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# ORDINAL INVARIANTS AND EPIMORPHISMS IN SOME CATEGORIES OF WEAK HAUSDORFF SPACES

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Abstract. In some categories $\underline{A}$ of weak Hausdorff spaces the epimorphisms are characterized as maps with dense image with respect to the $\underline{A}$-closure. In many cases the $\underline{A}$-closure is represented as the idempotent hull of another closure operator and the (ordinal) number of iterations is related to the tightness of the underlying space and the co-well-poweredness of the category $\underline{A}$.

Key words: Weak Hausdorff spaces, epimorphisms, ordinal invariants, $\underline{A}$-closure.

Classification: 54B30, 54D10, 18B30

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0. Introduction. Various versions of weak Hausdorffness are spread in the literature. An extensive bibliography and many interesting results can be found in the survey of Hoffmann ([13]). The present paper is a continuation of the study of epimorphisms and co-well-poweredness of epireflective subcategories of the category Top of topological spaces and continuous maps (see [4],[5], [6],[11] and [24]). For this purpose we focus on two types of weak Hausdorffness studied in [13].

Let $\underline{P}$ be a class of topological spaces and let $\underline{Haus}(\underline{P})$ denote the category of topological spaces X such that for every $P \in \underline{P}$ and for every continuous map $f:P \longrightarrow X$, f(P) is a Hausdorff subspace of X (in the notation of [13] this is $\underline{P}_2$). For $\underline{P} = \{\mathbb{N}_\infty\}$, where $\mathbb{N}_\infty$ denotes the one-point Alexandrov's compactification of the discrete space of the natural numbers $\mathbb{N}$, $\underline{Haus}(\underline{P})$ coincides with

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the category SUS of topological spaces in which every convergent sequence has precisely one accumulation point, namely its limit point. Tozzi ([24]) showed that SUS is co-well-powered describing explicitly the epimorphisms in SUS. Let Comp denote the class of all compact spaces and let $P_m$ denote the class of all topological spaces of cardinality less or equal to a given cardinality m. Hušek and the second author ([11]) studied the epimorphisms in Haus($P_m$) and Haus(Comp) showing that the latter category is not co-well powered. In [11] and [24] the epimorphisms were described by means of a closure operator which "measures" epimorphisms, introduced by Salbany ([21]) and examined in various subcategories of Top by the authors ([4],[5],[6],[10]). This operator was represented as the idempotent hull of more explicit closures - the sequential closure in [24] and the compactly determined closure in [11] (see [1] for these closures).

The aim of the present paper is to give a unified approach to all these cases. In Section 1 the epimorphisms in Haus(P) are characterized. As a corollary it is shown that the category Haus(HComp), where HComp is the class of all compact Hausdorff spaces, is not co-well-powered (Theorem 1.14). It is established also that the inclusion Haus ⟶ Haus(HComp), where Haus is the class of all Hausdorff spaces, does not preserve epimorphisms. The codomains for which any epimorphism in Haus(P) is surjective are also characterized (Proposition 1.11).

In Section 2 an ordinal invariant co(X) is introduced which, for P = $\{N_\infty\}$, gives the sequential order and for P = Comp the k-order both introduced by Arhangel´skii and Franklin ([1]). We show that the cardinality of co(X) is always less or equal to $t(X)^+$, where t(X) is the tightness of the space X in the sense of Arhangel´skii (Theorem 2.2). It is shown that the category Haus(P) is co-well-powered whenever the ordinal co(X) is bounded for X $\in$ $\in$ Haus(P)

In Section 3 another type of weak Hausdorffness is considered For a class P of topological spaces and $(X,\tau)\in$ Top, we denote by $(X,\tau^c)$ the coreflection of $(X,\tau)$ into the bicoreflective hull c(P) of P in Top, i.e. $M\subset X$ is closed in $\tau^c$ iff, for every $P\in$ P and for every continuous map $f:P\to X$, $f^{-1}(M)$ is closed in P. The space $(X,\tau)$ is called c-space if $\tau=\tau^c$ (for P = Comp these are the well known compactly generated spaces (k-spaces); for P = $\{N_\infty\}$ the c-spaces are the sequential spaces). Denote by $P_3$ the category of all topological spaces $(X,\tau)$ such that the diagonal $\Delta_X$ is closed in $(X\times X,(\tau\times\tau)^c)$. This weak version of the Hausdorff separation axiom was introduced by McCord ([20]) for P = HComp and by Lawson and Madison ([19]) for P = Comp. Further information can be found in [13]. In Theorem 3.4 we describe the $P_3$-closed sets by means of the c(P)-coreflection (for the definition of A-closed set see Section 1). In particular we recover the characterization from [24] of the epimorphisms in the category US of topological spaces in which every convergent sequence has the unique limit point (US = $(\{N_\infty\})_3$, see [13]). We prove that, under a mild condition on P, Haus(P)$\subset P_3$ and finish the description of an example given in [13], 2.9.9.

It is shown in [7] that the inclusion Haus(HComp) $\subset$ $(\text{HComp})_3$ is proper and that $(\text{HComp})_3 \not\subset$ SUS. This answers some related questions posed by Hoffmann in [13], 4.2.

1. Epimorphisms in Haus(P). It was proved in [13], 1.9, that the category Haus(P) and $P_3$ are quotient-reflective in Top (i.e they are closed under the formation of products, subspaces and refinements). Here we recall some necessary definitions and

results from [4]. If $f,g:X \longrightarrow Y$ are continuous maps $Eq(f,g)$ will denote the equalizer in Top of f and g, i.e. $Eq(f,g) = \{x \in X: :f(x) = g(x)\}$.

1.1. Definitions. Let $\underline{A}$ be an epireflective subcategory of Top.

(1) A subset F of a space X is said to be A-closed in X iff there exist $A \in \underline{A}$ and continuous maps $f,g:X \longrightarrow A$ such that $F = = Eq(f,g)$.

(2) The A-closure of a subset M of X, denoted by $[M]_{\underline{A}}$ is the intersection of all the $\underline{A}$-closed subsets of X containing M.

(3) A subset D of X is said to be A-dense iff $[D]_{\underline{A}} = X$.

The $\underline{A}$-closure is an extensive, monotone and idempotent operator, in general not additive. For a topological space $(X,\tau)$ we denote by $\tau_{\underline{A}}$ the coarsest topology on X which contains all the $\underline{A}$-closed sets as closed subsets.

To define $\underline{A}$-closure it is not necessary to have subcategories $\underline{A}$ of Top. For categories of algebras Isbell ([15]) introduced the $\underline{A}$-closure in the same way (it is called dominion there). It is clear then that a morphism $f:X \longrightarrow Y$ is an epimorphism in $\underline{A}$ iff $f(X)$ is $\underline{A}$-dense in Y ([10],[15]). The difficulties come when one has to calculate explicitly the $\underline{A}$-closure (see the zig-zags in [15], or the various cases of epireflective subcategories of Top in [4],[5],[6],[11] and [24]. For an exhaustive bibliography about the problem of the epimorphisms, see [17].

For $X \in$ Top and $M \subset X$ we denote by $X \sqcup_M X$ the adjunction space determined by the inclusion $M \subset X$, i.e., $X \sqcup_M X$ is the quotient of $X \sqcup X = X \times \{0,1\}$ obtained by identifying each $(m,0)$, $m \in M$, with $(m,1)$. $q:X \sqcup X \longrightarrow X \sqcup_M X$ denotes the natural quotient map. The maps $k_i:X \longrightarrow X \sqcup_M X$, $p:X \sqcup_M X \longrightarrow X$ are defined by $k_i(x)=q(x,i)$ and $p(x,i)=x$,

$i = 0,1$, respectively. The adjunction space plays an important rôle in the computation of the $\underline{A}$-closure. We stress on the fact that each retract of a space $X \in \underline{A}$ is $\underline{A}$-closed (a retract is the equalizer of the retraction and the identity map of X). It is easily seen also that $\underline{A}$-closed sets go into $\underline{A}$-closed sets under homeomorphisms of X.

1.2, Proposition. Let $\underline{A}$ be a quotient-reflective subcategory of Top. Then for every $X \in \underline{A}$ and $M \subset X$ the following conditions are equivalent:

(i) $M = [M]_{\underline{A}}$,

(ii) $X \sqcup_M X \in \underline{A}$,

(iii) $q(X \times \{1\}) = k_1(X)$ is $\underline{A}$-closed in $X \sqcup_M X$.

Proof: The equivalence (i)$\Longleftrightarrow$(ii) is proved in [4]. Since $k_1(X)$ is a retract of $X \sqcup_M X$ (ii)$\Longleftrightarrow$(iii). To prove (iii)$\Longrightarrow$(i) let $s: X \sqcup_M X \longrightarrow X \sqcup_M X$ be the symmetry; then s is a homeomorphism, so $s(k_1(X)) = k_0(X)$ is $\underline{A}$-closed. Thus $q(M \sqcup M) = k_1(X) \cap k_0(X)$ is $\underline{A}$-closed and $M \sqcup M$ is $\underline{A}$-closed in $X \sqcup X$. Now $M \times \{1\} = q^{-1}(M) \cap$ $\cap (X \times \{1\})$ is $\underline{A}$-closed in $X \times \{1\}$ since $X \times \{1\}$ is $\underline{A}$-closed in $X \sqcup X$ being clopen.

1.3. Lemma. Let $\underline{A}$ be a quotient-reflective subcategory of Top and let $X \in \underline{A}$ and $M \subset X$. Then the following hold:

(1) $q([M] \times \{0,1\}) = [q(M \times \{0,1\})]$,

(2) $[k_i(X)] = p^{-1}([M]) \cup k_i(X)$, $i=0,1$; in particular $[k_0(X)] \cap [k_1(X)] = p^{-1}([M])$,

(3) $[M] = p([k_0(X)] \cap [k_1(X)]) = p(k_0(X) \cap [k_1(X)]) =$ $= p([k_0(X)] \cap k_1(X))$.

Proof: (1) Consider the adjunction space $X \sqcup_{[M]} (X)$ and denote by $\tilde{p}$, $\tilde{q}$, $\tilde{k}_0$ and $\tilde{k}_1$ the related maps and denote by $t:X \sqcup_M X \longrightarrow$ $\longrightarrow X \sqcup_{[M]} X$ the quotient map. Then $\tilde{k}_0([M]) = \tilde{k}_1([M])$ and $[M] \times \{0,1\} = q^{-1}t^{-1}(\tilde{k}_0([M]))$, so $q([M] \times \{0,1\}) = t^{-1}(\tilde{k}_0([M]))$, hence it is $\underline{A}$-closed. (2) and (3) now follow easily from $t \circ q = \tilde{q}$, 1.2(iii) and the fact that $X \sqcup_M X \xrightarrow{t} X \sqcup_{[M]} X$ is the $\underline{A}$-reflection of $X \sqcup_M X$.

The following condition for the class $\underline{P}$ was considered in [13]:

($*$) If $P \in \underline{P}$, $P \neq \emptyset$ then $P \sqcup Q \in \underline{P}$ for some non-empty space Q.

Roughly spoken, ($*$) ensures that we can add a finite number of points to images of spaces from $\underline{P}$, i.e., if $P \in \underline{P}$ and $f:P \longrightarrow X$ is a continuous map, then for every finite subset F of X there exist $P_1 \in \underline{P}$ and a continuous map $f_1:P_1 \longrightarrow X$ such that $f_1(P_1) = f(P) \cup F$.

1.4. Lemma. If $\underline{P}$ satisfies ($*$) and $\underline{P} \neq \emptyset$, $\underline{P} \neq \{\emptyset\}$, then $\underline{\text{Haus}} \subset \underline{\text{Haus}}(\underline{P}) \subset \underline{\text{Top}}_1$.

Proof: By ($*$) every finite subspace of $X \in \underline{\text{Haus}}(\underline{P})$ is a continuous image of a space from $\underline{P}$; this yields the second inclusion, the first is trivial.

Since $\underline{\text{Haus}}(\underline{P})$ is quotient reflective it is easy to see that $\underline{\text{Haus}}(\underline{P}) = \underline{\text{Top}}$ iff $\underline{P} = \{$spaces with at most one point$\}$, so in all other cases $\underline{\text{Haus}}(\underline{P}) \subset \underline{\text{Top}}_0$. However, it may happen $\underline{\text{Haus}}(\underline{P}) \not\subset \underline{\text{Top}}_1$, for example if $\underline{P}$ is the class of all indiscrete spaces ([13], 2.9.4). Finally, it is easy to see that $\underline{\text{Haus}}(\underline{P}) \subset \underline{\text{Top}}_1$ (this is equivalent to $\underline{\text{Haus}}(\underline{P}) \neq \underline{\text{Top}}_0$) iff there exists a non-indiscrete space $P \in \underline{P}$; in fact in this case there would exist a continuous surjection of P onto the Sierpiński two-point space. So to prove 1.4 we do not need in fact ($*$); it is sufficient to have non-indiscrete spaces in $\underline{P}$.

The next two lemmas repeat in this more general set the arguments from [11], Prop. 3.3 and Lemma 3.4 and [24], Theorem 2.13.

1.5. Lemma. Let $\underline{P}$ be an arbitrary class of topological spaces, then for every $(X,\tau) \in \underline{\mathrm{Haus}}(\underline{P})$, $\tau_{\underline{\mathrm{Haus}}(\underline{P})} \geq \tau$ .

Proof: Let $X \in \underline{\mathrm{Haus}}(\underline{P})$ and let F be a closed set in X. To show that F is $\underline{\mathrm{Haus}}(\underline{P})$-closed it is enough to show that $X \sqcup_F X \in \underline{\mathrm{Haus}}(\underline{P})$ according to 1.2. Let $P \in \underline{P}$ and let $f:P \longrightarrow X \sqcup_F X$ be a continuous map. Take two distinct points x and y in f(P). If $p(x) \neq p(y)$ in X, then consider the composition $p \circ f:P \longrightarrow X$. By $X \in \underline{\mathrm{Haus}}(\underline{P})$ there exist disjoint open neighborhoods of p(x) and p(y), thus their preimages will be disjoint open neighborhoods of x and y. In the case p(x) = p(y) we can assume without loss of generality that there exists $z \in X \setminus F$ such that x = q((z,0)) and y = q((z,1)). Now, for $U = X \setminus F$, $q(U \times \{0\})$ and $q(U \times \{1\})$ are disjoint open neighborhoods of x and y in $X \sqcup_F X$.

It follows from 1.5 that for any class $\underline{P}$ the functor $F_{\underline{\mathrm{Haus}}(\underline{P})}$: $:\underline{\mathrm{Top}} \longrightarrow \underline{\mathrm{Top}}$ defined by $F_{\underline{\mathrm{Haus}}(\underline{P})}((X,\tau)) = (X,\tau_{\underline{\mathrm{Haus}}(\underline{P})})$ is a pre-monocoreflection in the sense of [23].

1.6. Lemma. Let $\underline{P}$ be a class of spaces satisfying ($*$) and let $X \in \underline{\mathrm{Haus}}(\underline{P})$. Then for every $P \in \underline{P}$ and for every continuous map $f:P \longrightarrow X$, $\overline{f(P)}$ is a Hausdorff subspace of X and for every $M \subset \overline{f(P)}$, $\overline{M} = [M]_{\underline{\mathrm{Haus}}(\underline{P})}$.

Proof: Let x and y be two distinct points of $\overline{f(P)}$. By ($*$) $f(P) \cup \{x,y\}$ is the continuous image of a space in $\underline{P}$, so it is a Hausdorff subspace of X. Let U and V be open neighborhoods of x and y such that $U \cap V \cap f(P) = \emptyset$. Then clearly $U \cap V \cap \overline{f(P)} = \emptyset$ and x and y are separated in $\overline{f(P)}$. Let $M \subset \overline{f(P)}$. By the previous lemma $[M]_{\underline{\mathrm{Haus}}(\underline{P})} \subset \overline{M}$, so it suffices to show that $\overline{M} \subset [M]_{\underline{\mathrm{Haus}}(\underline{P})}$. Take an element $x \in M$ and two continuous maps $h,g:X \longrightarrow Y$ with $Y \in \underline{\mathrm{Haus}}(\underline{P})$

and $M \subset Eq(h,g)$. Then $h(M) = g(M)$ and $x \in \overline{M}$ yields $h(x) \in \overline{h(M)}$ and $g(x) \in \overline{g(M)}$. On the other hand $M \subset \overline{f(P)}$ implies $h(M) \subset \overline{h(f(P))}$ and $g(M) \in \overline{g(f(P))}$, so $\overline{h(M)} = \overline{g(M)} \subset \overline{h(f(P))}$ and $\overline{h(M)} \subset \overline{g(f(P))}$. By the first part of the lemma the left-hand side is a Hausdorff subspace of Y, so if $h(x) \neq g(x)$, then they can be separated in $\overline{h(M)}$. By virtue of $x \in \overline{M}$ and $M \subset Eq(h,g)$ this does not occur. So $h(x) = g(x)$, therefore $x \in Eq(g,h)$. This proves $x \in [M]_{\underline{Haus}(\underline{P})}$.

Next we define a closure operator $cl_{\underline{P}}$ associated with $\underline{P}$ following the idea from [11].

1.7. <u>Definition</u>. For $X \in \underline{Top}$ and $M \subset X$ define $cl_{\underline{P}}(M) = \bigcup\{M \cap \overline{f(P)} : P \in \underline{P},\ f: P \longrightarrow X\}$.

In the following lemma we give some properties of this closure.

1.8. <u>Lemma</u>. (1) $cl_{\underline{P}}$ is an expansive, monotone and additive closure operator satisfying $cl_{\underline{P}}(M) \subset \overline{M}$ for every $X \in \underline{Top}$ and $M \subset X$.

(2) For $X \in \underline{Top}$ and $M \subset X$, $x \in cl_{\underline{P}}(M)$ iff there exists $P \in \underline{P}$ and a continuous map $f: P \longrightarrow X$ such that for every open neighborhood U of x and for every open subset V of U satisfying $V \cap M = U \cap M$, $V \cap f(P) \neq \emptyset$ holds.

(3) If $\underline{P}$ satisfies $(*)$ then for every $X \in \underline{Haus}(\underline{P})$ and $M \subset X$, $cl_{\underline{P}}(M) \subset [M]_{\underline{Haus}(\underline{P})}$.

<u>Proof</u>: (1) is trivial. To prove (2) observe that $x \in cl_{\underline{P}}(M)$ iff there exist $P \in \underline{P}$ and $f: P \longrightarrow X$ such that $x \in \overline{M \cap \overline{f(P)}}$, i.e., for every open neighborhood U of x, $U \cap M \cap \overline{f(P)}) \neq \emptyset$. Suppose that the condition in (2) does not hold; then there exists an open subset V of U such that $V \cap M = U \cap M$ and $V \cap f(P) = \emptyset$. Then clearly $V \cap \overline{f(P)} = \emptyset$ and $\emptyset = V \cap \overline{f(P)} \supset V \cap M \cap \overline{f(P)} = U \cap M \cap \overline{f(P)} \neq \emptyset$ - a contradiction. Now assume that $x \notin cl_{\underline{P}}(M)$; then there exist $P \in \underline{P}$, $f: P \longrightarrow X$ and an open neighborhood U of x such that $U \cap M \cap \overline{f(P)} = \emptyset$.

For every $z \in U \cap M$ choose an open neighborhood $V_z$ of $z$ contained in $U$ such that $V_z \cap f(P) = \emptyset$; then $V = \bigcup\{V_z : z \in U \setminus M\}$ is an open subset of $U$ satisfying $V \cap M = U \cap M$ and $V \cap f(P) = \emptyset$. This proves (2).

(3) Let $X \in$ Haus(P) and $M \subset X$; if $P \in$ P and $f:P \to X$ is a continuous map, then $M' = M \cap \overline{f(P)} \subset \overline{f(P)}$ so by Lemma 1.6 $\overline{M'} = c[M']_{\mathrm{Haus}(P)} \subset [M]_{\mathrm{Haus}(P)}$. This proves (3).

1.9. Theorem. Let P satisfy (*) and $X \in$ Haus(P). Then, for any $M \subset X$, the following conditions are equivalent:

(i) $X \sqcup_M X \in$ Haus(P);

(ii) $M$ is Haus(P)-closed;

(iii) $M = cl_{P}(M)$.

Proof: The equivalence of (i) and (ii) follows from 1.2, while the implication (ii) $\Rightarrow$ (iii) follows from 1.8 (3). To prove (iii) $\Rightarrow$ (i) take a space $P \in$ P and a continuous map $f:P \to X \sqcup_M X$. Let $x$ and $y$ be two distinct points in $f(P)$. If $p(x) \neq p(y)$ then $x$ and $y$ can be separated as in the proof of 1.5, i.e., projecting on $X$ by $p$. Assume that $p(x) = p(y)$; then $p(x) \notin M$ since $x \neq y$ in $X \sqcup_M X$. By (iii) $x \notin cl_{P}(M)$, thus by 1.8 (2) there exist an open neighborhood $U$ of $p(x)$ and an open subset $V$ of $U$ with $V \cap M = U \cap M$ and $V \cap p(f(P)) = \emptyset$. Then $W = q(V \times \{0\} \sqcup U \times \{1\})$ and $W' = q(U \times \{0\} \sqcup V \times \{1\})$ are open sets in $X \sqcup_M X$ which contain $x$ and $y$ since $p(x) = p(y) \in U = p(W) = p(W')$ and $p(x) \notin M$; on the other hand $W \cap W' \cap f(P) = \emptyset$ since $q^{-1}(W \cap W' \cap f(P)) = (U \sqcup V) \cap (V \sqcup U) \cap (p(f(P)) \sqcup p(f(P))) = (V \sqcup V) \cap (p(f(P)) \sqcup p(f(P))) = \emptyset$.

1.10. Corollary. Let P be a class satisfying (*). Then for spaces in Haus(P) the Haus(P)-closure is the idempotent hull of the closure $cl_{P}$. In particular for every $(X,\tau) \in$ Haus(P) the Haus(P)-closure is a Kuratowski operator. Moreover, for every

$(X,\tau)\in \underline{\mathrm{Haus}}(\underline{P})$, $(\tau_{\underline{\mathrm{Haus}}(\underline{P})})_{\underline{\mathrm{Haus}}(\underline{P})} = \tau_{\underline{\mathrm{Haus}}(\underline{P})}$.

Proof: The first part follows immediately from 1.9 and 1.8 (1), since the idempotent hull of an additive operator is additive. Let $(X,\tau)\in \underline{\mathrm{Haus}}(\underline{P})$ and $M\subset X$, then $\mathrm{cl}_{\underline{P}}(M) = \cup\{\overline{M\cap f(P)}: P\in \underline{P},\ f:P\to X\} = \bigcup_{S,f}\{[M\cap[f(P)]_{\underline{\mathrm{Haus}}(\underline{P})}]_{\underline{\mathrm{Haus}}(\underline{P})}\}$ by virtue of 1.6, so $\mathrm{cl}_{\underline{P}}(M)$ with respect to $\tau$ coincides with $\mathrm{cl}_{\underline{P}}(M)$ with respect to $\tau_{\underline{\mathrm{Haus}}(\underline{P})}$. Thus $(\tau_{\underline{\mathrm{Haus}}(\underline{P})})_{\underline{\mathrm{Haus}}(\underline{P})} = \tau_{\underline{\mathrm{Haus}}(\underline{P})}$.

For any $\underline{P}\subset \underline{\mathrm{Haus}}$ denote by $\underline{\mathrm{Dis}}(\underline{P})$ the category of all spaces X such that, for every $P\in \underline{P}$ and for every continuous map $f:P\to X$, $f(P)$ is a closed discrete subspace of X; obviously $\underline{\mathrm{Dis}}(\underline{P})\subset \underline{\mathrm{Haus}}(\underline{P})$ and if $\underline{P}\subset \underline{\mathrm{Comp}}$ $\underline{\mathrm{Dis}}(\underline{P})$ consists of all spaces X such that every continuous map $f:P\to X$ with $P\in \underline{P}$ has a finite closed image. For $\underline{P} = \underline{P}_{\alpha}$, $\underline{\mathrm{Dis}}(\underline{P})$ consists of all spaces $(X,\tau)$ such that $\tau$ is finer than the co-$\alpha$-topology, i.e., every subset of cardinality less or equal to $\alpha$ of X is closed.

1.11. Proposition. Let $(X,\tau)\in \underline{\mathrm{Haus}}(\underline{P})$ and let $\underline{P}$ satisfy $(*)$. Then $\tau_{\underline{\mathrm{Haus}}(\underline{P})}$ is discrete iff $(X,\tau)\in \underline{\mathrm{Dis}}(\underline{P})$. Consequently a space X in $\underline{\mathrm{Haus}}(\underline{P})$ belongs to $\underline{\mathrm{Dis}}(\underline{P})$ iff every epimorphism $Y\to X$ in $\underline{\mathrm{Haus}}(\underline{P})$ is surjective.

Proof: By virtue of 1.9 and 1.6 $\tau_{\underline{\mathrm{Haus}}(\underline{P})}$ is discrete iff every image $f(P)$ of a space $P\in \underline{P}$ is closed and discrete in X.

1.12. Examples. (a) By 1.11 if for some class $\underline{P}$ satisfying $(*)$, $\underline{\mathrm{Haus}}(\underline{P}) = \underline{\mathrm{Dis}}(\underline{P})$, then the epimorphisms in $\underline{\mathrm{Haus}}(\underline{P})$ are the surjective continuous maps. If $\underline{P}\subset \underline{\mathrm{Comp}}$ and consists of connected spaces, then $\underline{\mathrm{Dis}}(\underline{P})$ becomes a "disconnectedness" in the sense of Arhangel´skii and Wiegandt [2] and Herrlich [12], i.e., $\underline{\mathrm{Dis}}(\underline{P})$ consists of all spaces X such that every continuous map $f:P\to X$ with $P\in \underline{P}$ is constant. Such an example can be found in [13], 2.7

(if m is an infinite cardinal number and $X_m$ is a $T_1$-space with cofinite topology and cardinality m, take $\underline{P} = \{X_m\}$). More about this category can be found in Section 3; it will be denoted by $\mathcal{C}_m$.

(b) Let m be an arbitrary infinite cardinal number. It is known that for a space X, $t(X) \leqq m$ iff for every $M \subset X$, $\overline{M} = \overline{cl_{\underline{Haus}(\underline{P}_m)} M} = \cup\{\overline{S} : S \subset M$, and card $S \leqq m\}$ in fact, being $S \subset M$, $\overline{S} = \overline{S} \cap M$. This is why $t(X) \leqq m$ always implies $\tau = \tau_{\underline{Haus}(\underline{P}_m)}$. On the other hand $d(X) \leqq m$ also implies $\tau = \tau_{\underline{Haus}(\underline{P}_m)}$ obviously (the converse is not true: if $(X,\tau)$ is any non-separable metric space then $\tau = \tau_{\underline{Haus}(\underline{P}_{\aleph_0})}$ while $d(X) > \aleph_0$). Let X be the power $\{0,1\}^{2^m}$ where $\{0,1\}$ has the discrete topology, then $d(X) \leqq m$ so $\tau = \tau_{\underline{Haus}(\underline{P}_m)}$ while $t(X) = 2^m$. Finally observe that if $(X,\tau) \in \underline{Haus}(\underline{P}_m)$ and $\tau = \tau_{\underline{Haus}(\underline{P}_m)}$, then $t(X) \leqq 2^{2^m}$. In fact, if $t(X) > 2^{2^m}$ then there exists a non-closed subset M of X such that, for every $S \subset M$ with card $S \leqq 2^{2^m}$, $\overline{S} \subset M$, i.e., M is $\underline{Haus}(\underline{P}_m)$-closed by virtue of 1.9 and M is not closed, so $\tau < \tau_{\underline{Haus}(\underline{P}_m)}$.

Let $\underline{A} \supset \underline{B}$ be two quotient reflective subcategories of $\underline{Top}$. If for every $X \in \underline{A}$ and for every $M \subset X$, $[M]_{\underline{A}} = [M]_{\underline{B}}$ then $\underline{A} = \underline{B}$ by the diagonal theorem proved in [11], Theorem 2.2. However, it is possible to have $[M]_{\underline{A}} = [M]_{\underline{B}}$ for every $X \in \underline{B}$ and $M \subset X$ as the following lemma shows.

For a class $\underline{P}$ let $\underline{P}'$ denote the class of all continuous images of spaces from $\underline{P}$, then $\underline{Haus}(\underline{P}) = \underline{Haus}(\underline{P}')$. In general $\underline{Haus}(\underline{P}) \subset \underline{Haus}(\underline{P}' \cap \underline{Haus})$.

1.13. Lemma. Let $\underline{P}$ be a class satisfying $(*)$ and closed under continuous images and let $X \in \underline{Haus}(\underline{P} \cap \underline{Haus})$. Then for every $M \subset X$, $[M]_{\underline{Haus}(\underline{P})} = [M]_{\underline{Haus}(\underline{P} \cap \underline{Haus})}$.

Proof: It follows from the definitions.

1.14. Theorem. Let $\underline{P}$ be a class of topological spaces satisfying ($*$) and closed with respect to continuous images. Then $\underline{\mathrm{Haus}}(\underline{P}) \longrightarrow \underline{\mathrm{Haus}}(\underline{P} \cap \underline{\mathrm{Haus}})$ preserves epimorphisms. In particular $\underline{\mathrm{Haus}}(\underline{\mathrm{Comp}}) \longrightarrow \underline{\mathrm{Haus}}(\underline{\mathrm{HComp}})$ preserves epimorphisms and $\underline{\mathrm{Haus}}(\underline{\mathrm{HComp}})$ is not co-well-powered.

Proof: By virtue of 1.13 it remains to prove only the last statement. The category $\underline{\mathrm{Haus}}(\underline{\mathrm{Comp}})$ is not co-well-powered (see [11] Theorem 4.3); this is why the fact that $\underline{\mathrm{Haus}}(\underline{\mathrm{Comp}}) \longrightarrow \underline{\mathrm{Haus}}(\underline{\mathrm{HComp}})$ preserves epimorphisms implies that $\underline{\mathrm{Haus}}(\underline{\mathrm{HComp}})$ is not co-well-powered according to Corollary 3.3 from [5].

The next example is given in [13], 3.5 to show that $\underline{\mathrm{Haus}}(\underline{\mathrm{Comp}}) \subsetneqq \underline{\mathrm{Haus}}(\underline{\mathrm{HComp}})$. Let $Q_\infty$ be the one-point Alexandroff´s compactification of the rationals provided with the usual topology. Then every quasi-compact set in $Q_\infty^+$ is closed, so $Q_\infty \in \underline{\mathrm{Haus}}(\underline{\mathrm{HComp}})$. On the other hand every continuous map $f: Q_\infty \longrightarrow X$, with $X \in \underline{\mathrm{Haus}}(\underline{\mathrm{Comp}})$ is constant, i.e. the reflection of $Q_\infty$ in $\underline{\mathrm{Haus}}(\underline{\mathrm{Comp}})$ is a single point. In fact in $Q_\infty$ $a \neq \infty$ cannot be separated since $a$ has not compact neighborhoods in $Q$. This is why $f(a)$ and $f(\infty)$ cannot be separated in $X$. Since $f(Q_\infty)$ is Hausdorff this means $f(a) = f(\infty)$.

2. The $\beta$-order. Let $\underline{A}$ be a quotient-reflective subcategory of $\underline{\mathrm{Top}}$ such that, for any $(X,\tau) \in \underline{A}$, $\tau_{\underline{A}} \geq \tau$ . Then $(X,\tau_{\underline{A}}) \in \underline{A}$, $(X,(\tau_{\underline{A}})_{\underline{A}}) \in \underline{A}$ and so on for all iterations. Denote by $\underline{A}_0$ the subcategory of $\underline{A}$ consisting of spaces $(X,\tau)$ such that $\tau = \tau_{\underline{A}}$. It was proved in [5], 4.12, that $\underline{A}_0$ is a coreflective subcategory of $\underline{A}$. In particular, $(\underline{\mathrm{Haus}}(\underline{P}))_0$ is a coreflective subcategory of $\underline{\mathrm{Haus}}(\underline{P})$ and the coreflection is given by $(X,\tau_{\underline{\mathrm{Haus}}(\underline{P})}) \longrightarrow (X,\tau)$ according to 1.10. By 1.9, for $(X,\tau) \in \underline{\mathrm{Haus}}(\underline{P})$, $\tau_{\underline{\mathrm{Haus}}(\underline{P})}$ has as closed sets all subsets $M$ of $X$ which satisfv $M = \mathrm{cl}_\tau(M)$, i.e. all sets $M$ which

:e closed with respect to the family of subspaces $\{\overline{f(P)}\}$ where $\in \underline{P}$ and $f:P \to X$ is a continuous map. Clearly $\tau_{\underline{Haus}(\underline{P})}$ is coar-,er than $\tau^c$ (see the introduction). If every continuous image in $(X,\tau)$ of the spaces from $\underline{P}$ is closed (the subcategory of such spaces is denoted by $\underline{P}_1$ in [13], then clearly these topologies coincide. It was proved in [13], 1.5, that if $\underline{P} \subset \underline{Comp}$ and satisfies $(*)$, then $\underline{Haus}(\underline{P}) \subset P_1$. This remains true also for categories $\underline{P}$ satisfying $(*)$ and consisting of quasi-H-closed spaces, i.e. spaces X for which every open cover of X admits a finite subfamily whose closures cover X.

Next we define an ordinal invariant and a cardinal function for the category $(\underline{Haus}(\underline{P}))_0$ first and then to all $\underline{Haus}(\underline{P})$ by means of the $(\underline{Haus}(P))_0$-coreflection.

2.1. Definition. Let $X \in \underline{Haus}(\underline{P}))_0$ and $M \subset X$; then $\widetilde{M} = [M]_{\underline{Haus}(\underline{P})}$. By 1.10 $\widetilde{M}$ can be obtained by iterations of $cl_{\underline{P}}$. Set $M^0 = M$ and if $M^\beta$ has been defined for any ordinal $\beta$ less than an ordinal number $\alpha$ set $M^\alpha = cl_{\underline{P}}(M^\beta)$ if $\alpha = \beta + 1$ for some $\beta < \alpha$, otherwise $M^\alpha = \bigcup_{\beta < \alpha} M^\beta$. Denote by co(M) the least ordinal $\alpha$ with $M^{\alpha+1} = M^\alpha$. Clearly $M^{co(M)} = \widetilde{M}$ and co(M) is the least ordinal with this property. Denote by CO(M) the cardinality of co(M) and set $PCO(M) = \sup\{card(\alpha):M^{\alpha+1} \neq M^\alpha\}$ (obviously $PCO(M) \leq CO(M)$). Finally set $co(X) = \sup\{co(M):M \subset X\}$, $PCO(X) = \sup\{PCO(M):M \subset X\}$ and $CO(X) = \sup\{CO(M):M \subset X\}$. The ordinal co(X) will be called <u>c-order</u>, the cardinal PCO(X) will be called the <u>point-wise c-cardinal</u> of X and CO(X) the <u>c-cardinal</u> of X.

For $(X,\tau) \in \underline{Haus}(\underline{P})$ set $co(X,\tau) = co(X,\tau_{\underline{Haus}(\underline{P})})$, so $co(X,\tau)$ is defined for $(X,\tau) \in \underline{Haus}(\underline{P})$, too.

The c-order generalizes the sequential order and the k-order introduced by Arhangel'skii and Franklin [1]. If X is a c-space , then the c-order coincides with the $\underline{P}$-order defined by Kannan [16] (as a matter of fact the only case when this does not occur is the category $\underline{\text{Haus}}(\underline{P}_m)_0$; however in spite of $\tau_{\underline{\text{Haus}}(\underline{P}_m)} \neq \tau^c$ in general for a space $(X,\tau) \in \underline{\text{Haus}}(\underline{P}_m)$, the $\underline{P}$-order of Kannan coincides with the c-order in this case, too; it is in fact $\leq 1$ ([16], Ex. 5.4.1). For an example of a space $(X,\tau) \in \underline{\text{Haus}}(\underline{P}_m)$ with $\tau_{\underline{\text{Haus}}(\underline{P}_m)} \neq \tau^c$ take $X = \{0,1\}^{2^m}$. Then $\tau = \tau_{\underline{\text{Haus}}(\underline{P}_m)}$; on the other hand $t(X) = 2^m$, so there exists a non-closed subset M of X with $t(M) > m$, then M is closed in $\tau^c$.

For $\underline{P}$ = $\underline{\text{Comp}}$, PCO(X) and CO(X) were introduced in [3]. Clearly $PCO(X) \leq CO(X) \leq PCO(X)^+$ and $CO(X) \leq card(co(X))$. It was proved in [3] that for $\underline{P}$ = $\underline{\text{Comp}}$, $PCO(X) \leq t(X)$. The next theorem shows it for any class $\underline{P}$ satisfying (∗) (a similar result can be found in [25].

2.2. Theorem. Let $\underline{P}$ be a class satisfying (∗) and $X \in \underline{\text{Haus}}(\underline{P})$. Then $PCO(X) \leq t(X)$; in particular if $CO(X) = PCO(X)$, then $CO(X) = t(X)$ otherwise $CO(X) = t(X)^+$.

Proof: Suppose that $PCO(X) > t(X)$; then there exists $A \subset X$ such that $PCO(A) > t(X)$. Let $\alpha$ be the least ordinal with cardinality $t(X)^+$; then $A^{\alpha+1} \neq A^{\alpha}$, otherwise $A^{\alpha+1} = A^{\alpha} = \bigcup_{\beta<\alpha} A^{\beta}$ which would imply $PC(X) < t(X)^+$. Hence there exists an element $x \in A^{\alpha+1} \setminus A^{\alpha}$, so for some $P \in \underline{P}$ and a continuous map $f: P \longrightarrow X$, $x \in \overline{(\overline{f(P)} \cap A^{\alpha})} \setminus A^{\alpha}$. Then there exists $C \subset \overline{f(P)} \cap A^{\alpha}$ such that $x \in \overline{C}$ and card $C \leq t(X)$. Since $A^{\alpha} = \bigcup_{\beta<\alpha} A^{\beta}$, every element of C is contained in some $A^{\beta}$, $\beta < \alpha$. Since $t(X)^+$ is a regular cardinal, C is contained in some $A^{\beta}$. Thus $x \in \overline{C \subset \overline{f(P)} \cap A^{\beta}} \subset A^{\beta+1} \subset A^{\alpha}$ - a contradiction.

The next theorem shows that the c-order is related to the co-well-poweredness of the category $\underline{\text{Haus}}(\underline{P})$.

2.3. Theorem. Let $\underline{P}$ be a class satisfying ($*$) and such that for every $X \in \underline{Haus}(\underline{P})$, $co(X) \leq \alpha$ for a fixed ordinal $\alpha$. Then $\underline{Haus}(\underline{P})$ is co-well-powered.

Proof: Let $X \in \underline{Haus}(\underline{P})$ and $M \subset X$. It is enough to show that the cardinality of $[M]_{\underline{Haus}(\underline{P})}$ is bounded by a cardinality which does not depend on the space X. By 1.9 $[M]_{\underline{Haus}(\underline{P})}$ is the idempotent hull of $cl_{\underline{P}}$ and the number of iterations is less or equal to $\alpha$. So it suffices to see that card $cl_{\underline{P}}(M)$ is limited by a cardinality which does not depend on X. For every $P \in \underline{P}$ and every continuous map $f:P \longrightarrow X$, $\overline{f(P)}$ is Hausdorff, so $\mathrm{card}(M \cap \overline{f(P)}) \leq 2^{2^{\mathrm{card}\, M}}$. On the other hand the different subsets of M of the type $M \cap \overline{f(P)}$ are at most $2^{\mathrm{card}\, M}$, so card $cl_{\underline{P}}(M) \leq 2^{2^{\mathrm{card}\, M}}$

We do not know if the converse of 2.3 is true. For the category $\underline{Ury}$ of Urysohn spaces the $\underline{Ury}$-closure is the idempotent hull of the known $\Theta$-closure and the number of iterations is unbounded in $\underline{Ury}$. This was used by Schröder [22] to prove that the category $\underline{Ury}$ is not co-well-powered.

The next corollary covers 3.6 (d) from [11] and 2.18 from [24]

2.4. Corollary. Let $\underline{P}$ be a class satisfying ($*$) and such that d(P) is bounded for $P \in \underline{P}$. Then $\underline{Haus}(\underline{P})$ is co-well-powered. In particular, if all spaces of $\underline{P}$ have bounded cardinality, then $\underline{Haus}(\underline{P})$ is cowell-powered.

Proof: In view of 2.3 it suffices to show that the c-order is bounded in $\underline{Haus}(\underline{P})$. Assume that for every $P \in \underline{P}$ $d(P) \leq m$. Denote by $\alpha$, the least ordinal of cardinality $(2^{2^{m}})^{+}$; then $co(X) \leq \alpha$ for every $X \in \underline{Haus}(\underline{P})$. In fact, it suffices to see that for every $M \subset X$ $M^{\alpha+1} = M^{\alpha}$. If $x \in M^{\alpha+1}$, then for some $P \in \underline{P}$ and $f:P \longrightarrow X$ $x \in \overline{M^{\alpha} \cap f(P)}$. Now $d(f(P)) \leq d(P) \leq m$, so card $\overline{f(P)} \leq 2^{2^{m}}$. Now by card($\alpha$) > card $\overline{f(P)}$ there exists $\beta < \alpha$ such that $f(P) \cap M^{\alpha} \subset M^{\beta}$, so $x \in \overline{M^{\beta} \cap \overline{f(P)}} \subset M^{\beta+1}$, hence $x \in M^{\alpha}$.

Observe that if $\underline{P} \not\subset \underline{Haus}$ the restriction on the density of the spaces of $\underline{P}$ gives no restriction on their cardinality, so $\underline{P}$ may have arbitrarily large spaces (see 1.12(a) above for such spaces).

2.5. Examples. (a) Let m be a fixed cardinality; denote by $\underline{dP}_m$ the class of spaces X with $d(X) \leq m$. Then $\underline{Haus}(\underline{dP}_m) \subset \underline{Haus}(\underline{P}_m)$ and both categories are cowell-powered by virtue of 2.4. The intersection of all $\underline{Haus}(\underline{P}_m)$ when m varies is $\underline{Haus}$.

(b) Let for every cardinal m, $D_m$ denote a discrete space of cardinality m. Then $\underline{Haus}(\{\beta D_m\}) = \underline{Haus}(\underline{Comp} \cap \underline{dP}_m)$ since every compact Hausdorff space X with $d(X) \leq m$ is a continuous image of $\beta D_m$. Denote by $\mathcal{D}_m$ this category; it is co-well-powered by 2.4. The intersection of all $\mathcal{D}_m$ is $\underline{Haus}(\underline{HComp})$ which is not co-well-powered. Finally denote by $\mathcal{D}'_m$ the category $\underline{Haus}(\{0,1\}^{2^m})$; clearly $\mathcal{D}_m \subset \mathcal{D}'_m$ and the intersection of all $\mathcal{D}'_m$ is the category $\underline{Haus}(\underline{D})$ where $\underline{D}$ is the class of all dyadic compact Hausdorff spaces. We do not know if this category is co-well-powered (observe that $\underline{P} = \{\beta D_m\}$ satisfies (*) since $\beta D_m \cong \beta D_m \sqcup \beta D_m$; the same holds for $\{0,1\}^{2^m}$).

(c) Let $\underline{P}$ be the class of all compact metrizable spaces; then every $P \in \underline{P}$ is a continuous image of $\{0,1\}^{\aleph_0}$ so $\underline{Haus}(\underline{P}) = \underline{Haus}(\{0,1\}^{\aleph_0})$. It contains $\mathcal{D}'_m$ for every m.

If a class $\underline{P}$ does not satisfy (*) we can form the class $\underline{P}^*$ of spaces of the form $P \sqcup F$ where $P \in \underline{P}$ and F is a finite discrete space. Obviously $\underline{P}^*$ satisfies (*).

(d) $\underline{Haus}(\{0,1\}^{\aleph_0}) \subsetneq \underline{Haus}(\{I\}^*) \cap \underline{SUS}$ (here I is the unit interval). To show it we use the one-point extension ${}^{n}X$ of a topological space X defined in [13]: for $X \in \underline{SUS}$ take for a base of neighborhoods of the point $\infty$ in ${}^nX$ the complements of finite unions of convergent sequences in X. If X is sequential then ${}^nX \in \underline{SUS}$ ([13], 3.8). Now set $X = \{0,1\}^{\aleph_0}$; then ${}^nX \in \underline{SUS}$ and ${}^nX \notin \mathrm{Haus}(\{0,1\}^{\aleph_0})$ but ${}^nX \in \underline{\mathrm{Haus}}(\{I\}^*)$ since every continuous map $f: I \longrightarrow {}^nX$ is constant (if $F = f^{-1}(\infty)$, then F is a closed subset of I. If $F = \emptyset$ then f is constant since I is connected and X is totally disconnected. Assume that $F \neq \emptyset$; we shall prove that $F = I$. If $F \neq I$ and $(a,b)$ is an open interval with $\{a,b\} \subset F$ and $(a,b) \cap F = \emptyset$, then $f((a,b))$ is connected in X, so it is a single point, say z. Then $f^{-1}(\{z\})$ is closed in I and contains $(a,b)$, hence intersects F - a contradiction.)

(e) $\underline{SUS} \not\subset \underline{\mathrm{Haus}}(\{I\}^*)$: by 3.8 from [13] ${}^nI \in \underline{SUS}$; on the other hand ${}^nI$ is a continuous image of $I \sqcup \{\infty\}$ so ${}^nI \notin \underline{\mathrm{Haus}}(\{I\}^*)$. Finally note that $\mathrm{Haus}(\{I\}^*) \subset C_{2^{\aleph_0}}$ since $X_{2^{\aleph_0}}$ is a continuous image of I. On the other hand $X_{\aleph_0} \in \underline{\mathrm{Haus}}(\{I\}^*)$ since every continuous map $f: I \longrightarrow X$ is constant. In fact, assume that $f(I)$ is not a single point; then $I = \bigcup \{f^{-1}(x): x \in X\}$ is a disjoint countable union of closed sets. This contradicts the Theorem of Sierpiński ([8], 6.1.2).

(f) Let X be a connected Hausdorff space with $d(X) = m > 1$. Then $\underline{\mathrm{Haus}}(\{\beta D_m\}) \subset \underline{\mathrm{Haus}}(\{X\}^*) \subset \underline{\mathrm{Haus}}(\{I\}^*)$, since there exists a continuous map of X on I.

3. Epimorphisms in $\underline{P}_3$. Let $\underline{P}$ be a class of topological spaces. For $(X,\tau) \in \underline{Top}$ and $M \subset X$ define $cl^{\underline{P}}(M) = \bigcup \{f(f^{-1}(M)): P \in \underline{P}$ and $f: P \longrightarrow X\}$.

3.1. Lemma. The $cl^{\underline{P}}$ has the following properties:
(1) it is monotone, expansive and additive; moreover,

$cl^{\underline{P}}(M) \subset cl_{\underline{P}}(M)$ for every $M \subset X$;

(2) for every continuous map $f:X \longrightarrow Y$ $f(cl^{\underline{P}}(M)) \subset cl^{\underline{P}}(f(M))$;

(3) for every $(X,\tau) \in \underline{Top}$ and $M \subset X$, $cl^{\underline{P}}(M) = M$ iff $M$ is $\tau^c$-closed;

(4) if $(X,\tau)$ is a c-space then the ordinary closure of $(X,\tau)$ is the idempotent hull of $cl^{\underline{P}}$ and the number of iterations is given by the $\underline{E}$-order (for $\underline{E} = \underline{P}$) defined by Kannan ([16]);

(5) if $Y \in \underline{P}_3$ and $f,g:X \longrightarrow Y$, then $Eq(f,g)$ is $cl^{\underline{P}}$ -closed in $X$; in particular every $\underline{P}_3$-closed set is $cl^{\underline{P}}$ -closed.

Note that the converse in (5) is not true. Take for example $\underline{P} = \{N_\infty\}$ ; then $\underline{P}_3 = \underline{US}$ and $cl^{\underline{P}}$ -closed sets are the sequentially closed sets. There exists a space $X \in \underline{US}$ and $M \subset X$ which is sequentially closed and not $\underline{US}$-closed (see [24], Ex. 2.12).

3.2. Definition. For $X \in \underline{Top}$ and $M \subset X$ denote $\langle M \rangle_{\underline{P}} = p(cl^{\underline{P}}(k_0(X)) \cap k_1(X)) = p(k_0(X) \cap cl^{\underline{P}}(k_1(X))) = p(cl^{\underline{P}}(k_0(X)) \cap \cap cl^{\underline{P}}(k_1(X)))$.

It is easy to establish that the equalities hold and that for every $X \in \underline{Top}$ and $M \subset X$ $cl^{\underline{P}}(M) \subset \langle M \rangle_{\underline{P}}$ by virtue of 3.1 (2).

3.3. Lemma. $\langle M \rangle_{\underline{P}}$ is an expansive, monotone operator and for every $X \in \underline{P}_3$ and $M \subset X$ $\langle M \rangle_{\underline{P}} \subset [M]_{\underline{P}_3}$.

Proof: In fact by 3.1 (5) $cl^{\underline{P}}(k_i(X)) \subset [k_i(X)]_{\underline{P}_3}$, $i = 0,1$, hence $cl^{\underline{P}}(k_0(X)) \cap cl^{\underline{P}}(k_1(X)) \subset ([k_0(X)]_{\underline{P}_3} \cap [k_1(X)]_{\underline{P}_3})$ and by 1.3 (3), $p([k_0(X)]_{\underline{P}_3} \cap [k_1(X)]_{\underline{P}_3}) = [M]_{\underline{P}_3}$.

3.4. Theorem. Let $\underline{P}$ be any class of spaces closed with respect to closed subspaces, $X \in \underline{P}_3$ and $M \subset X$. Then the following conditions are equivalent:

(i) $M = \langle M\rangle_{\underline{P}}$;

(ii) $M = [M]_{\underline{P}_3}$;

(iii) $k_0(X) = cl^{\underline{P}}(k_0(X))$ in $X \sqcup_M X$;

(iv) $k_0(X) = cl^{\underline{P}}(k_0(X))$ and $k_1(X) = cl^{\underline{P}}(k_1(X))$.

Proof: (ii) $\Rightarrow$ (i) by 3.3 and (iii) $\Leftrightarrow$ (iv) since $k_0(X)$ and $k_1(X)$ are exchanged by the homeomorphism $s$ of $X \sqcup_M X$. On the other hand (iv) $\Leftrightarrow$ (i) by the definition. To finish the proof we have to prove (iv) $\Rightarrow$ (ii). By virtue of 1.2 it is enough to prove that $X \sqcup_M X \in \underline{P}_3$, i.e., that for every $P \in \underline{P}$ and every continuous maps $f,g: P \longrightarrow X \sqcup_M X$, $Eq(f,g)$ is closed in $P$. For $i = 0,1$ denote $P_f^i = f^{-1}(k_i(X)) \subset P$. By the hypothesis $k_i(X)$ is $cl^{\underline{P}}$ -closed in $X \sqcup_M X$, hence $P_f^i$ is closed in $P$. Define $P_g^i (i = 0,1)$ in the same way, then $P_g^i$ is closed in $P$. Finally define $f_i: P_f^i \longrightarrow k_i(X)$ and $g_i: P_g^i \longrightarrow k_i(X)$ as the restrictions of $f$ and $g$, respectively. For $i = 0,1$, $P_f^i \cap P_g^i$ is a closed subspace of $P$, so it belongs to $\underline{P}$, hence for $p \circ f_i$, $p \circ g_i: P_f^i \cap P_g^i \longrightarrow X$, $Eq(f_i,g_i)$ is closed in $P_f^i \cap P_g^i$, so it is closed in $P$. In the same way one sees that $Eq(f_i,g_i) \cap P_f^i \cap P_g^i$ is closed in $P$. Therefore $Eq(f,g) = \bigcup_{i=0}^{1} P_f^i \cap P_g^i \cap Eq(f_i,g_i)$ is closed in $P$.

3.5. Corollary. The $\underline{P}_3$-closure in each $X \in \underline{P}_3$ is the idempotent hull of $\langle\ \rangle_{\underline{P}}$.

3.6. Example. For $\underline{P} = \{N_\omega\}$, for every $X \in \underline{Top}$ and $M \subset X$, $\langle M\rangle_{\underline{P}} = \{x \in X$: there exists $x_n \longrightarrow x$ in $X$ such that for every open neighborhood $U$ of $x$ and for every open set $V \subset U$ with $V \cap M = U \cap M$, $x_n \notin V$ only for finitely many $n\}$.

In fact, by 3.2 $x \in \langle M\rangle_{\underline{P}}$ iff there exists a sequence $\{x_n\}$ in $X$ such that $k_1(x_n) \longrightarrow k_0(x)$ in $X \sqcup_M X$. Since the basic neighborhoods of $k_0(x)$ in $X \sqcup_M X$ are $q(U \times \{0\} \cup V \times \{1\})$ where $U$ is an open neighborhood of $x$ in $X$ and $V$ is an open subset of $U$ with $V \cap M =$

$= U \cap M$, there is nothing to prove.

Observe that $\langle M\rangle_{\underline{P}} \subset cl_{\underline{P}}(M)$, in this particular case it follows from 1.8 (2). On the other hand it is easy to see that $x \in \langle M\rangle_{\underline{P}}$ iff there exists a sequence $x_n \longrightarrow x$ in X such that for any subsequence $\{x_{n_k}\}$, $x \in \overline{\{x_{n_k}\} \cap M}$ (this form of the closure was given in [24], 2.19; see also 2.21). Assume, in fact, that there exists a subsequence $\{x_{n_k}\}$ such that $x \notin \overline{\{x_{n_k}\} \cap M}$; then there exists an open neighborhood U of x such that $U \cap M \cap \overline{\{x_{n_k}\}} = \emptyset$. In the same way as in 1.8 (2) we find an open subset V of U with $V \cap M = U \cap M$ such that $x_{n_k} \notin V$ for every k. Conversely, assume that there exists an open neighborhood U of x such that, for some open subset V of U with $V \cap M = U \cap M$, $x_{n_k} \notin V$ for infinitely many k. Then clearly $V \cap \overline{\{x_{n_k}\}} = \emptyset$ and $U \cap M \cap \overline{\{x_{n_k}\}} = V \cap M \cap \overline{\{x_{n_k}\}} = \emptyset$, so $x \notin \overline{\{x_{n_k}\} \cap M}$.

The following theorem, for $\underline{P} \subset \underline{Comp}$, can be obtained also from 1.4 and 1.5 of [13].

3.7. Theorem. If $\underline{P}$ is a class of topological spaces satisfying $(*)$ then $\underline{Haus}(\underline{P}) \subset \underline{P}_3$.

Proof: Let $X \in \underline{Haus}(\underline{P})$, then by the diagonal theorem (Theorem 2.1 of [11]) the diagonal $\Delta_X$ in $X \times X$ is $\underline{Haus}(\underline{P})$-closed. Since, for every $(Y,\tau) \in \underline{Haus}(\underline{P})$, $\tau_{\underline{Haus}(\underline{P})} \leq \tau^c$ it follows that $\Delta_X$ is also $\tau^c$-closed. By the definition of $\underline{P}_3$ this means that $X \in \underline{P}_3$.

3.8. Corollary. $\underline{SUS} \hookrightarrow \underline{US}$ preserves epimorphisms. While SUS is co-well-powered, we do not know if $\underline{US}$ is.

3.9. Examples. (a) $\underline{Haus}(\{I\}^*) \not\subset \underline{US}$. In fact, take any convergent sequence $x_n \longrightarrow x$ such that the space $\{x\} \cup \{x_n : n = 1,2,\dots\}$ is $T_1$ and blow up the point x. The space X obtained in this way is $T_1$

and $X \notin \underline{US}$. On the other hand $X \in \underline{Haus}(\{I\}^*)$ because of the theorem of Sierpiński.

(b) Let m be a cardinal and $\underline{P} = \{X_m\}$ be as in 1.12 (a). We discuss first the topology $\tau^c$ for an arbitrary $(Y,\tau) \in \underline{Top_1}$. Consider first the case $m = \aleph_0$. A sequence $\{x_n\}$ in a $T_1$-space is said to be a <u>0-sequence</u> if it is homeomorphic to $X_{\aleph_0}$ provided with the relative topology (see [18]). Now $cl^{\underline{P}}(M)$ is exactly the 0-sequential closure of M, i.e., the limits of 0-sequences in M. The c-spaces are exactly the 0-sequential spaces, i.e., the spaces in which every 0-sequentially closed set is closed. Let us observe that $cl^{\underline{P}}$ is an extensive, monotone and additive operator, in general non idempotent, as the following example shows. Set $Y_2 = \{\infty\} \cup \cup \{x_n : n = 1,2,\dots\} \cup \{x_{mn} : m,n = 1,2,\dots\}$ where each $Z_n = \{x_n\} \cup \{x_{mn} : m = 1,2,\dots\}$ is open in $Y_2$ and has the cofinite topology, a basic neighborhood of $\{\infty\}$ has the form $\bigcup_{n=k}^{\infty} Z_n \setminus F_n$, where for $n \geq k$ $F_n$ is a finite subset of $Z_n \setminus \{x_n\}$. Now for $M = \{x_{mn} : m,n = 1,2,\dots\}$, $\infty \notin cl^{\underline{P}}(M) = \bigcup_{n=1}^{\infty} Z_n$ and $\infty \in cl^{\underline{P}}(\{x_n : n = 1,2,\dots\}) \subset cl^{\underline{P}}(cl^{\underline{P}}(M))$. The space $Y_2$ is in fact a slight modification of the space $S_2$ considered in [1] to show that the sequential order is not idempotent.

Let us mention that for $m > \aleph_0$ an analogous description of the reflection $\tau^c$ can be given by means of 0-nets (here by 0-net we mean a net whose relative topology in the whole space is the cofinite topology. This definition differs from [9]). It is obvious in all cases that, for any $(X,\tau) \in \underline{Top}$, $X \in \mathcal{C}_m$ (cf. 1.12 (a)) iff $\tau^c$ is the discrete topology. This is why for $\underline{P} = \{X_m\}$, $X \notin \underline{P}_3$ implies that $X \times X \notin \mathcal{C}_m$ ( $\Delta_X$ is not closed in the coreflection). It is easy to see that for $Y \in \underline{Top_1}$, $Y \in \mathcal{C}_m$ iff $Y \times Y \in \mathcal{C}_m$. Hence we get $X \notin \mathcal{C}_m$. On the other hand if $X \in \mathcal{C}_m$ then $X \times X \in \mathcal{C}_m$, so $\Delta_X$

is closed in the c-coreflection of $X \rtimes X$. We have proved in this way

3.10. Theorem. For every cardinal m and $\underline{P} = \{X_m\}$, $\underline{P}_3 = \mathcal{C}_m$.

This answers a question of Hoffmann and completes 2.2.9 of [13].

Question. Is the $\underline{P}_3$-closure a Kuratowski operator?

To answer this question it suffices, in account of 3.5, to prove that $\langle\ \rangle_{\underline{P}}$ is an additive operator.

Hušek remarked to us that in all statements where (∗) is needed, it suffices to take $\overline{\text{Haus}}(\underline{P}) = \{X \in \underline{\text{Top}}$: for each $f: P \longrightarrow X$, $P \in \underline{P}$, $\overline{f(P)}$ is Hausdorff$\}$ instead of Haus($\underline{P}$), which includes also other classes.

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