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# A NOTE ON NONUNIFORM NONRESONANCE FOR JUMPING NONLINEARITIES

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Abstract: We prove some lemmas as technical bases for existence results for nonlinear noncoercive problems with jumping nonlinearities and nonuniform nonresonance conditions.

Key words: jumping nonlinearities, nonuniform nonresonance, BVP's for ODE's.

Classification: 34 B 15, 34 C 25, 47 H 12.

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0. We consider a positively homogeneous scalar real ODE:

$$u'' + g_+(t)u^+ - g_-(t)u^- = 0, \tag{1}$$

a.e. on an interval $[0,T]$, $T > 0$, where $'=d/dt$, $u^{\pm} = \max(\pm u, 0)$, and where $g_{\pm}$ are measurable mappings from $[0,T]$ into the realline $\mathbb{R}$. Equ.(1) is one of the simplest examples of an ODE with jumping nonlinearity. We recall Fučík's classical book [4] as main reference for nonlinear noncoercive problems with jumping nonlinearities. See Drábek [2] for a survey of recent results in this field. We confine here our attention to (1) because, in the framework of the so-called nonlinear Fredholm alternative, the problem of the existence of solutions for the periodic BVP on $[0,T]$ for an ODE like

$$u'' + cu' + f(t,u) = h(t), \tag{2}$$

where f is jumping (in the sense that there are measurable functions $\alpha_+$, $\alpha_-$, $\beta_+$, $\beta_-$ such that the inequalities

$$\alpha_{\pm}(t) \le \liminf_{u\to\pm\infty} f(t,u)/u \le \limsup_{u\to\pm\infty} f(t,u)/u \le \beta_{\pm}(t) \tag{3}$$

hold uniformly a.e. on $[0,T]$), can be reduced by degree arguments to the uniqueness of the trivial solution of (1) joint with the following boundary conditions:

$$u(0) = u(T) = 0, \qquad \operatorname{sign} u'(0) = \operatorname{sign} u'(T). \tag{4}$$

See Dancer [1] for a particular case; see Drábek and the author [3] for a more general one. In the last mentioned paper the authors prove the uniqueness for (1)-(4) assuming that the range of the map $g = (g_+,g_-)\colon [0,T] \to \mathbb{R}^2$ is contained into some compact subset having empty intersection with a closed set $A_{-1}$.

This set $A_{-1}$ is the set of all pairs $(\mu,\nu)$ such that the problem $u''+\mu u^{+}-\nu u^{-}=0$, joint with boundary conditions (4), has nontrivial solutions: it can be completely described; see [4], or [3]. Thus the main result of [3] is based on a uniform nonresonance condition.

Therefore, the recent successful application of nonresonance conditions of nonuniform type (Mawhin and Ward [5-6], Mawhin [7],..) to existence problems for BVP's, suggests the study of (1)-(4) allowing, in a controlled way, nonempty intersection of the range of g with $A_{-1}$. We give our pertinent result in Sect.1. In Sect.2. we exemplify the possible applications of the preceding results considering the periodic BVP for equation $u'' + f(t,u) = h(t)$ on $[0,T]$. However, it is possible to give existence results, using the same methods, for the periodic BVP for (2), and for some BVP's for suitable PDE's, as the periodic-Dirichlet problem for the telegraph equation, as well. We will not discuss here in details these further applications.

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1. Let m be the Lebesgue measure on the real line $\mathbb{R}$. Let K and $H \subseteq K$ be closed subsets of $\mathbb{R}^N$, and let $g:[0,T] \to \mathbb{R}^N$ be a measurable map. We shall write

$$g(t) \in K \sim H \text{ on } [0,T]$$

when: (*) $g(t) \in K$ a.e. on $[0,T]$, but there is a subset J of $[0,T]$ with $m(J) > 0$ such that $g(t) \in K \setminus H$ for every $t \in J$. It is important to remark that condition (*) imply that an inequality $\operatorname{dist}(g(t),H) \geq \varepsilon$ holds true for some $\varepsilon > 0$ and for all t in a subset of J having positive measure too. In fact, (*) implies $J_n = \{t \in J \mid \operatorname{dist}(g(t),H) \geq 1/n\} \uparrow J$ ; the continuity of m from below gives $m(J_n) > 0$ for sufficiently large n. In the sequel, for short, $I = [0,T]$.

The condition (*) with N=1 was first introduced in the study of BVP's for differential equations by Mawhin and Ward [5-6]. See also Mawhin [7]. In these cases a typical choice for K is a compact interval $[\lambda_k,\lambda_{k+1}]$, or a closed half-line $(-\infty,\lambda_1]$, where $\lambda_1 < \lambda_2 < \ldots$ are the distinct eigenvalues of a linear problem associated to the considered BVP, and H is the boundary $\partial K$ of K. Assuming the terminology of these authors, we will call (*) a nonuniform nonresonance condition. We will apply a condition of this type to a case where N=2. Let us consider the BVP (1)-(4). We introduce a 'singular set' $A_{-1}$ (corresponding to the spectrum $\{\lambda_1, \lambda_2, \ldots\}$ in the 1-dimensional case), defined as the union of a sequence $\{C_1, C_2, \ldots\}$ of curves, where, for any $k \geq 1$,

$$C_k = \{(\mu,\nu) \in \mathbb{R}^2 \mid \mu\nu > 0,\ 2\sqrt{\mu}\sqrt{\nu}/(\sqrt{\mu}+\sqrt{\nu}) = k(2\pi/T)\}.$$

Then we introduce the set K, closed and with nonempty interior, of three possibly different types: the product of two compact intervals, of a compact interval and a closed half-line, of two closed half lines. We fix the position of K in $\mathbb{R}^2$ in such a way it intersects $A_{-1}$ only at some of its vertexes. We define H as the set of all boundary points of K having at least one coordinate in common with some of these vertexes. Then we prove that the condition $g(t) \in K \sim H$ on I (provided g is integrable) implies that (1)-(4) admits only the trivial solution. We will consider separately each possible form of K in the following lemmas.

Lemma 1. Let $R = [r_+, s_+] \times [r_-, s_-]$, $r_\pm < s_\pm$, $(r_+, r_-) \in C_k$, $(s_+, s_-) \in C_{k+1}$, for some fixed $k \geq 1$. Let $g=(g_+, g_-)$ be a measurable map $I \to \mathbb{R}^2$ such that

$$g(t) \in R \sim \partial R \text{ on } I.$$

Then the BVP

$$u'' + g_+(t)u^+ - g_-(t)u^- = 0 \text{ a.e. on } I,$$
$$u(0) = u(T) = 0, \quad \operatorname{sign} u'(0) = \operatorname{sign} u'(T),$$

admits only the trivial solution.

Proof. Let u be a possible nontrivial solution. Then (by Uniqueness) u vanishes only at a finite number of points. Let $I_+^{(i)}$ $(i=1,\ldots,P)$ (resp. $I_-^{(i)}$ $(i=1,\ldots,M)$) be all the different connected components (open intervals) - if any - of the subset of I where $u > 0$ (resp. $u < 0$). Then the boundary conditions imply P=M. We claim that the 2P relations

$$\pi/\sqrt{s_+} \leq m(I_+^{(i)}) \leq \pi/\sqrt{r_+} \qquad (i=1,\ldots,P) \tag{5}$$

$$\pi/\sqrt{s_-} \leq m(I_-^{(i)}) \leq \pi/\sqrt{r_-} \qquad (i=1,\ldots,P) \tag{6}$$

hold, and that there are strict inequality signs in at least one of them (more precisely in any relation corresponding to an interval $I_\pm^{(i)}$ having intersection of positive measure with a subset of I where $\operatorname{dist}(g(t),\partial R) \geq \varepsilon > 0$ holds). This is sufficient to get a contradiction. Namely, adding (5) and (6) and taking into account of the strict inequality signs in at least one relation, we get

$$P(\pi/\sqrt{s_+} + \pi/\sqrt{s_-}) < T < P(\pi/\sqrt{r_+} + \pi/\sqrt{r_-}).$$

But the definition of $C_k$ and $C_{k+1}$ gives

$$k(\pi/\sqrt{s_+} + \pi/\sqrt{s_-}) = T = (k+1)(\pi/\sqrt{r_+} + \pi/\sqrt{r_-}).$$

Thus we deduce $P > k$ and $P < k+1$.

To prove the claim we consider only the inequality $m(I_+^{(i)}) \le \pi/\sqrt{r_+}$ for some value of i, since the remaining inequalities can be proved in the same way. Suppose $I_+^{(i)} = ]a,b[$ , so that $m(I_+^{(i)}) = b-a = \rho$. Assume $\rho > \pi/\sqrt{r_+}$, i.e. $r_+ > (\pi/\rho)^2$. Define the sphere

$$\Sigma = \{w \in W_0^{1,2}(a,b;\mathbb{R}) \mid \int_a^b |w'|^2 = 1\},$$

and let $w^*$ be a non-negative eigenfunction for the Picard problem

$$w'' + (\pi/\rho)^2 w = 0, \quad w(a) = w(b) = 0.$$

We can assume that for all t in a set J with $m(J) > 0$ the inequalities

$$r_\pm + \varepsilon \le g_\pm(t) \le s_\pm - \varepsilon \tag{7}$$

hold with some $\varepsilon > 0$. To simplify the notations, let $A = ]a,b[ \cap J$, $B = ]a,b[ \setminus J$. The minimum principle for eigenvalues implies that

$$\begin{aligned} 1 &= \sup_{w\in\Sigma} \int_a^b g_+|w|^2 \ge \int_a^b g_+|w^*|^2 = \int_A g_+|w^*|^2 + \int_B g_+|w^*|^2 \\ &\ge \int_A (r_+ + \varepsilon)|w^*|^2 + \int_B r_+|w^*|^2 = \int_a^b r_+|w^*|^2 + \varepsilon\int_A |w^*|^2 \\ &> \int_a^b (\pi/\rho)^2|w^*|^2 + \varepsilon\int_A |w^*|^2 = 1 + \varepsilon\int_A |w^*|^2, \end{aligned}$$

a contradiction, even if $m(A) = 0$. If $r_+ = (\pi/\rho)^2$, we get a contradiction as soon as $m(A) > 0$.

In a similar manner one can prove the following

<u>Lemma 2</u>. Let $R = (-\infty,s_+] \times (-\infty,s_-]$, with $(s_+,s_-) \in C_1$. If $g: I \to \mathbb{R}^2$ is integrable and $g(t) \in R \setminus \partial R$ on I, then the conclusion of Lemma 1 holds.

One easily realizes that the nonresonance condition considered in Lemma 1 (resp. in Lemma 2) corresponds to a situation 'between two consecutive eigenvalues' (resp. 'on the left of the spectrum') for the case N=1. Here, being $N > 1$, a slightly different situation can occur. Let $(\mu,\nu)$ be the generic point in $\mathbb{R}^2$. Each $C_k$ $(k \ge 1)$ intersect the asymptotes $\mu = a_{k+1}^2$, $\nu = a_{k+1}^2$ ($a_{k+1} = (k+1)\pi/T$ for short) of $C_{k+1}$ at points $(k^2 a_{k+1}^2, a_{k+1}^2)$, $(a_{k+1}^2, k^2 a_{k+1}^2)$. Let us consider the case $\mu > \nu$ only (for $\mu < \nu$ we have symmetric results). Let $(r_+,r_-)$ be any point in $C_k$ with $1^{st}$ coordinate so large that $r_+ > k^2 a_{k+1}^2$. Then the unbounded strip $S = [r_+, +\infty) \times [r_-, a_{k+1}^2]$ intersects the singular set $A_{-1}$ only at $(r_+, r_-)$. Moreover, let $\partial_1 S$ be the set of all boundary points of $S$ having at least one coordinate in common with $(r_+, r_-)$, i.e. let

$$\partial_1 S = (\{r_+\} \times [r_-, a_{k+1}^2]) \cup ([r_+, +\infty) \times \{r_-\}).$$

We have the following

Lemma 3. Let $(r_+, r_-)$, $S$, $\partial_1 S$ be given as above. If $g\colon I \to \mathbb{R}^2$ is integrable and $g(t) \in S \sim \partial_1 S$ on $I$, then the conclusion of Lemma 1 holds.

Proof. Let $u$ be a possible nontrivial solution to (1)-(4). Following the proof of Lemma 1 we get the inequalities

$$T/(k+1) = \pi/a_{k+1} \leq m(I_-^{(i)}) \leq \pi/\sqrt{r_-} \qquad (i=1,\ldots,P),$$

$$m(I_+^{(i)}) \leq \pi/\sqrt{r_+} \qquad (i=1,\ldots,P),$$

where a strict inequality sign holds at the right hand side in at least one case.

Therefore we get $P > k \geq 1$, i.e. $P \geq 2$. But evaluating the measure of the subset of $I$ where $u$ is negative we obtain

$$m\{u < 0\} = \sum_{i=1,P} m(I_-^{(i)}) \geq \sum_{i=1,P} T/(k+1)$$

$$\geq \sum_{i=1,k+1} T/(k+1) = T,$$

i.e. $u$ is negative a.e., and so $P=1$, a contradiction.

2. To illustrate the results of Sect. 1., we consider the periodic BVP

$$u'' + f(t,u) = h(t) \quad \text{a.e. on } I, \tag{8}$$

$$u(0) = u(T), \quad u'(0) = u'(T). \tag{9}$$

The right-hand side $h$ in (8) is arbitrary in $L^1(I;\mathbb{R})$. The map $f\colon I \times \mathbb{R} \to \mathbb{R}$ satisfies the usual Carathéodory conditions, and has linear growth, i.e. we have $|f(t,u)| \leq A(t) + B|u|$ a.e. on $I$ with $A \in L^1(I;\mathbb{R})$ and $B \geq 0$. We assume that there are measurable mappings $\alpha_+, \alpha_-, \beta_+, \beta_- : I \to \mathbb{R}$ such that, a.e. on $I$,

$$\alpha_\pm(t) \leq \liminf_{u \to \pm\infty} f(t,u)/u \leq \limsup_{u \to \pm\infty} f(t,u)/u \leq \beta_\pm(t).$$

Theorem 1. Suppose that there are real numbers $r_+ < s_+$, $r_- < s_-$ such that $r_\pm \leq \alpha_\pm(t)$ and $\beta_\pm(t) \leq s_\pm$ a.e. on $I$ with strict inequality signs for $t$ in a subset of positive measure. Assume either (i) $(r_+, r_-) \in C_k$ and $(s_+, s_-) \in C_{k+1}$ for a fixed $k \geq 1$, or (ii) $r_\pm \geq 0$ and $(s_+, s_-) \in C_1$, or (iii) $s_\pm \leq 0$.

Then the BVP (8)-(9) has a solution.

Theorem 2. Suppose that there are real numbers $r_+ > k^2(k+1)^2\pi^2/T^2$, and $r_-$ such that $r_\pm \leq \alpha_\pm(t)$ a.e. on I with strict inequality signs for t in a subset of positive measure. Suppose also $\beta_- \leq (k+1)^2\pi^2/T^2$ a.e. on I. Assume for some $k \geq 1$, that $(r_+,r_-) \in C_k$. Then, provided $\alpha_\pm$ and $\beta_\pm$ are integrable, the BVP (8)-(9) has a solution.

We will only outline the proof of Theorem 1. The proof of Theorem 2 is similar. To prove Theorem 1. we follow the argument in [3]. Let R be the rectangle $[r_+,s_+] \times [r_-,s_-]$, and let $(c_+,c_-)$ be the centre of R. Consider the homotopy

$$u'' + \lambda f(t,u) + (1-\lambda)(c_+u^+ - c_-u^-) = \lambda h(t) \tag{10}$$

$(0 \leq \lambda \leq 1)$. If (10)-(9) possesses an unbounded set of solutions, then there exists a nontrivial solution v of the BVP

$$v'' + g_+(t)v^+ - g_-(t)v^- = 0 \text{ a.e. on I},$$

$$v(0) = v(T) = 0, \quad v'(0) = v'(T) = 1,$$

where $g = (g_+,g_-)$ is a suitable map which verifies $g(t) \in R \sim \partial R$ on I. This can be proved by a mainly technical modification of the argument used in [3] , and it is a contradiction with the results of Lemma 1. and Lemma 2.

Since (10)-(9) can be rewritten as a homotopy of compact perturbations of the identity on a ball with centre 0 in $L^1(I;\mathbb{R})$, the Leray-Schauder degree is defined for our problem. We can see directly that this degree is odd when $\lambda = 0$.

The reader can easily obtain versions of the preceding theorems for BVP (2)-(9) following, for example, the method used in [3] to 'eliminate' the first derivative u' from the linear part of the equation.

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