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OSCILLATION THEOREM FOR A SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION WITH DAMPING TERM

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Abstract: A new oscillation criterion for the equation

$$(a(t)x'(t))' + p(t)x'(t) + c(t)x(t) + q(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0, \quad 0 < \gamma < 1,$$

is established.

Key words: Differential equation, oscillatory solutions, nonoscillatory, sublinear.

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Consider the second order nonlinear differential equation

$$(1) \quad (a(t)x'(t))' + p(t)x'(t) + c(t)x(t) + q(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0, \quad 0 < \gamma < 1,$$

where a, p, c, $q:[t_0,\infty) \longrightarrow R = (-\infty,\infty)$ are continuous and $a(t) > 0$

We shall restrict our attention to solutions of equation (1) which exist on some ray $[t_0,\infty)$. A solution of equation (1) is called oscillatory if it has no largest zero; otherwise it is called nonoscillatory. An equation is said to be oscillatory if every solution is oscillatory.

Recently Kwong and Wong [3] considered the sublinear ordinary differential equation

$$(*) \quad x''(t) + q(t)|x(t)|^{\gamma} \operatorname{sgn} x(t) = 0, \quad 0 < \gamma < 1,$$

and proved the following theorem:

Theorem A. If there exists a positive function ρ such that

$\varphi \geq 0$ and $\ddot{\varphi} \leq 0$ that satisfies

$$(**) \quad \lim_{t\to\infty} \int_0^t \varphi^{\gamma}(s)q(s)ds = \infty$$

then Eq. (1) is oscillatory.

Theorem A extended and unified Belohorec Theorem [1].

The purpose of this paper is to proceed further in this direction and to present a new oscillation theorem for Eq. (1) which extends Theorem A of Kwong and Wong.

Our main result is the following theorem:

Theorem 1. Let $c(t) \geq \frac{p^2(t)}{4\gamma a(t)}$ and φ be a positive twice differentiable function on the interval $[t_0,\infty)$ such that:

(2) $p(t)\varphi^{\cdot}(t) \geq 0$, and $(a(t)\varphi^{\cdot}(t))^{\cdot} \leq 0$ for $t \geq t_0$;

and

$$(3) \quad \limsup_{t\to\infty} \frac{1}{\int_{t_0}^t \frac{1}{a(s)}ds} \int_{t_0}^t \frac{1}{a(s)} \int_{t_0}^s \varphi^{\gamma}(\tau)q(\tau)d\tau ds = \infty;$$

then Eq. (1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \geq t_0$. For $t \geq t_0$, define

$$(4) \quad w(t) = \left(\frac{x(t)}{\varphi(t)}\right)^{\gamma},$$

which is again positive. Let $\beta = \frac{1}{\gamma} > 1$, then

$$x(t) = \varphi(t)w^{\beta}(t).$$

Differentiating (4), we obtain

$$\frac{1}{w(t)}(a(t)(\varphi(t)w^{\beta}(t))^{\cdot})^{\cdot} = \frac{\beta}{\beta-1}(a(t)(\varphi(t)w^{\beta-1}(t))^{\cdot})^{\cdot} +$$

$$+ \frac{1}{1-\beta}(a(t)\varphi^{\cdot}(t))^{\cdot}w^{\beta-1}(t) + \beta a(t)\varphi(t)w^{\beta-3}(t)w^{\cdot 2}(t).$$

From equation (1) and (4) we have

$$\frac{(a(t)x^{\cdot}(t))^{\cdot}}{w(t)} = \frac{(a(t)(\varphi(t)w^{\beta}(t))^{\cdot})^{\cdot}}{w(t)} = -\frac{p(t)x^{\cdot}(t)}{w(t)} - \frac{c(t)x(t)}{w(t)} -$$

$$- \frac{q(t)x^{\gamma}(t)}{w(t)} = - \frac{p(t)}{w(t)}\left[\varphi^{\cdot}(t)w^{\beta}(t) + \beta\varphi(t)w^{\beta-1}(t)w^{\cdot}(t)\right] -$$

$$- \frac{c(t)}{w(t)}\varphi(t)w^{\beta}(t) - \varphi^{\gamma}(t)q(t) = - \varphi^{\gamma}(t)q(t) - p(t)\varphi^{\cdot}(t)w^{\beta-1}(t) -$$

$$- c(t)\varphi(t)w^{\beta-1}(t) - \beta p(t)\varphi(t)w^{\beta-2}(t)w^{\cdot}(t).$$

Thus,

$$\frac{\beta}{\beta-1}\,(a(t)(\varphi(t)w^{\beta-1}(t))^{\cdot})^{\cdot} + \frac{1}{1-\beta}\,(a(t)\varphi^{\cdot}(t))^{\cdot}w^{\beta-1}(t) +$$

$$+ \beta a(t)\varphi(t)w^{\beta-3}(t)w^{\cdot 2}(t) + p(t)\varphi^{\cdot}(t)w^{\beta-1}(t) + \beta p(t)\varphi(t)w^{\beta-2}(t)w^{\cdot}(t) +$$

$$+ c(t)\varphi(t)w^{\beta-1}(t) = - \varphi^{\gamma}(t)q(t).$$

Using (2) we get

$$\frac{\beta}{\beta-1}\,(a(t)(\varphi(t)w^{\beta-1}(t))^{\cdot})^{\cdot} + \beta a(t)\varphi(t)w^{\beta-3}(t)w^{\cdot 2}(t) +$$

$$+ \beta p(t)\varphi(t)w^{\beta-2}(t)w^{\cdot}(t) + c(t)\varphi(t)w^{\beta-1}(t) \leq - \varphi^{\gamma}(t)q(t).$$

Now

$$\frac{\beta}{\beta-1}\,(a(t)(\varphi(t)w^{\beta-1}(t))^{\cdot})^{\cdot} + c(t)\varphi(t)w^{\beta-1}(t) - \frac{\beta p^2(t)\varphi(t)w^{\beta-1}(t)}{4a(t)}$$

$$+ \left[(\beta a(t)\varphi(t)w^{\beta-3}(t))^{1/2}w^{\cdot}(t) + \frac{\beta p(t)\varphi(t)w^{\beta-2}(t)}{2(\beta a(t)\varphi(t)w^{\beta-3})^{1/2}}\right]^2 \leq$$

$$\leq - \varphi^{\gamma}(t)q(t).$$

Using the fact that $c(t) \geq \frac{p^2(t)}{4\gamma a(t)}$, we obtain

$$(5)\quad (a(t)(\varphi(t)w^{\beta-1}(t))^{\cdot})^{\cdot} \leq - \frac{\beta-1}{\beta}\,\varphi^{\gamma}(t)q(t).$$

Integrating (5) twice from t_0 to t we get

$$(6)\quad (\varphi(t)w^{\beta-1}(t)) \leq C_1 + C_0 \int_{t_c}^{t} \frac{1}{a(s)}\,ds -$$

$$- \frac{\beta-1}{\beta}\int_{t_c}^{t} \frac{1}{a(s)} \int_{t_c}^{s} \varphi^{\gamma}(\tau)q(\tau)d\tau\,ds,$$

where C_0 and C_1 are appropriate integration constants. Obviously $\int_{t_c}^{\infty} \frac{1}{a(s)}\,ds$ exists in $(0,\infty) \cup \{\infty\}$ and consequently $\lim_{t\to\infty} \left(\int_{t_c}^{t} \frac{1}{a(s)}\,ds\right)^{-1} = L$ for some $L \in [0,\infty)$.

So we derive

$$\limsup_{t\to\infty} \frac{1}{\int_{t_0}^{t} \frac{1}{a(s)}\,ds} \int_{t_0}^{t} \frac{1}{a(s)} \int_{t_0}^{s} \rho^{\gamma}(\tau)q(\tau)d\tau\, ds \le$$

$$\le \frac{\beta}{\beta-1} [C_0 + C_1 L - \liminf_{t\to\infty} \frac{1}{\int_{t_0}^{t} \frac{1}{a(s)}\,ds} (\rho(t)w^{\beta-1}(t)) < \infty ,$$

which contradicts (3). This completes the proof.

In Theorem 1 no assumption is made on $\int_{t_0}^{\infty} \frac{1}{a(s)}\,ds$. Therefore, its conclusion holds in both cases where (I) or (II) below is satisfied:

(I) $\int_{t_0}^{\infty} \frac{1}{a(s)}\,ds = \infty$,

(II) $\int_{t_0}^{\infty} \frac{1}{a(s)}\,ds < \infty$.

In the second case, i.e. when (II) is satisfied, the condition (3) is clearly equivalent to the following one:

(7) $\limsup_{t\to\omega} \int_{t_0}^{t} \frac{1}{a(s)} \int_{t_0}^{s} \rho^{\gamma}(\tau)q(\tau)d\tau\, ds = \infty$.

Remarks: 1. Our Theorem 1 improves and includes Theorem 1 of Kwong and Wong [3] (take $a(t) = 1$, $c(t) = p(t) = 0$). Also, it includes the sufficiency part of Belohorec Theorem in [1], for $a(t) = 1$, $c(t) = p(t) = 0$ and $\rho(t) = t$.

2. Theorem 1 can be extended to more general nonlinear equations of the form

(8) $(a(t)x'(t))' + p(t)x'(t) + c(t)x(t) + f(t,x(t)) = 0$,

where a, p, c are as above, $f: [t_0,\infty)\times R \to R$ is continuous such that $xf(t,x) > 0$ for $x \neq 0$, and

$$\frac{f(t,x)}{|x|^{\gamma}} \ge q(t), \quad 0 < \gamma < 1,$$

where $q: [t_0,\infty) \to R$ is a continuous function.

3. It is clear that the oscillatory behavior of Eq. (1) or

(8) (with $a(t) = 1$ and $c(t) \geq \frac{p^2(t)}{4\gamma}$) and equation ($*$) are exactly the same.

For illustration we consider the following example:

Example 1. Consider the differential equation

$$(9)\quad (t^{\alpha_1}x^{\cdot})^{\cdot} + t^{\alpha_2}x^{\cdot} + t^{\alpha_3}x + (t^{\lambda}\sin t)|x|^{\gamma}\operatorname{sgn} x = 0,$$

$$0 < \gamma < 1,\ t \geq 1,$$

where $\alpha_1, \alpha_2, \alpha_3$ and λ are constants. Let $\rho(t) = t^{\theta}$, where θ is any nonnegative constant such that

$$\alpha_1 + \theta - 1 \leq 0,\quad \alpha_1 + \alpha_3 = 2\alpha_2,\ 4\gamma \geq 1.$$

If

$$\gamma\theta + \lambda > 1,$$

then all solutions of equation (9) are oscillatory. One can easily check that none of the known criteria [1-6] is applicable to Eq. (9).

References

[1] S. BELOHOREC: Oscillatory solutions of certain nonlinear differential equations of second order, Mat.Fyz.Časopis Sloven.Akad.Vied 11(1961),250-255.(In Czech.)

[2] I.V. KAMENEV: Certain specifically nonlinear oscillation theorems, Mat.Zametki 10(1971),129-134. (In Russian.)

[3] M.K. KWONG and J.W.W. WONG: On an oscillation theorem of Belohorec, Siam J.Math.Anal.14(1983), 474-476.

[4] A. WINTNER: A criterion of oscillatory stability, Quart.Appl. Math.7(1949), 115-117.

[5] J.S.W. WONG: A second order nonlinear oscillation theorem, Proc.Amer.Math.Soc.40(1973), 487-491.

[6] J.W.W. WONG: On the generalized Emden-Fowler equations, Siam Rev.17(1975), 339-360.

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