Darko Žubrinić

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ON THE OPTIMAL CONTROL OF NONRESONANT ELLIPTIC EQUATIONS

Darko ŽUBRINIĆ

Abstract: In this note we obtain a result on the existence of optimal controls for nonresonant equations describing a distributed parameter system.

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1. Introduction. In this paper we shall study the optimal control problem for nonresonant elliptic equations defined on a bounded domain in R^n. Similar problems have been studied for instance in [2] and [4]. Here we obtain the existence result of optimal controls.

Throughout the paper Ω will be a bounded domain in R^n. We shall consider the optimal control problem whose state equation is described by the nonlinear elliptic equation of the following form

$$(1) \qquad \begin{aligned} -\Delta u &= g(x,u,\nabla u) + p(x) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

We impose on g the so called nonresonance condition, insuring thus the unique solvability of the state equation for any given $p \in L_2(\Omega)$. Roughly speaking, this means that the growth rate of the function g with respect to the second variable should be away from the spectrum of $-\Delta$.

Next, let a nonempty set U in $L_2(\Omega)$ be given, which we interpret as the set of admissible controls. To formulate the optimal control problem we must also define the cost functional

$$J: U \longrightarrow R_+$$

where R_+ is the sèt of nonnegative reals. The functional J will be of the form

$$J(p) = J_1(p) + J_2(u), \quad J_1, J_2 \geq 0$$

where u is a solution of the state equation (1), generated by the control p. A typical example would be for instance

$$J_1(p) = \nu \int_\Omega p^2 dx, \quad \nu > 0$$

$$J_2(u) = \int_\Omega (u - z)^2 dx$$

where z is a given target function in $L_2(\Omega)$ (see [2]). Now the optimal control problem consists in finding the control $p \in U$, generating the state u via (1), such that J(p) is minimal , i.e.

$$\min_{p \in U} (J_1(p) + J_2(u)) \tag{2}$$

We shall denote this problem in short as the problem (1),(2).

2. <u>The unique solvability result</u>. Let us now formulate a result which is an immediate consequence of the main result in [3].

<u>Theorem 1</u>. Let $g: \Omega \times R \times R^n \longrightarrow R$ be a mapping of the Carathéodory type, i.e. measurable with respect to the first variable and continuous with respect to the remaining 1+n variables. Assume also that $g(\cdot,0,0) \in L_2(\Omega)$, and let there exist $\varepsilon > 0$ and $k \in N$ such that for two distinct consecutive eigenvalues λ_k, λ_{k+1} of $-\Delta$ with the Dirichlet boundary condition, we have the following nonresonance condition:

$$\lambda_k + \varepsilon \leq (g(x,u_1,p) - g(x,u_2,p))/(u_1 - u_2) \leq \lambda_{k+1} - \varepsilon \tag{3}$$

Furthermore, let

$$(4) \qquad |g(x,u,p_1) - g(x,u,p_2)| \leq c\ |p_1 - p_2|_2$$

$$(5) \qquad c < \varepsilon / \lambda_{k+1}^{1/2}$$

where $|\ \ |_2$ is an l_2-norm in R^n.

Then for every $p \in L_2(\Omega)$ there exists a unique weak solution of (1) in the Sobolev space $H_0^1(\Omega)$.

We shall denote the usual scalar product and the norm in $L_2(\Omega)$ by $(\ ,\ )_0$ and $\|\ \|_0$ respectively, and for the Sobolev space $H_0^1(\Omega)$ we define

$$(u,v)_1 = \int_\Omega \nabla u \nabla v dx, \quad \|u\|_1 = (u,v)_1^{1/2}$$

In the proof of the existence of optimal controls, a crucial role is played by the following fundamental lemma, which describes how the solutions of (1) depend on p.

<u>Lemma</u>. Let the conditions of the theorem 1 be satisfied. There exist effectively computable constants $C_1, C_2 > 0$, independent of p and u, such that for any p and u satisfying (1) we have

$$(6) \qquad \|u\|_1 \leq C_1 + C_2 \|p\|_0$$

Moreover, for any two solutions (p_1,u_1) and (p_2,u_2) of (1) we have

$$(7) \qquad \|u_1 - u_2\|_1 \leq C_2 \|p_1 - p_2\|_0$$

<u>Proof</u>. For the sake of simplicity we shall prove (6) only, because (7) can be obtained in a similar way.

We shall use the method of the proof analogous to that in [3].

So let (φ_i) be a sequence of eigenfunctions of $-\Delta$, normalized by $\|\varphi_i\|_0 = 1$, with the corresponding eigenvalues λ_i. As is well known, the sequence (λ_i) is such that

$$0<\lambda_1<\lambda_2\leq\lambda_3\leq\ldots\leq\lambda_i\longrightarrow\infty$$

and it is easy to check that

(8) $$\|\varphi_i\|_1 = \lambda_i^{1/2}$$

The sequence (φ_i) is complete and orthogonal in both $L_2(\Omega)$ and $H_0^1(\Omega)$. Let

$$H^- = \text{span}\{\varphi_1,\ldots,\varphi_k\}$$
$$H^+ = \overline{\text{span}}\{\varphi_i : i>k\}$$

We thus have $H_0^1(\Omega) = H^-\oplus H^+$ and consequently, a solution u of (1) has the form

$$u = u^- + u^+$$

with $u^\pm\in H^\pm$ respectively. Let

(9) $$r = (\lambda_k + \lambda_{k+1})/2$$
$$\tilde{u} = -u^- + u^+$$

Defining

$$A = g(x,u,\nabla u) - g(x,0,\nabla u) - rz$$
$$B = g(x,0,\nabla u) - g(x,0,0)$$
$$C = g(x,0,0) + p(x)$$

we can rewrite (1) in the following way

(10) $$-\Delta u - ru = A + B + C$$

Let $u = \sum_{i=1}^{\infty} z_i\varphi_i$ be the Fourier series of u with respect to the basis (φ_i) in $L_2(\Omega)$. Note that

$$\|u\|_0^2 = \sum_{i=1}^{\infty} z_i^2,\ \|u\|_1^2 = \sum_{i=1}^{\infty}\lambda_i z_i^2$$

(see (8)). Multiplying (10) by $\tilde{u}$ and integrating over Ω, we get

(11) $$\sum_{i=1}^{\infty}|\lambda_i - (\lambda_{k+1}+\lambda_k)/2|z_i^2 \leq \int_\Omega (A+B+C)\tilde{u}\,dx$$

Using the Cauchy's inequality, $\|u\|_0 = \|\tilde{u}\|_0$ and

$$|A|\leq((\lambda_{k+1}-\lambda_k)/2-\varepsilon)|u|$$
$$|B|\leq c|\nabla u|_2$$

$$ab \leq a^2/4s + sb^2, \quad s > 0$$

with a suitable choice of a and b, we arrive to the following inequalities:

$$\int_\Omega (A+B)\tilde{u}\, dx \leq ((\lambda_{k+1} - \lambda_k)/2 - \varepsilon + s) \sum_{i=1}^{\infty} z_i^2 + (c^2/4s) \sum_{i=1}^{\infty} \lambda_i z_i^2$$

$$\int_\Omega C\tilde{u}\, dx \leq (s_1+s_2) \sum_{i=1}^{\infty} z_i^2 + (\|g(\cdot,0,0)\|_0^2/4s_1 + \|p\|_0^2/4s_2)$$

Note that $t = s_1 + s_2$ can be made as small as we want. Let

$$d = \max\,(1/4s_1, 1/4s_2)$$

Now from (11) we obtain

$$\sum_{i=1}^{\infty} a_i \lambda_i z_i^2 \leq d(\|g(\cdot,0,0)\|_0^2 + \|p\|_0^2) \tag{12}$$

where

$$a_i = (|1-(\lambda_{k+1}+\lambda_k)/2| - (\lambda_{k+1}-\lambda_k) + (\varepsilon - s) - t)/\lambda_i - c^2/4s$$

As a consequence of

$$i \leq k \Rightarrow a_i \geq (\varepsilon - s)/\lambda_k - c^2/4s - t/\lambda_1$$

$$i > k \Rightarrow a_i \geq (\varepsilon - s)/\lambda_{k+1} - c^2/4s - t/\lambda_1$$

we see that by putting $s = \varepsilon/2$, using (5) and choosing t (i.e. s_1 and s_2) small enough, it is possible to achieve that

$$m = \min\{a_i : i \in N\} > 0$$

So (12) implies

$$m\|u\|_1^2 \leq d(\|g(\cdot,0,0)\|_0^2 + \|p\|_0^2)$$

and (6) follows.

QED

2. <u>The optimal control problem</u>. Here we would like to formulate the main result of this paper, concerning the existence of optimal controls. It extends our previous result in [4]. We assume that the domain Ω is sufficiently regular, in order to be able to apply L_2- regularity theory of the Dirichlet problem.

Theorem 2. Let Ω be a bounded, regular domain, $g: \Omega \times R \times R^n \longrightarrow R$ as in Theorem 1 and U a bounded, closed, convex subset in $L_2(\Omega)$. Moreover, let the function

$$J_1 : U \longrightarrow R_+$$

be weakly lower semicontinuous and

$$J_2 : H_0^1(\Omega) \longrightarrow R_+$$

lower semicontinuous. Then the optimal control problem (1), (2) has at least one solution.

Remark. It is easy to see from the proof which follows that if J is coercive, i.e. $J(p) \longrightarrow \infty$, $\|p\|_0 \longrightarrow \infty$, then we can drop the boundedness condition on U.

Also note that the condition on J_2 in the theorem is weaker than to impose a lower semicontinuity on the functional $J_2 : L_2(\Omega) \rightarrow \longrightarrow R_+$.

Proof of the theorem 2. First, as Ω is regular, we can use a well known regularity result from [1] to obtain that (1) implies $u \in H^2(\Omega)$ and

$$\|u\|_2 \leq C_3(\|g(\cdot, u, \nabla u)\|_0 + \|p\|_0) \tag{13}$$

where $\| \; \|_2$ is the usual norm in $H^2(\Omega)$, and C_3 independent of u and p. From the conditions (3),(4) it follows easily that there exist constants C_4, C_5 and C_6 such that

$$\|g(\cdot, u, \nabla u)\|_0 \leq C_4 + C_5\|u\|_0 + C_6\|u\|_1 \leq$$
$$\leq C_4 + (C_5/\lambda_1^{1/2} + C_6)\|u\|_1$$

where in the last inequality we used the Poincaré inequality. Combining this with (13), and using the lemma, we obtain that there exist constants C_7, C_8 such that

$$\|u\|_2 \leq C_7 + C_8\|p\|_0$$

The rest of the proof is the same as the proof of our main result in [4] (see the steps 2 and 3).

QED

Let us add finally that in the case when g is independent of ∇u, we can drop the regularity condition on Ω . This corresponds to the following distributed parameter control system

$$(14) \qquad \begin{aligned} -\Delta u &= f(x,u) + p(x) \text{ in } \Omega \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{aligned}$$

Theorem 3. Let Ω be a bounded domain, $f: \Omega \times R \longrightarrow R$ a Carathéodory function such that $f(\cdot,0) \in L_2(\Omega)$ and the nonresonance condition has the form

$$\lambda_k + \varepsilon \leq (f(x,u_1)-f(x,u_2))/(u_1-u_2) \leq \lambda_{k+1} - \varepsilon \ , \quad \varepsilon > 0$$

where $\varepsilon > 0$ and λ_k, λ_{k+1} are two distinct, consecutive eigenvalues of $-\Delta$. If U is a bounded, closed, convex subset in $L_2(\Omega)$, $J_1: U \longrightarrow R_+$ weakly lower semicontinuous and $J_2: L_2(\Omega) \longrightarrow R_+$ lower semicontinuous, then the optimal control problem (14),(2) possesses at least one solution.

The idea of the proof consists essentially in proving the complete continuity of the nonlinear operator

$$L_2(\Omega) \ni p \longmapsto u \in L_2(\Omega)$$

defined by (14). The detailed proof of this fact can be found in the final remark in [4].

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Elektrotehnički fakultet, Zavod za primjenjenu matematiku, Unska 3, 41000 Zagreb, Yugoslavia

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