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CONVEX SETS AND HARNACK INEQUALITY

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Abstract: Given a locally compact part $\mathfrak{D}$ of a convex set, let H(U) be a linear space of continuous locally affine functions on an open set $U \subset \mathfrak{D}$. It is proved that the map $H: U \longrightarrow H(U)$ is a harmonic sheaf of functions possessing the Brelot convergence property. Some properties of parts and faces of compact sets and of Choquet simplexes are discussed.

Key words: Faces and parts of convex sets, harmonic sheaf, Harnack inequality, Choquet simplex.

Classification: 31D05, 46A55

Introduction. For any convex set L with induced topology from locally convex Hausdorff space E we shall introduce the following notations: A(L) - space of all continuous affine real-valued functions on L;

$$A^+(L) = \{a \in A(L): a \geq 0\};$$

face (x) will denote the smallest (not necessarily closed) face of L containing x ($y \in$ face $(x) \Longleftrightarrow \exists\, r > 0: x + r(x-y) \in L$, it means that the point x is surrounded in the set face (x)).

Let $\mathfrak{D} \subseteq L$ be a locally compact Gleason part (we shall note that in this locally compact topology every point $y \in \mathfrak{D}$ will have the basis of compact convex neighbourhoods). The necessary and sufficient condition of the local compactness of $\mathfrak{D}$ which is contained in the convex compact S, is given in the proposition 7. The condition is: $\mathfrak{D}$ must have at least one point which has a compact neighbourhood belonging to $\mathfrak{D}$.

Definition. Let $\mathcal{U}$ be an open subset of a part. The function $f: \mathcal{U} \rightarrow R$ is called locally affine, if every point in $\mathcal{U}$ has such a convex neighbourhood $\mathcal{V}$ that the restriction $f/_{\mathcal{V}}$ is an affine function.

Let $H(\mathcal{U})$ be a linear space of all continuous and locally affine on $\mathcal{U}$ functions (only the real-valued functions are considered in this paper).

It will be proved in Theorem 6 that the map $H: \mathcal{U} \longrightarrow H(\mathcal{U})$ is a harmonic sheaf of functions possessing the Brelot convergence property. As shown in [1, p. 16] Harnack inequality will be valid for such sheaves, i.e. if $\mathcal{U}$ is connected then for an arbitrary compact set $K \subset \mathcal{U}$ there exists a positive real number $\alpha_K \geq 1$ such that for any positive function $f \in H(\mathcal{U})$ and for any $x, y \in K$ we have $f(x) \leq \alpha_K f(y)$.

In the process of preparation for the construction of the sheaf we shall prove that if the point $x \in S$ has face (x) of the second category then for an arbitrary compact set $K \subset$ face (x) a positive real number $\alpha_K \geq 1$ exists such that for any positive function $f \in A^+$(face (x)) we have

$$\sup_{y \in K} f(y) \leq \alpha_K f(x).$$

From this we shall have that if face (x) is a locally compact face then any lower semi-continuous affine function $f: S \longrightarrow]-\infty\ ; +\infty]$ has a continuous restriction on face (x).

In the last paragraph of the paper it is proved that if S is Choquet simplex, for any of its part P the solving of the Dirichlet problem with an arbitrary continuous boundary function will be continuous on P in the part metric. Besides, it is proved that for any point $x \in S$ any bounded set of the affine functions of the first Baire class contains a sequence which is on face (x) converging pointwise to some affine function f:face $(x) \longrightarrow R$.

1. Properties of the facesof the second category of convex compacts

Proposition 1. Assume that L is bounded and the function $f: L \longrightarrow]-\infty ; +\infty]$ is a supremum of an increasing net of functions from A(L). Assume that its effective set dom $f = \{x \in L : f(x) < +\infty\}$ is non-empty. Then:

1) dom f is convex and is F_σ ;

2) if dom f is a topological space of the second category, the restriction $f/_{\text{dom } f}$ is upper bounded.

Proof. We shall not prove the evident statement 1).

2) It is clearly seen that $g = f/_{\text{dom } f}$ is a lower semicontinuous affine function and that is why it has an upper bound m in R on some non-empty intersection dom f with the open subset $\mathcal{V} \subset E$ (the set of discontinuity points of g is of first category in dom f by Osgood's Theorem and, by Baire's Theorem g has a continuity point in dom f). We shall choose a point y from this intersection. As dom f is bounded in E, it can be absorbed by the neighbourhood $\mathcal{V} - y$ of zero point. That is why the inclusion

$$\text{dom } f \subset y + n(\mathcal{V} - y)$$

is valid for some natural n. As g is an affine function, it must be upper bounded on dom f. Indeed, let the point $z \in \text{dom } f$, then there exists a point $t \in \mathcal{V} \cap \text{dom } f$ such that

$$z = y + n(t - y),$$

then

$$t = \frac{1}{n} z + \frac{n-1}{n} y,$$

$$g(t) = \frac{1}{n} g(z) + \frac{n-1}{n} g(y)$$

and

$$g(z) = n\, g(t) + (1-n)g(y) \leq n\, m + (1-n)g(y).$$

Remark. The proof of the statement 2) is a repetition of the proof of the first part of the Choquet theorem for a case when dom f is compact (see [3],Theorem 1.2.6).

Corollary. 1.1. If condition 2) of the proposition 1 is fulfilled, the effective set of the function f is closed.

Theorem 2 (Bear H.S. [2]). Let S be a convex compact set and we shall consider the sequence $\{a_n\}$ belonging to the space A(S) of all continuous affine real-valued functions on S which satisfy the requirement $a_n \leq a_{n+1}$, $n \in N$.
If $\{a_n(x)\}$ converges for some point $x \in S$ then $\{a_n(y)\}$ converges for all $y \in$ face (x).

Corollary 2.1. Let the sequence $\{h_n\}$ belong to the space A(face(x)) of all continuous affine real-valued functions on face (x) and $h_n \leq h_{n+1}$ for all $n \in N$. Let us consider the function $h = \sup h_n$.
If $h(x) < +\infty$ then the inequality $h(z) < +\infty$ is valid for any point $z \in$ face (x).

Proof. For a point $z \in$ face (x) we shall choose a point $y \in$ face (x), so that $x \in]y;z[$. As the sequence $\{h_n|_{[y;z]}\}$ satisfies the theorem 2 (the point x is surrounded in [y;z] and $h(x) < +\infty$) then $h(z) < +\infty$.

Theorem 3. Assume that the point $x \in S$ has the face (x) of the second category; then for any compact

$$K \subset \text{face}\ (x)$$

there exists such a number $\alpha_K \geq 1$ that

$$\sup_{y \in K} f(y) \leq \alpha_K f(x)$$

for all $f \in A^+(\text{face}(x))$.

Proof. Assume the contrary, then there exist two sequences of points $\{x_n\} \subset K$ and functions $\{f_n\} \subset A^+(\text{face}(x))$ such that $f_n(x_n) \geq n^3 f_n(x)$ for all $n \in N$.
Consider the function

$$f(y) = \sum_{t=1}^{\infty} \frac{f_t(y)}{t^2 f_t(x)}$$

It is evident that the sequence of the continuous affine functions

$$f^{(m)} = \sum_{t=1}^{m} \frac{f_t}{t^2 f_t(x)}, \quad m \in N$$

is increasing to f. As $f(x) < +\infty$ then by Corollary 2.1 $f(y) < +\infty$ for all $y \in \text{face}\ (x)$. However,

$$f(x) = \sum_{t=1}^{\infty} \frac{f_t(x_n)}{t^2 f_t(x)} \geq \frac{f_n(x_n)}{n^2 f_n(x)} \geq \frac{n^3 f_n(x)}{n^2 f_n(x)} = n.$$

But it contradicts Proposition 1.

Corollary 3.1. Assume that the point $x \in S$ has face (x) of the second category. Consider the sequence $\{h_n\} \subset A\ (\text{face}(x))$ with the property $h_n \leq h_{n+1}$ for all $n \in N$ and the function $h = \sup h_n$. So if $h(x) < +\infty$ then the sequence $\{h_n\}$ uniformly converges to h on every compact $K \subset \text{face}(x)$. In particular, if a set face(x) is a metrizable or locally compact then $h \in A(\text{face}(x))$.
As $h_{n+p} - h_n \in A^+(\text{face}(x))$, the proof follows from the fact that for any compact $K \subset \text{face}(x)$ we have the following inequality:

$$0 \leq h_{n+p} - h_n \leq \alpha_K (h_{n+p}(x) - h_n(x)).$$

Corollary 3.2. Let $f:S \longrightarrow]-\infty ; +\infty]$ be a lower semicontinuous affine function. Assume that the point $x \in S$ has a locally compact face face(x). Then if $h(x) < +\infty$ then

$$f/_{face(x)} \in A(face(x)).$$

Proof. By the corollary 1.1.4 [3], f is a pointwise limit of the increasing net $\{a_\alpha\} \subset A(S)$. by the corollary 3.1, for any increasing sequence $\{b_n\} \subset \{a_\alpha\}$ the following inclusion is valid:

$$\sup_n b_n/_{face(x)} \in A(face(x)).$$

That is why by the topological lemma by A. Cornea (see [1], p.10) $f/_{face(x)}$ will be a continuous function.

2. Locally compact Gleason parts and the harmonic Brelot's sheaves

Definition. Let x and y be two points of the convex set $\mathcal{U}$. It is said that the segment $[x;y]$ extends in $\mathcal{U}$ beyond the point x by a positive number $r>0$ if $x + r(x-y) \in \mathcal{U}$.

Theorem 4 (H.S. Bear [2]). Let x and y belong to S. The segment $[x;y]$ extends in S beyond the point x by the positive number $r>0$ if and only if

$$a(y) \leq (1 + \frac{1}{r})\, a(x)$$

for all $a \in A^+(S)$.

Definition. Two points x, y are said to be included in one part of the convex set $\mathcal{U}$, if face(x) = face(y), or which is equivalent to the line segment $[x;y]$ extends in $\mathcal{U}$ beyond x and y.

Proposition 5. Let $\mathcal{U}$ now be a convex subset of E with an induced topology from E. Assume that $\mathcal{U} = \bigcup_{\alpha \in J} \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ are convex open subsets of $\mathcal{U}$. Let $f: \mathcal{U} \longrightarrow R$. Then from

$f/_{\mathcal{U}_\alpha} \in A(\mathcal{U}_\alpha)$, $\alpha \in J$ it follows $f \in A(\mathcal{U})$ (as in the introduction $A(\mathcal{U}_\alpha)$, $A(\mathcal{U})$ denote the spaces of all continuous affine real-valued functions on $\mathcal{U}_\alpha$ and on $\mathcal{U}$).

Proof. Let us check that f is an affine function. Let $a, b \in \mathcal{U}$ and consider the affine isomorphism $\varphi: \alpha \longrightarrow (1-\alpha)a + \alpha b$ where $\alpha \in [0;1]$. Then the sets of the affine functions on segments $[a;b]$ and $[0;1]$ are isomorphic. As a restriction on $[a;b]$ of a locally convex topology from E is the topology, defined by the image of the topology on $[0;1]$ after mapping φ, then the function $f \circ \varphi$ will be locally affine on $[0;1]$ and hence affine on $[0;1]$. Hence we have that f is an affine function on $[a;b]$.

Definition. Let Y be a topological space. A sheaf of functions on Y is the map $\mathcal{F}$ defined on the set of open sets of Y such that:

a) for any open set $\mathcal{U}$ of Y $\mathcal{F}(\mathcal{U})$ is the set of functions on $\mathcal{U}$;

b) for any two open sets $\mathcal{U}, \mathcal{V}$ of Y such that $\mathcal{U} \subset \mathcal{V}$ the restriction of any function from $\mathcal{F}(\mathcal{V})$ to $\mathcal{U}$ belongs to $\mathcal{F}(\mathcal{U})$;

c) for any family $(\mathcal{U}_i)_{i \in J}$ of open sets of Y a function on $\bigcup_{i \in J} \mathcal{U}_i$ belongs to $\mathcal{F}(\bigcup_{i \in J} \mathcal{U}_i)$ if for any $i \in J$ its restriction to $\mathcal{U}_i$ belongs to $\mathcal{F}(\mathcal{U}_i)$.

Definition. A sheaf of functions H on a locally compact space Y is called a harmonic sheaf, if for any open set $\mathcal{U}$ of Y $H(\mathcal{U})$ is a real vector space of real continuous functions on $\mathcal{U}$. A function defined on the set containing an open set $\mathcal{U}$ is called an H-function on $\mathcal{U}$ if its restriction to $\mathcal{U}$ belongs to $H(\mathcal{U})$.

Definition. We shall say that a harmonic sheaf H on a locally compact space Y possesses the Brelot convergence property, if the limit function of any increasing sequence of H-functions on any open connected set of Y is an H-function whenever it is finite at a point.

Let $\mathfrak{D}$ be a locally compact part of L. By the proposition 5 the map H, defined in the introduction on the set of open subsets $\mathfrak{D}$ satisfies the axioms of the harmonic sheaf.

Theorem 6. The sheaf H on part $\mathfrak{D}$ satisfies the Brelot convergence property.

Proof. Let $\mathcal{U}$ be an open and connected subset of $\mathfrak{D}$. Each point $x \in \mathcal{U}$ has a convex compact neighbourhood $\mathcal{V} = \mathcal{V}(x) \subset \mathcal{U}$. That is why, if the increasing sequence of functions from the space $A(\mathcal{V})$ of all continuous affine functions on $\mathcal{V}$ is bounded even in one interior point $y \in \mathcal{V}$ then by the corollary 3.1, its limit will belong to $A(\mathcal{V})$.

Now let $\{h_n\}$ be an increasing sequence of H-functions on $\mathcal{U}$, $h = \sup h_n$. Assume that in the point $x \in \mathcal{U}$ the function $h(x) < +\infty$. Let us prove that $\mathcal{U}$ = dom h. If it is not so, then by the theorem 2 the set $T = \{y \in \mathcal{U} : h(y) = +\infty\}$ is open. By connection $\mathcal{U}$ at the bound of the set dom h there exists a point, the convex compact neighbourhood of which has a non-empty intersection both with T and with dom h, which is impossible by the theorem 2. So we have proved that h is bounded in every point $z \in \mathcal{U}$. Besides, we have proved earlier that h is a locally affine and continuous function. From this we have that $h \in H(\mathcal{U})$. A necessary and sufficient condition that the part $\mathfrak{D} \subset S$ is locally compact, will be obtained in the following proposition.

Proposition 7. Assume that some point $x \in \mathfrak{D} \subset S$ has a compact neighbourhood $K(x) \subset \mathfrak{D}$, then $\mathfrak{D}$ is locally compact.

$\underline{\text{Proof}}$. Let $y \in \mathfrak{D}$. We can consider the neighbourhood K(x) convex without limiting generality. The point x is surrounded in K(x) (i.e. for every point $y \in K(x)$ and $y \neq x$ the segment $[x;y]$ may be extended in K(x) beyond the point x). By the theorem 3 there exists such a number $\alpha > 1$ that the following inequalities are fulfilled:

$$\sup_{z \in K(x)} a(z) \leq \alpha\, a(x) \quad (\forall a \in A^+(K(x)))$$

and

$$\sup_{z \in K(x)} a(z) \leq \alpha\, a(y) \quad (\forall a \in A^+(S)).$$

Let $r > 0$ be such a number that $\alpha = 1 + \frac{1}{r}$. By the theorem 4 $(\forall z \in K(x))$ we obtain the following inclusions:

$$x + r(x-z) \in K(x), \quad y + r(y-z) \in S.$$

Let $t \in]0;r[$. It is obvious that $(\forall z \in K(x))$ we have the following inclusions:

$$x + t(x-z) \in K(x), \quad y + t(y-z) \in \mathfrak{D}.$$

We shall consider the map

$$\varphi : z \longrightarrow \tilde{z} = y + t(y-z), \quad z \in K(x).$$

It is clearly seen that φ is continuous and hence $K(\tilde{x}) = \varphi(K(x))$ is a compact neighbourhood of the point $\tilde{x}$. We shall consider the set

$$V(y) = K(\tilde{x}) + y - \tilde{x}.$$

If we can prove the inclusion $V(y) \subset \mathfrak{D}$ then V(y) will be a compact neighbourhood of the point y.

Indeed, let $k \in K(\tilde{x}) + y - \tilde{x}$, then

$$k = \tilde{z} + y - \tilde{x} = y + t(x-z).$$

We remark that for every $z \in K(x)$ we have the equalities

$$\tilde{z} - \tilde{x} = t(x-z), \quad \tilde{x} + (x-z)t = \tilde{z}, \quad \frac{\tilde{z}}{1+t} + \frac{t}{1+t} z = y.$$

Now we have

$$k = y + (x-z)t = \frac{1}{1+t}[\tilde{x} + (x-z)t] + \frac{t}{1+t}[x + t(x-z)] =$$

$$= \frac{\tilde{z}}{1+t} + \frac{t}{1+t}[x + t(x-z)] \in \mathcal{D}.$$

The proof is complete.

3. On some characteristics of parts and faces of simplexes.

Let S be now Choquet simplex, E(S) will denote the extreme boundary, i.e. the set of the extreme points of S, μ_x may denote the boundary measure (see [3]) which represents $x \in S$. On the linear space $C(\overline{E(S)})$ of all continuous real-valued functions on $\overline{E(S)}$ we define the Dirichlet operator $f \longrightarrow u_f$, where

$$u_f(x) = \mu_x(f), \quad x \in S.$$

Let us consider the part P of S and any two points $x, y \in P$. For the part P of S we can define a function $\alpha_P : P \times P \longrightarrow [1; +\infty[$ as follows:

$$\alpha_P(x;y) = \inf\{1 + \Delta^{-1} : [x;y] \text{ extends by } \Delta \text{ in } S\}.$$

Let us define as in [3] the part metric on P

$$\rho(x;y) = \ln \alpha_P(x;y).$$

Proposition 8. The affine function $u_f|_P$ is continuous in the part metric.

Proof. Let us consider the σ-neighbourhood of the point x:

$$V(x;\sigma) = \{y \in P : \rho(x;y) < \sigma\} = \{y \in P : \alpha_P(x;y) < e^{\sigma}\}.$$

There exists such a number $\Delta > 0$ that $1 + \Delta^{-1} < e^{\sigma}$ and the segment $[x;y]$ may be extended in S by the number Δ, that is why by the theorem 11.5.25 [3] the following inequality will be valid:

$$\mu_y \le (1 + \Delta^{-1})\mu_x.$$

For a positive function $f \in C(\overline{E(S)})$ we obtain

$$u_f(y) - u_f(x) = \mu_y(f) - \mu_x(f) \le \Delta^{-1}\mu_x(f) \le \Delta^{-1}\|f\| < (e^{\sigma} - 1)\|f\|.$$

As

$$u_f(x) - u_f(y) < (e^{\sigma} - 1) \| f \|$$

then

$$|u_f(x) - u_f(y)| < (e^{\sigma} - 1) \| f \| .$$

Let now be $\varepsilon > 0$, we consider the number $\sigma = \ell n(\frac{\varepsilon}{\|f\|} + 1)$. It is obvious that as soon as $\rho(x;y) < \sigma$ the inequality $|u_f(x)-u_f(y)| < \varepsilon$ will be fulfilled. If $f \not> 0$ we take such a number $c > 0$ that $f + c > 0$. In this case

$$\mu_y(f) - \mu_x(f) = \mu_y(f+c) - \mu_x(f+c)$$

and $\sigma = \ell n(\frac{\varepsilon}{\|f+c\|} + 1)$. The proposition is complete.

Theorem 9. Let $\{f_n\}$ be a sequence of the affine functions of the first Baire class defined on S and $\|f_n\| < c$, $n \in N$ for some number $c > 0$. Let $x \in S$, then the sequence $\{f_n\}$ has the subsequence $\{f_{n_k}\}$ which on face face(x) converges pointwise to some affine function $f: \text{face}(x) \longrightarrow R$.

Proof. The sequence $\{f_n\}$ may be considered as the bounded sequence of continuous linear functionals on the space $L_2 = L_2(\mu_x)$ as

$$|\int f_n h \, d\mu_x| \le c \sqrt{\int |h|^2 \, d\mu_x} \quad c\|h\|_{L_2}.$$

As the unit ball in L_2 is weakly compact then from $\{f_n\}$ we may choose the subsequence $\{f_{n_k}\}$ which converges on every function $h \in L_2$, i.e.

$$\int f_{n_k} h \, d\mu_x \longrightarrow \cdot$$

For every point $y \in \text{face}(x)$ the measure μ_y is absolutely continuous with respect to the measure μ_x ($\mu_y << \mu_x$) and by the theorem 11.5.15 [3] we have

$$\left\| \frac{d\mu_y}{d\mu_x} \right\|_{L_\infty} \leq \text{const.}$$

Therefore the density function $\frac{d\mu_y}{d\mu_x} \in L_2(\mu_x)$. By the Choquet theorem [3, p. 16] for every function f_n, $n \in N$ the barycentric formulae are valid:

$$\mu_y(f_n) = f_n(y).$$

Hence
$$f_{n_k}(y) = \int f_{n_k} d\mu_y = \int f_{n_k} \left(\frac{d\mu_y}{d\mu_x}\right) d\mu_x \longrightarrow$$

It is obvious that the limit function is affine.

Let S be a metrizable simplex, B is a set of all bounded measurable Borel functions defined on E(S). As above on B we define the Dirichlet operator. Let us denote by A_0 the set of continuity points of u_f for all $f \in B$. As follows from [4] if $A_0 \neq \emptyset$ then with every point x the set A_0 contains face(x). If $\{f_n\}$ is a sequence function from the theorem 9 then the limit function f for its subsequence $\{f_{n_k}\}$ will belong to the first Baire class.

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