(a,b,b)+(b,a,a)+(a,b,b)=0$, $e=-((b,a,a),a,a)-((a,b,b),a,a)+((b,a,a),a,b)+((a,b,b),a,b)=0$, $f=-(a,b,(b,a,a))-(a,b,(a,b,b))=0$ and $g=(a,(a,b,a-b),(a,b,a-b))=((a,b,b),a,(a,b,a-b))+((b,a,a),a,(a,b,(a-b))=0$. Hence $c=b$.

In the rest of this section, let $a,b,c\in G$. Put

$p=(a*b)*(a*c)$,

$d=-(b,a,a)-(a,b,b)-(c,a,a)-(a,c,c)+(a,c,b)-$
$-(a,c,c)-(a,b,b)-(b,a,c)-(c,b,b)-(b,c,c)$,

$e=-((b,a,a),a,b)-((a,b,b),a,b)+((b,a,a),a,c)+((a,b,b),a,c)-((b,a,a),c,b)-((a,b,b),c,b)+((b,a,a),c,c)+((a,b,b),c,c)+((c,a,a),a,b)+((a,c,c),a,b)-((c,a,a),a,c)-((a,c,c),a,c)+((c,a,a),b,b)+((a,c,c),b,b)-((c,a,a),b,c)-((a,c,c),b,c,$

$f=-(a,c,(b,a,a))-(a,c,(a,b,b))+(a,c,(c,a,a))+(a,c,(a,c,c))-$
$-(b,a,(b,a,a))-(b,a,(a,b,b))+(b,a,(c,a,a))+(b,a,(a,c,c))-$
$-(b,c,(b,a,a))-(b,c,(a,b,b))+(b,c,(c,a,a))+(b,c,(a,c,c))$,

$g=((a,b,a-b),(a,c,a-c),b-c)$,

$h=-(a+c,(a,b,a-b),(a,b,a-b))+(a+c,(a,b,a-b),(a,c,a-c))$,

$i = (a+b,(a,c,a-c),(a,b,a-b))-(a+b,(a,c,a-c),(a,c,a-c)),$

$j = ((a,b,a-b),(a,c,a-c),(a,b,a-b))-((a,b,a-b),(a,c,a-c),$
$(a,c,a-c)),$

$q = (a*a)*(b*c),$

$k = -(c,b,b)-(b,c,c)-(b,a,a)+(a,b,b)-2(a,b,c)-(c,a,a)-$
$-2(a,c,b)+(a,c,c),$

$m = ((c,b,b),a,a)+((b,c,c)a,a)+((c,b,b),a,b)+((b,c,c),a,b)+$
$+((c,b,b),a,c)+((b,c,c),a,c),$

$n = -(a,b,(c,b,b))-(a,b,(b,c,c))-(a,c,(c,b,b))-(a,c,(b,c,c)),$

$r = (a,(b,c,b-c),(b,c,b-c)).$

3.3. Lemma. $p = 2a+b+c+d+e+f+g+h+i+j$ and $q = 2a+b+c+k+m+n+r$.

Proof. Easy.

3.4. Lemma. $d-k = 0$.

Proof. Easy (use (a2)).

3.5. Lemma. Suppose that (c1) and (d1) are satisfied. Then $e+f-m-n = 0$.

Proof. Using (c1), 2.5 and 2.2, we see that
$e+f-m-n = s_1+s_2+s_3+s_4+s_5+s_6$ where

$s_1 = -((b,a,a),c,b)+((c,a,a),b,b)-((c,b,b),a,a)-(a,c,(a,b,b))-$
$-(b,c,(b,a,a)),$

$s_2 = -((a,b,b),c,b)-((c,b,b),a,b)-(b,c,(a,b,b))+(a,b,(c,b,b)),$

$s_3 = ((b,a,a),c,c)-((c,a,a),b,c)-((b,c,c),a,a)+(b,a,(a,c,c))+$
$+(b,c(c,a,a)),$

$s_4 = ((a, b,b),c,c)+((a,c,c),b,b)-((b,c,c),a,b)-((c,b,b),a,c)+$
$+(a,b,(b,c,c))+(a,c,(c,b,b)),$

$s_5 = -((a,c,c),b,c)-((b,c,c),a,c)+(b,c,(a,c,c))+(a,c,(b,c,c)),$

$s_6 = -(a,c,(b,a,a))+(b,a,(c,a,a)).$

But $-((b,a,a),c,b)=((c,a,a),b,b)$ and $s_1 = -((c,a,a),b,b)+$

$+(c,a(a,b,b))-((c,b,b),a,a)+(c,b,(b,a,a))$ by (c2), (a1) and (b2).

Consequently, $s_1 = 0$ by (d1). Further, $s_2 = -(b,c,(a,b,b))-$ $-(b,a,(c,b,b))$ by (c2) and (b2), and hence $s_2 = 0$ by 2.4. Similarly, $s_3 = -((b,a,a),c,c)+(b,a,(a,c,c))-((b,c,c),a,a)+(b,c,(c,a,a))=$ $= 0$ by (c2),(a1) and (d1). As for s_4, by (c2) and (a1), we have $s_4 = -((a,b,b),c,c)+(a,b,(b,c,c))-((a,c,c),b,b)+(a,c,(c,b,b))=0$ by (d1). On the other hand , by (c2) and (b2), $s_5 = (b,c,(a,c,c))+$ $+(a,c,(b,c,c))= -(c,b,(a,c,c))-(c,a,(b,c,c))$. Hence, by 2.4, $s_5 =$ $= -(c,b,(a,c,c))+(c,b,(a,c,c))=0$. Finally, $s_6 = (a,b,(c,a,a))-$ $-(a,b,(c,a,a))=0$ by (b2) and 2.4.

3.6. Lemma. Suppose that (c1) and (d1) are satisfied. Then $g = ((a,b,b),(a,c,c),b)-((a,b,b),(a,c,c),c)+((a,b,b),(c,a,a), b)-$ $-((a,b,b),(c,a,a),c)$.

Proof. Use 2.3.

3.7. Lemma. Suppose that (c1) and (d1) are satisfied. Then

$h = (c,(a,b,a-b),(a,c,a-c))$,

$i = (b,(a, c,a-c),(a,b,a-b))$,

$r = 0$.

Proof. We have $(a,(a,b,a-b),(a,c,a-c))= -((a,b,a-b),a,$ $(a,c,a-c))=0$ by (c1),(c2) and (b1). The rest follows from 2.8 and 2.9.

3.8. Lemma. Suppose that (c1) and (d1) are satisfied. Then $g+h+i-r=0$.

Proof. We have $g+h+i-r=t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9$ where

$t_1 = ((a,b,b),(a,c,c),b)+(b,(a,c,c),(a,b,b))=0$ by 2.6,

$t_2 = ((a,c,c),(a,b,b),c)+(c,(a,b,b),(a,c,c))=0$ by 2.6,

$t_3 = -((c,a,a),(a,b,b),b)=0$ and $t_4 =((c,a,a),(a,b,b),c)=0$ by 2.3,

$t_5 = (b,(c,a,a),(a,b,b))= -((c,a,a),b,(a,b,b))=((b,a,a),c,(a,b,b))=$ $= -(c,(b,a,a),(a,b,b))=0$ by 2.8,

t_6 = (c,(b,a,a),(a,c,c))= -(b,(c,a,a),(a,c,c))=0 by 2.8,

t_7 = (c,(b,a,a),(c,a,a))= -(b,(c,a,a),(c,a,a))=0 by 2.9,

t_8 = (b,(c,a,a),(b,a,a))= -(c,(b,a,a),(b,a,a))=0 by 2.9,

and t_9 = (c(a,b,b),(c,a,a)+(b,(a,c,c),(b,a,a))=0 by 2.12.

3.9. Lemma. $j = 0$.

Proof. This follows from 2.9 and 2.10.

3.10. Lemma. $p = q$.

Proof. See 3.3,...,3.9.

4. Some identities for ternary algebras. Here, let R be an associative ring and A = A(+,rx,T) a ternary algebra. Further, let F be a homomorphism of R into Z_3. Consider the following two identities:

(q1) $(r^3-r)T(x,y,y)=0$ for all $r \in R$ and $x,y \in A$.

(q2) $rT(x,y,z)=0$ for all $r \in Ker(F)$ and $x,y,z \in A$.

4.1. Lemma. (q2) implies (q1).

Proof. We have $x^3 = x$ for every $x \in Z_3$, and so $r^3-r \in Ker(F)$ for every $r \in R$.

5. The construction.

5.1. Theorem. Let G = G(+,T) be a ternary ring satisfying (a1),(b1),(c1) and (d1). Put $x * y = x+y+T(x,y,x-y)$ for all $x,y \in G$. Then G(*) is a commutative Moufang loop.

Proof. The result follows from 3.1, 3.2 and 3.10.

5.2. Theorem. Let R be an associative ring and A = A(+,rx,T) a ternary algebra satisfying (a1),(b1),(c1),(d1) and (q1). Put $x * y = x+y+T(x,y,x-y)$ for all $x,y \in A$. Then A(*,rx) is a quasimodule. Moreover, if F is a homomorphism of R into Z_3 and A satisfies (q2) then the quasimodule is F-special.

Proof. By 5.1, $A(*)$ is a commutative Moufang loop. Further, $rx * sx = rx+sx+(rx,sx,rx-sx) = rx+sx+rs(r-s)(x,x,x) = rx+sx = (r+s)x$, $r(x*y) = rx+ry+r(x,y,x-y) = rx+ry+r^3(x,y,x-y) = rx+ry+(rx,ry,rx-ry) = rx*ry$. The rest is easy.

6. Auxiliary results. In this section, let G=G(+,T) be a ternary ring satisfying the conditions (a1),(b1),(c1) and (d1).

6.1. Lemma. For all $x,y, z \in G$,

$$T(T(x,y,y),z,y)+T(x,y,T(z,y,y))=0.$$

Proof. By (d4) and 2.2, $-((x,y,y),z,y)+(x,y,(y,z,y))-((x,y, y),y,z) +(x,y,(y,y,z))=0$, and hence $((x,y,y),z,y)+(x,y,(z,y,y))=0$.

6.2. Lemma. For all $x,y,z \in G$,

$$T(T(x,y,y),z,y)+T(y,z, T(x,y,y))=0.$$

Proof. By (c2) and 6.1, $((x,y,y),z ,y)= -((z,y,y),x,y)= =(z,y,(x,y,y))= -(y, z,(x,y,y))$.

6.3. Lemma. For all $x,y,z \in G$,

$$T(T(x,y,z),z,x)+T(T(x,z,y),z,x)-T(T(y,x,x),z,z)-$$
$$-T(x,z,T(y,z,x))-T(x,z,T(y,x,z))=0.$$

Proof. By (d4), $((x,y,z),z,x)+((x,z,y),z,x)+((x,y,z),x,z)-(x,y,(z,z,x))-(x,z,(y,z,x))-(x,y,(z,x,z))+((x,z,y),x,z)-(x, z,(y,x,z))=0$. But $(x,y,(z,z,x))=0$, $((x,y,z),x,z)+((x,z,y),x,z)=0$ by (c3) and $-(x,y,(z,x,z))= -(y,x,(x,z,z))= = -((y,x,x),z,z)$.

6.4. Lemma. For all $x,y,z \in G$,

$$T(T(x,y,z).z,z)+T(T(x,z,y),z,z)-T(x,z,T(y,z,z))=0.$$

Proof. By (d4), $-((x,y,z),z,z)+(x,y,(z,z,z))-((x,z,y),z,z)+$ $+(x,z,(y,z,z))=0$.

6.5. Lemma. For all $x,y,z \in G$,

$$T(T(x,y,y),z,z)+T(z,y,T(y,x,z))+T(z,y,(T(y,z,x))=0.$$

Proof. By (d3), $((z,y,y),x,z)+((z,y,y),z,x)-(z,y,(y,x,z))-$ $-(z,y,(y,z,x))=0$. But $((z,y,y),z,x)=0$ by (c1) and $((z,y,y),x,z)=$ $= -((x,y,y),z,z)$ by (c2).

6.6. Lemma. For all $x,y,z \in G$,

$$T(T(x,y,y),z,z)-T(T(x,z,z),y,y)+T(z,y,T(y,x,z))-$$
$$-T(y,z,T(z,x,y))=0.$$

Proof. By 6.5, $((x,y,y),z,z)= -(z,y,(y,x,z))-(z,y,(y,z,x))$ and $((x,z,z),y,y)= -(y,z,(z,x,y))-(y,z,(z,y,x))$. The rest is clear.

6.7. Lemma. For all $x,y,z \in G$,

$$T(T(x,y,y),z,z)+T(T(x,y,z),y,z))+T(T(x,z,y),y,z)=0.$$

Proof. By 6.3, $(z,y,(y,x,z))= -(z,y,(x,y,z))=(z,y,(x,z,y))+$ $+((x,z,z),y,y)-((z,y,x),y,z)-((z,x,y),y,z)$. Hence, by 6.6,
$((x,y,y),z,z)-((x,z,z),y,y)+(z,y,(x,z,y))+((x,z,z),y,y)-$
$-((z,y,x),y,z)-((z,x,y),y,z)-(y,z,(z,x,y))=0$. From this,
$((x,y,y),z,z)+((y,z,x),y,z)+((x,z,y),y,z)=0$.
But $((y,z,x),y,z)= -((y,x,z),y,z)=((x,y,z),y,z)$ by (c2).

6.8. Lemma. For all $x,y,z,u \in G$,

$$T(x,T(x,y,z),u)= -T(x,T(y,x,z),u)= -T(x,T(x,z,y),u)=$$
$$=T(x,T(z,x,y),u).$$

Proof. Use (b2) and (c3).

6.9. Lemma. For all $x,y,z,u \in G$,

$$T(z,T(x,y,z),u)+T(z,T(x,z,y),u)-T(x,T(y,z,z),u)=0.$$

Proof. Use (c4).

6.10. Lemma. For all $x,y,z,u \in G$,

$$T(x,T(x,y,z),u)+T(x,T(y,z,x),u)+T(x,T(z,x,y),u)=$$
$$=3T(x,T(x,y,z),u)+T(y,T(z,x,x),u).$$

Proof. Apply 6.8 and 6.9.

6.11. Lemma. For all $x,y,z \in G$,

$$T(x,T(y,z,z),T(x,y,y))-T(y,T(z,x,x),T(z,y,y))=0.$$

Proof. By 2.14 and (c2), $(x,(y,z,z),(x,y,y))= -(z,(y,x,x),(z,y,y))=((y,x,x),z,(z,y,y))= -((z,x,x),y,(z,y,y))=(y,(z,x,x),(z,y,y))$.

6.12. Lemma. For all $x,y,z \in G$,

$$T(x,T(y,z,z),T(z,x,y)+T(z,y,x))=0.$$

Proof. By (d2), $((y,z,z),x,u)+((y,z,z),u,x)-(y,z,(z,x,u))-(y,z,(z,u,x))=0$, $u=(z,x,y)+(z,y,x)$. But $((y,z,z),u,x)= -((u,z,z),y,x)$ by (c2) and $(u,z,z)=0$ by (c3). Similarly, $(z,u,x)= -(u,z,x)=0$ by (c3). Hence $((y,z,z),x,u)=(y,z,(z,x,u))$. Further, $(z,x,u)= -(x,z,(z,x,y))-(x,z,(z,y,x))=((y,z,z),x,x)$ by 6.5. Consequently, $((y,z,z),x,u)=(y,z,((y,z,z)x,x))$. On the other hand, by (d3), $(y,z,((y,z,z),x,x)+(y,(y,z,z),(z,x,x))-((y,z,(y,z,z)),x,x)-((y,(y,z,z),z),x,x)=0$. Since $(y,(y,z,z),(z,x,x))=(y,(y,z,z),z)=0$ by (c2), we have $(y,z,((y,z,z),x,x)=((y,z,(y,z,z)),x,x)=0$ by 2.5.

6.13. Lemma. For all $x,y,z \in G$,

$$T(x,T(y,z,z),T(T(y,x,x),z,z))=0.$$

Proof. By (d4), $((y,z,z),x,((y,x,x),z,z))-(y,z,(z,x,((y,x,x),z,z)))+((y,z,z),((y,x,x),z,z),x)-(y,z,(z,((y,x,x),z,z),x))=0$. But $(z,x,((y,x,x),z,z))= -(z,(y,x,x),(x,z,z))=0$ by 2.4 and 2,7, $((y,z,z),((y,x,x),z,z),x)= -((((y,x,x),z,z),z,z),y,x)=0$ $(z,((y,x,x),z,z),x))= -((((y,x,x)z,z),z),x)=0$.

6.14. Lemma. For all $x,y,z \in G$,

$$T(x,T(y,z,z),T(T(y,z,z),x,x))=0.$$

Proof. Use (d4), 2.3 and 2.7.

6.15. Lemma. For all $x,y,z \in G$,

$$T(x,T(y,z,z),T(T(x,z,z),y,y))=0.$$

Proof. We have $(x,(y,z,z),((x,z,z),y,y))-((y,z,z),x,((x,z,z),y,y))=((x,z,z),y,((x,z,z),y,y))= -(y,(x,z,z),((x,z,z),y,y))=0$ by 2.4.

6.16. Lemma. For all $x,y,z \in G$,

$$T(x,T(y,z,z),T(T(x,y,y),z,z))=0.$$

Proof. We have $(x,(y,z,z),((x,y,y),z,z))= -((y,z,z),x,((x,y,y),z,z))=((x,z,z),y,((x,y,y),z,z))= -(y,(x,z,z),((x,y,y),z,z))=0$ by 6.13.

6.17. Lemma. For all $x,y,z \in G$,

$$T(x,y,T(T(x,y,y),z,z))=0.$$

Proof. By (d4), $(x,y,((x,y,y),z,z))+(x,(x,y,y),(y,z,z))-((x,y,(x,y,y)),z,z)-((x,(x,y,y),y),z,z)=0$. However, $(x,(x,y,y),(y,z,z))= -((x,y,y),x,(y,z,z))=0$ by (c2), $(x, y,(x,y,y))=0$ by 2.5, $(x,(x,y,y)= -((x,y,y),x,y)=0$ by (c2).

7. Auxiliary results. In this section, let $G=G(+,T)$ be a ternary ring satisfying (a1),(b1),(c1) and (d1). Put $x * y=x+y+T(x,y,x-y)$ for all $x,y \in G$, so that $G(*)$ is a commutative Moufang loop (see 5.1).

Let $a,b,c \in G$ and

$d = (a * b) * c,$

$e = -(b,a,a)-(a,b,b)-(c,a,a)-(a,c,c)-(b,c,c)-(c,b,b)+(a,c,b)+ +(b,c,a),$

$f = -((b,a,a),c,a)-((a,b,b),c,a)-((b,a,a),c,b)-((a,b,b),c,b)+$

$+((b,a,a),c,c)+((a,b,b),c,c)$,

$g = -(a,c,(b,a,a))-(a,c,(a,b,b))-(b,c,(b,a,a))-(b,c,(a,b,b))$,

$h = -(c,(a,b,a-b),(a,b,a-b))$,

$i = a * (b * c)$,

$j = -(c,b,b)-(b,c,c)-(b,a,a)-(a,b,b)-(c,a,a)-(a,c,c)-(a,b,c)-$
$-(a,c,b)$,

$k = ((c,b,b),a,a)+((b,c,c),a,a)-((c,b,b),a,b)-((b,c,c),a,b)-$
$-((c,b,b),a,c)-((b,c,c),a,c)$,

$m = (a,b,(c,b,b))+(a,b,(b,c,c))+(a,c,(c,b,b))+(a,c,(b,c,c))$,

$n = -(a,(b,c,b-c),(b,c,b-c))$.

7.1. Lemma. $d=a+b+c+e+f+g+h$ and $i=a+b+c+j+k+m+n$.

Proof. Easy.

7.2. Lemma. $e-j=(a,b,c)+(b,c,a)-2(c,a,b)$.

Proof. Easy.

7.3. Lemma. $f+g-k-m=((a,c,c),b,b)-((a,b,b),c,c)+((b,a,a)c,c)-$
$-((b,c,c),a,a)+((c,b,b),a,a)-((c,a,a),b,b)$.

Proof. We have $f+g-k-m=p_1+p_2+p_3+p_4+p_5+p_6$ where

$p_1= -((b,a,a),c,a)-(a,c,(b,a,a))$,

$p_2= -((a,b,b),c,a)-((b,a,a),c,b)-(a,c,(a,b,b))-(b,c,(b,a,a))-$
$-((c,b,b),a,a)$,

$p_3 = -((a,b,b),c,b)-(b,c,(a,b,b))+((c,b,b),a,b)-(a,b,(c,b,b))$,

$p_4 = ((b,a,a),c,c)-((b,c,c),a,a)$,

$p_5 = ((a,b,b),c,c)+((b,c,c),a,b)+((c,b,b),a,c)-(a,b,(b,c,c))-$
$(a,c,(c,b,b)$,

$p_6 = ((b,c,c),a,c)-(a,c,(b,c,c))$.

Now, $p_1=p_3=p_6=0$ by 6.2, $p_2=((c,b,b),a,a)+((c,a,a),b,b)+$
$+((c,a,a),b,b)+((c,b,b),a,a)-((c,b,b),a,a)= - ((c,a,a),b,b)+$
$+((c,b,b),a,a)$, $p_5= -((a,b,b),c,c)+((a,c,c),b,b)$.

7.4. Lemma. $h = n = 0$.

Proof. Use 2.8 and 2.9.

Let $q = d - i$.

7.5. Lemma. $q=(a,b,c)+(b,c,a)-2(c,a,b)+((b,a,a),c,c)-$
$-((b,c,c),a,a)+((c,b,b),a,a)-((c,a,a),b,b)+((a,c,c),b,b)-$
$-((a,b,b),c,c)$.

Proof. Use 7.1,...,7.4.

7.6. Lemma. $d*(-i)=q+(q,i,q-i)$.

Proof. $d*(-i)=d-i+(d,-i,d+i)=q+(d,-i,d+i)$. But $d=q+i$ and $(q+i,-i,q+2i)=(q,-i,q+2i)= -(q,i,q)+(q,i,i)=(q,i,i-q)$.

Put

$w = a+b+c$,

$r = -(a,b,b)-(b,a,a)-(a,c,c)-(c,a,a)-(b,c,c)-(c,b,b)$,

$s = -(a,b,c)-(a, c,b)$,

$t = (a,b,c)+(b,c,a)-2(c,a,b)$,

$u = ((c,b,b),a,a)+((b,c,c),a,a)-((a,c,c),b,b)-((a,b,b),c,c)$,

$v = ((b,a,a),c,c)-((b,c,c),a,a)+((c,b,b),a,a)-((c,a,a),b,b)+$
$+((a,c,c),b,b)-((a,b,b),c,c)$.

7.7. Lemma. (i) $i=w+r+s+u$.

(ii) $q=t+v$.

(iii) $i-q=w+r+s-t+u-v$.

(iv) $r+s-t= -(a,b,b)-(b,a,a)-(a,c,c)-(c,a,a)-(b,c,c)-$
$-(c,b,b)-2(a,b,c)+3(c,a,b)-(b,c,a)$.

(v) $u-v= -((b,a,a),c,c)-((b,c,c),a,a)+((a,c,c),b,b)+$
$+((c,a,a),b,b)$.

Proof. Easy.

8. Auxiliary results.

8.1. Lemma. $(t,w,w)=((a,b,c),c,w)+((b,c,a),a,w)+$

$+((c,a,b),b,w)+2((a,b,c)a,w)+2((b,c,a),b,w)+2((c,a,b,),c,w)$.

Proof. By (c3), $((c,a,b),a,w)= -((a,c,b),a,w)=((a,b,c),a,w)$, $((a,b,c),b,w)=((b,c,a)b,w)$ and $((b,c,a),c,w)=((c,a, b),c,w)$.

Now, let

$q_1 = ((a,b,c),c,a)+((b,c,a),a,c)+2((a,b,c),a,c)+2((c,a,b),c,a)$,

$q_2 = ((a,b,c),c,b)+((c,a,b),b,c)+2((b,c,a),b,c)+2((c,a,b),c,b)$,

$q_3 = ((b,c,a),a,b)+((c,a,b),b,a)+2((a,b,c),a,b)+2((b,c,a),b,a)$,

$q_4 = ((a,b,c),c,c)+2((c,a,b),c,c)$,

$q_5 = ((b,c,a),a,a)+2((a,b,c),a,a)$,

$q_6 = ((c,a,b),b,b)+2((b,c,a),b,b)$.

8.2. <u>Lemma</u>. $(t,w,w)=q_1+q_2+q_3+q_4+q_5+q_6$.

Proof. Use 8.1.

8.3. <u>Lemma</u>. $q_1+((b,a,a),c,c)-((b,c,c),a,a)=((a,b,c) a,c)+$ $+3((c,a,b),c,a)$.

Proof. By 6.7, $q_1=((a,b,c),c,a)+2((c,a,b),c,a)+((b,c,a),a,c)+$ $+2((a,b,c),a,c)= -((b, a,c),c,a)-((b,c,a),c,a)+3((c,a,b),c,a)+$ $+((b,c,a),a,c)+((b,a,c),a,c)+3((a,b,c),a,c)=((b,c,c),a,a)-$ $-((b,a,a),c,c)+3((a,b,c),a,c)+3((c,a,b),c,a)$.

8.4. <u>Lemma</u>. $q_2 +((a,c,c),b,b)-((a,b,b),c,c)=3((c,a,b),c,b)+$ $+3((b,c,a),b,c)$, $q_3+((c,b,b),a,a)-((c,a,a),b,b)=3((a,b,c),a,b)+$ $+3((b,c,a),b,a)$.

Proof. Similar to 8.3.

8.5. <u>Lemma</u>. $q_4=(c,b(a,c,c))$.

Proof. By 6.4 and 2.4, $q_4=((a,b,c),c,c)-((c,a,b),c,c)=$ $=((a,b,c),c,c)+((a,c,b),c,c)=(a,c,(b,c,c))= -(c,a,(b,c,c))=$ $=((c,b,(a,c,c))$.

8.6. <u>Lemma</u>. $q_5=(a,c,(b,a,a))$ and $q_6=(b,a,(c, b,b))$.

Proof. Similar to 8.5.

8.7. Lemma. $q+(t,w,w)=(a,b,c)+(b,c,a)-2(c,a,b)+3((a,b,c),a,c)+3((c,a,b),c,a)+3((c,a,b),c,b)+3((b,c,a),b,c)+3((a,b,c),a,b)+3((b,c,a),b,a)+(c,b,(a,c,c))+(a,c,(b,a,a))+(b,a,(c,b,b))$.

Proof. Use 7.5, 8.2,...,8.6.

8.8. Lemma. Suppose that (a4) is satisfied. Then $q+(t,w,w)=(a,b,c)+(b,c,a)+(c,a,b)+(a,c,(b,a,a))+(b,a,(c,b,b))+(c,b,(a,c,c))$.

Proof. This is an immediate consequence of 8.7.

9. Auxiliary results.

9.1. Lemma. $(q,i,i-q)=(t,w,w)+(t,w,r)+(t,w,s)-(t,w,t)+(t,w,u)-(t,w,v)+(t,r,w)+(t,r,r)+(t,r,s)-(t,r,t)+(t,r,u)-(t,r,v)+(t,s,w)+(t,s,r)+(t,s,s)-(t,s,t)+(t,s,u)-(t,s,v)+(t,u,w)+(t,u,r)+(t,u,s)-(t,u,t)+(t,u,u)-(t,u,v)+(v,w,w)+(v,w,r)+(v,w,s)-(v,w,t)+(v,w,u)-(v,w,v)+(v,r,w)+(v,r,r)+(v,r,s)-(v,r,t)+(v,r,u)-(v,r,v)+(v,s,w)+(v,s,r)+(v,s,s)-(v,s,t)+(v,s,u)-(v,s,v)+(v,u,w)+(v,u,r)+(v,u,s)-(v,u,t)+(v,u,u)-(v,u,v)$.

Proof. See 7.7.

10. Auxiliary results.

10.1. Lemma. $(t,w,r)=(w,z,-r)$, where $z=(a,b,c)+(b,c,a)+(c,a,b)$.

Proof. Easy.

10.2. Lemma. $(t,w,r)=(a,(b,c,c),-r)+(b,(c,a,a),-r)+(c,(a,b,b),-r)$.

Proof. Use 10.1 and 6.10.

10.3. Lemma. $(t,w,r)=(a,(b,c,c),(a,b,b))+(a,(b,c,c),(b,a,a))+(b,(c,a,a),(b,c,c))+(b,(c,a,a),(c,b,b))+(c,(a,b,b),(c,a,a))+(c,(a,b,b),(a,c,c))$.

Proof. Use 10.1, 2.7, 2.8, 2.9, and 2.10.

10.4. Lemma. $(t,w,r)= -(a,(b,c,c),(a,b,b))-(b,(c,a,a),(b,c,c))-(c,(a,b,b),(c,a,a))$.

Proof. Use 10.3 and 6.11.

10.5. Lemma. $(t,w,r)=(a,(c,b,b),(c,a,a))+(b,(a,c,c),(a,b,b))+ +(c,(b,a,a),(b,c,c))$.

Proof. Use 10.4 and (c2).

11. Auxiliary results.

11.1. Lemma. $(t,w,s-t)=(w,(a,b,c)+(b,c,a)+(c,a,b),t-s)+ +3(w,(c,a,b),s-t)$ and $t-s=(b,c,a)+2(a,b,c)-3(c,a,b)$.

Proof. Easy.

11.2. Lemma. $(t,w,s-t)=(a,(b,c,c),(b,c,a))-(a,(b,c,c),(a,b,c)+(b,(c,a,a),(b,c,a))-(b,(c,a,a),(a,b,c))+(c,(a,b,b),(b,c,a))- -(c,(a,b,b),(a,b,c))$.

Proof. By 11.1 and 6.10, $(t,w,s-t)=(b,(c,a,a),t-s)+ +(c,(a,b,b),t-s)+(a,(b,c,c),t-s)+3(a,(a,b,c),t- s)+3(b,(b,c,a),t-s)+3(c,(c,a,b),t-s)+3(w,(c,a,b),s-t)$. But, clearly, $3i=3w=3d$ and $3r=3u=3v=0$. Now, by 7.7(i),(ii), $3q=0$ and $3s=3t=0$. The rest is clear.

11.3. Lemma. $(t,w,s-t)=(a,(b,c,c),(b,c,a))-(a,(b,c,c),(a,b,c))+(b,(c,a,a),(b,c,a))-(b,(c,a,a),(a,b,c))$.

Proof. Apply 11.2 and 6.12.

12. Auxiliary results.

12.1. Lemma. $(t,w,u-v)=(a+b+c,(a,b,c)+(b,c,a)+(c,a,b),v-u)$, $v-u=((b,a,a),c,c)+((b,c,c),a,a)-((a,c,c),b,b)-((c,a,a),b,b)$.

Proof. Obvious.

12.2. Lemma. $(t,w,u-v)=(b,(c,a,a),v-u)+(c,(a,b,b),v-u)+ +(a,(b,c,c),v-u)$.

Proof. See 12.1 and 6.10.

12.3. <u>Lemma</u>. (t,w,u-v)= -(a,(b,c,c),((c,a,a),b,b))-(b,(c, a,a),((a,c,c),b,b))+(c,(a,b,b),((b,a,a),c,c))+(c,(a,b,b), ((b,c,c),a,a)).

Proof. Apply 12.2, 6.13, 6.14, 6.15, 6.16.

12.4. <u>Lemma</u>. (t,w,u-v)=(a,(b,c,c),((c,a,a),b,b))+ +(b,(c,a,a),((a,b,b),c,c))+(c,(a,b,b),((b,c,c),a,a)).

Proof. We have -(b,(c,a,a),((a,c,c),b,b))=(b,(a,c,c), ((c,a,a),b,b))= -((a,c,c),b,((c, a,a),b,b))=((b,c,c),a,((c,a.a), b,b))=-(a,(b,c,c),((c,a,a),b,b)) and (c,(a,b,b),((b,a,a),c,c))= = -(c,(b,a,a),((a,b,b)c,c))=((b,a,a),c,((a,b,b),c,c))= = -((c,a,a)b,((a,b,b),c,c))=(b,(c,a,a),((a,b,b),c,c)) by 2.4.

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(Oblatum 30.9. 1985)