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NON-LINEAR VARIATIONAL INEQUALITIES AND THE EXISTENCE OF EQUILIBRIUM IN ECONOMIES WITH A RIESZ SPACE OF COMMODITIES

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Abstract: Using the concept of a variational inequality, we give a new proof of the Aliprantis-Brown theorem on the existence of equilibria in economies with a Riesz space of commodities.

Key words: Variational inequalities, monotone, hemicontinuous, Riesz dual system, equilibrium, excess demand function.

Classification : Primary 90A14
Secondary 46N05

Introduction. The object of this paper is to investigate the existence of equilibrium in an economy with a Riesz space of commodities. Riesz space methods have been used in economics by Aliprantis and Brown (1983). Their proof is based on a theorem by Ky Fan (1961) which generalizes the classic Knaster-Kuratowski-Mazurkiewicz theorem. The main idea of the proof given by Aliprantis and Brown is to define a "revealed preference" relation on the space of prices and then to use Ky Fan's theorem to show the existence of a maximal element for this ordering. This maximal element is then proved to be an equilibrium point.

The proof given in this paper is not based on Ky Fan's theorem. Instead we have shown that the existence of an equilibrium point for the economy is equivalent to the solution of a non-linear variational inequality which was first proved by Hartman and Stampacchia (1966) and Browder (1965) independently (see also

Tarafdar, 1977 and Mosco, 1976). For applications of non-linear variational inequalities we refer the reader to Hartman and Stampacchia (1966) and Mosco (1976).

1. Preliminaries. An ordered set is a non-empty set X with a binary relation $\leq$ defined on it that is reflexive, transitive and anti-symmetric.

A lattice is an ordered set such that sup $\{x,y\}$ and inf $\{x,y\}$ exist for each pair , x,y in X.

An ordered vector space $(L,\leq)$ is a vector space L over the reals such that L is an ordered set and $f\leq g$ implies $f + h\leq g + h$ for all h in L and $\alpha f \leq \alpha g$ for all $\alpha \geq 0$.

An ordered vector space L which is also a lattice is said to be a Riesz space. The set $L^+ = \{f \in L/f\geq 0\}$ is called the positive cone of L.

Let L be a Riesz space. Then for $f\in L$ we put $f^+ = f\vee 0$, $f^- = (-f)\vee 0$ and $|f| = f \vee (-f)$ where $x \vee y$ is the supremum of the two elements x and y.

A linear functional $f:L \rightarrow$ Reals is said to be order-bounded whenever f maps order-intervals of the form $[-u,u] = \{a\in L/-u \leq a \leq u\}$, where $u\in L^+$, into bounded subsets of the real line. The vector space of all order-bounded linear functionals on L is called the order-dual of L and is denoted by $L^\sim$. In $L^\sim$, an ordering $\geq$ is introduced by saying $f\geq g$ whenever $f(u)\geq g(u)$ for all $u\in L^+$ The proof of the following result can be found in Aliprantis and Burkinshaw (1981, pp. 189-190).

Theorem 1.1 (Riesz) If L is a Riesz space, then its order dual $L^\sim$ is also a Riesz space. If $f\in L$ and $u\in L^+$ then

$$f^+(u) = \sup \{f(v)/0\leq v\leq u\}$$

$$f^-(u) = \sup \{-f(v)/0\leq v\leq u \quad \text{and} \quad |f|(u) = \sup \{f(v)/|v|\leq u\}.$$

Let $L_+^\sim$ denote the positive cone of $L^\sim$. Its members are called positive linear functionals on L. f belongs to $L_+^\sim$ if and only if $f(u) \geq 0$ for all u in L^+. f is strictly positive ($f >> 0$) if $u>0$ implies $f(u) > 0$.

A proof of the following theorem can be found in Aliprantis and Burkinshaw (1981, pp. 190-191).

Theorem 1.2. Let L be a Riesz space and let $f \in L^\sim$ such that $f \geq 0$. Then for every $x \in L$, $f(x^+) = \sup\{g(x)/g \in L^\sim, 0 \leq g \leq f)$, $f(x^-) = \sup\{-g(x)/g \in L^\sim, 0 \leq g \leq f\}$ and $f(|x|) = \sup\{g(x)/|g| \leq f\}$.

An ideal or order-ideal A of a Riesz space L is a vector subspace of L such that $|f| \leq |g|$ and $g \in A$ imply $f \in A$.

If L is a Riesz space and L' an ideal of $L^\sim$ separating the points of L, then the dual pair (L,L') is called a Riesz dual system.

Let (L,L') be a Riesz dual system. A price-simplex D for (L,L') is a non-empty, w^*-compact and convex subset of $L_+^\sim$. Here, w^* is the weak-star topology on $L^\sim$, i.e. it is the $w(L^\sim,L)$ topology. We assume that D satisfies the following condition:

($*$) The cone generated by $S = \{p \in L'_+ \cap D/p >> 0\}$ is w^*-dense in L'_+.

Let D be a price-simplex for a Riesz dual system (L,L'). An excess demand function E is a mapping $E: D \longrightarrow L$, satisfying the following condition (Walras'Law):

$$p\ E(p) = 0 \text{ for all } p \in D.$$

By an economy, we mean a Riesz dual system (L,L'), a price-simplex D for (L,L') and an excess demand function E defined on D. An economy is said to have an equilibrium price p if $E(p) \leq 0$ where $\leq$ is the Riesz order on the space L.

Let K be a subset of Hausdorff linear topological space F over the reals and T a single-valued (non-linear) mapping of K into F', the topological dual of F. Then a point u_0 is said to satisfy the <u>variational inequality</u> if

$$(T(u_0), v-u_0) \leq 0 \text{ for all } v \text{ in } K.$$

Here, (,) denotes the pairing between F' and F. u_0 is also called a solution of the variational inequality. The mapping T is said to be <u>monotone</u> if $(T(u) - T(v), u-v) \leq 0$ for all u, v in K. T is said to be <u>hemicontinuous</u> if T is continuous from the line segments in K to the weak topology of F'

2. Existence of equilibrium

<u>Lemma 2.1</u>. Let (L,L') be a Riesz dual system and let $u \in L$. Then $u \geq 0$ holds if and only if $f(u) \geq 0$ for all $f \geq 0$ in L'.

<u>Proof</u>. The proof of the lemma is based on the two Riesz theorems cited in the preliminary remarks (see Aliprantis and Brown (1983, Theorem 2.2)).

We now prove the following important result.

<u>Theorem 2.1</u>. Any point p in D is an equilibrium price for $((L,L'),D,E)$ if and only if p is a solution of the variational inequality.

<u>Proof</u>. Suppose that $E(p) \leq 0$ for some p. Then $qE(p) \leq 0$ for all q in D since q is a positive linear functional. This implies that $qE(p) \leq pE(p)$ since by Walras'law $pE(p) = 0$ for all p. Consequently, $(E(p),p-q) \leq 0$, or, equivalently, $(E(p),q-p) \leq 0$ for all q in D and p solves the variational inequality. Thus an equilibrium price p solves the variational inequality.

Conversely, suppose that p is a solution of the variational inequality. Then $(E(p),p-q) \geq 0$ which implies that $0 = E(p) \cdot p \geq E(p) \cdot q$

for all q in D, where the first equality holds by Walras´law. Hence, $E(p)\cdot q \leq 0$ for all q in L'_+ by the density condition (∗). Now Lemma 2.1 implies that $E(p) \leq 0$ so that p is an equilibrium price.

We now prove the existence of equilibrium.

Theorem 2.2. Let $((L,L'),D,E)$ be an economy. Then there exists an equilibrium price for this economy, if either one of the following conditions holds:

1) $E:(D,w^*) \to (L,w(L,L')$ is continuous.

2) E is hemicontinuous and monotone.

Proof. Suppose first that E is continuous. Since $E:(D,w^*) \to (L,w(L,L'))$ and D is w^*-compact and convex, E satisfies the conditions of Browder´s theorem 2 (Browder, 1968, p. 286). We conclude that there exists a p such that $(E(p),p-q) \geq 0$ for all q in D which implies that $(E(p),q-p) \leq 0$ for all q in D so that p solves the variational inequality. Theorem 2.1 now implies that p is an equilibrium price.

Suppose now that E is hemicontinuous and monotone. Again, since D is w^*-compact and convex, E satisfies the conditions of the corollary of Theorem 2 of Tarafdar (1977). We conclude as above that there exists an equilibrium price for this economy.

We now consider the existence of equilibrium prices for a more general class of economies. Suppose that (L,L') is a Riesz dual system and that D is a price-simplex for (L,L'). We now suppose that the domain D' of E is a subset of D. An excess demand function E is now defined to be a mapping $E:(D',w^*) \to (L,w(L,L')$ which satisfies the following properties:

a) Density condition. D' is a w^*-dense convex subset of D.

b) Walras´law. $pE(p) = 0$ for all p in D'

(c) Boundary condition. If p_n is a net in D' which converges to q in $D \setminus D'$, then there exists a p in D' such that the upper limit $\overline{\lim}\ p(E(p_n)) > 0$.

We now prove the following theorem of Aliprantis and Brown (1983) without using the concept of a maximal element for the "revealed preference" relation on the space of prices.

<u>Theorem 2.3</u>. Let $((L,L'),D,E)$ be an economy. Then there exists an equilibrium price for this economy if E is continuous.

<u>Proof</u>. Let A denote the collection of all the finite subsets of D'. For each $a \in A$, let D_a be the convex hull of a. Each D_a is w^*-compact, and the restriction of E to D_a is continuous so that Theorem 2.2 implies the existence of an equilibrium price p_a for the economy $((L,L'),D_a,E)$. Since p_a is an equilibrium price for D_a, p_a solves the variational inequality $(E(p_a),q-p_a) \leq 0$ by Theorem 2.1. This implies that $q\ E(p_a) \leq 0$ for all q in D_a.

Although the rest of the argument is similar to that in Aliprantis and Brown (1983, Theorem 3.6) we include it for the sake of completeness. Consider the net $\{p_a : a \in A\}$ where A is directed by inclusion. Since D is w^*-compact we may assume that $p_a \longrightarrow q$ in the w^*-topology.

We show first that $q \in D'$. If $q \in D \setminus D'$, then by the boundary condition on the excess demand function there exists a $p \in D$ with $\overline{\lim}\ p(E(p_a)) > 0$. Since $\bigcup_{a \in A} D_a = D'$, $p \in D_a$ for some a, so that there exists $b \in A$ such that $p \in D_a$ for all $a \gtrsim b$. But then for all $a \gtrsim b$ $p\ E(p_a) \leq 0$ since p_a is an equilibrium price so that $\overline{\lim}\ p\ E(p_a) \leq 0$, a contradiction. Thus $q \in D'$

We now show that q is an equilibrium price. To this end let $p \in D'$. The function $p \cdot E(r)$ from (D,w^*) to the reals is continuous as a composite of two continuous functions. It follows that

$$p\ E(q) = w^*\text{-lim}\ p\ E(p_a) \text{ since } p_a \xrightarrow{w^*} q.$$

As above, there exists $b \in A$ satisfying $p\ E(p_a) \leq 0$ for all $a \geq b$, and so $p\ E(q) \leq 0$. This is true for all $p \in D'$.

Now the density condition on the excess demand function implies that $p\ E(q) \leq 0$ for all $p \in D$ since D' is w^*-dense in D. Since D is a price simplex, the density condition $*$ implies that $p\ E(q) \leq 0$ for all $p \in L'_+$. The lemma now implies that $E(q) \leq 0$ so that q is an equilibrium price.

Remark. Theorem 2.3 can be obtained in the same manner under the assumption of monotonicity of the excess demand function E if the hemicontinuity condition on E is strengthened in the following way. Let $\{p_a : a \in A\}$ be a net in D' with $p_a \in D_a$ and D_a being as described in the above proof. Then we need to assume that if if $p_a \xrightarrow{w^*} p$ in D' and for each a, $(E(p_a), q-p_a) \leq 0$ for all q in D_a then it follows that $(E(p), q-p) \leq 0$ for all q in D'

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