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ATOMS IN THE FAMILY OF COREFLECTIVE SUBCATEGORIES OF UNIFORM SPACES

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Abstract: The present work complements and extends the work of Tashjian and Vilímovský by identifying two classes $\{P(m):m \geq \aleph_0\}$ and $\{T(m):m \geq \aleph_1\}$ which, in conjunction with the unit interval $[0,1]$, are fundamental for the study of coreflective subcategories of uniform spaces. Using these classes, the following results are established:

The class of zero-dimensionally proximally fine spaces is the largest coreflective subcategory of uniform spaces which contains only trivial proximally discrete spaces.

The class of Baire-Fine spaces is the largest coreflective subcategory of uniform spaces disjoint from the class $\{[0,1]\} \cup \cup \{P(m):m \geq \aleph_0\}$.

Every non-uniformly discrete space inductively generates one of the spaces $[0,1]$, $P(m)$, or $T(m)$.

Each one of the spaces $[0,1]$, $P(m)$, or $T(m)$ inductively generates a coreflective subcategory which is an atom: the only proper coreflective subcategory is the class of uniformly discrete spaces.

Key words: Uniform space, coreflective subcategory, proximally discrete, proximally fine, Baire-Fine, atom.

Classification: 18B30, 54E15

Section one: definitions and notation. $[0,1]$ (respectively R) will denote the unit interval (the real line) with the usual metric uniformity.

For each pair of uniform spaces X and Y, the collection of uniformly continuous mappings $X \longrightarrow Y$ will be denoted by $U(X,Y)$. If $Y = R$ with the usual uniformity, then $U(X,Y)$ is denoted by $U(X)$.

Unif will denote the category of separated uniform spaces and uniformly continuous mappings. X is <u>uniformly discrete</u> if $\{\{x\}:x\in X\}$ is a uniform cover. The class of uniformly discrete spaces will be denoted by Unif Discrete. X is <u>proximally discrete</u> if every finite cover is uniform. The class of proximally discrete spaces will be denoted by Prox Discrete.

Assume $\mathcal{A}\subset$ Unif. Then coref ($\mathcal{A}$) will denote the coreflective hull of the class $\mathcal{A}$, that is, coref ($\mathcal{A}$) consists of all uniform spaces which are quotients of sums of members from $\mathcal{A}$. It is well known that Y is a member of coref ($\mathcal{A}$) if and only if the following condition is satisfied: for any uniform space Z, $f:Y\to Z$ is uniformly continuous if $f\circ h:A\to Z$ is uniformly continuous for every uniformly continuous mapping $h:A\to X$, where A is a member of $\mathcal{A}$. Sometimes the phrase "X generates Y" will be used instead of writing $Y\in$ coref (X).

Assume that Y and Y' are uniform structures on the same set such that the identity mapping $Y'\to Y$ is uniformly continuous. Then $Y - Y'$ denotes the class of uniform spaces X such that $U(X,Y')=$ $= U(X,Y')$. One can verify that $Y - Y'$ is a coreflective subcategory of Unif.

Coz (X) denotes the family of sets of the form $\{x:f(x)\neq 0\}$, where f belongs to U(X). Baire (X) denotes the smallest σ-field containing Coz (X). The members of Baire (X) are called Baire sets. A family $\mathcal{S}$ of Baire sets is <u>completely additive</u> if the union of each subfamily is a Baire set. For uniform spaces X and Y, a mapping $X\to Y$ is <u>measurable</u> if the pre-image of every Baire set is a Baire set. X is <u>finitely measurable</u> if every finite Baire cover is a uniform cover. The class of finitely measurable spaces will be denoted by Fin-Meas. X is <u>Baire-Fine</u> if every measurable mapping to another uniform space is uniformly continuous. It is well known that X is Baire-Fine if and only if every completely additive

Baire partition is a uniform cover. The class of Baire-Fine spaces will be denoted by Baire-Fine.

Given a uniform space Y, pY denotes the uniform structure on Y with the basis consisting of all finite uniform covers of Y. A uniform space X is proximally fine if U(X,pY) = U(X,Y) for every space Y. We say that X is zero-dimensionally proximally fine if the preceding statement is valid for every uniform space Y with a basis of uniform partitions. It is clear that this property is weaker than being proximally fine; for example, every uniform space with a connected topology is zero- dimensionally proximally fine.

For each infinite cardinal m, P(m) denotes the uniform structure on the set [0,m) consisting of all partitions of power $< m$ and D(m) denotes the uniformly discrete structure on the set [0,m).

Section two: Largest coreflective subcategories

Theorem 1: ([TV]) Fin-Meas is the largest coreflective subcategory of Unif which does not contain [0,1].

Theorem 1 may be restated in the following form: X generates [0,1] if and only if X is not finitely measurable.

Theorem 2: For each $m \geq \aleph_0$, P(m)-D(m) is the largest coreflective subcategory of Unif which does not contain P(m).

Proof: Clearly, P(m) is not a member of P(m)-D(m). Assume that X is a uniform space such that $U(X,P(m)) \neq U(X,D(m))$. We will show that X generates P(m); then it will follow that $P(m) \notin$ coref (X) implies $X \in$ P(m)-D(m).

Assume that $f: P(m) \to Y$ is not uniformly continuous. By definition of P(m), there exists a uniformly discrete subset D = $= \{ y_j : j \in J\}$ of f[P(m)] such that $|J| = m$. For each $j \in J$, choose

x_j such that $f(x_j) = y_j$ and define $S = \{x_j : j \in J\}$. Since $|S| = m$, S is uniformly equivalent to P(m), so by assumption there exists a uniformly continuous mapping $g:X \to S$ such that $g:X \to D(S)$ is not uniformly continuous, where D(S) denotes the uniformly discrete space on the set S. Since $f|S:D(S) \to D$ is a uniform equivalence, $f \circ g:X \to Y$ is not uniformly continuous. It follows that X generates P(m).

Corollary 1: Let X be a uniform space. The following statements are equivalent for each $m \geq \aleph_0$:

(i) X generates P(m).

(ii) There exists a non-uniform partition $\{V_k : k \in K\}$ of X, $|K| = m$, satisfying the following condition:

(#) If $K = \bigcup\{K_t : t \in T\}$, $|T| < m$, then $\{B_t : t \in T\}$ is a uniform cover, where $B_t = \bigcup\{V_k : k \in K_t\}$.

Proof: We will first show $\neg$ (i) $\to$ $\neg$ (ii). Assume $P(m) \notin$ coref (X). Then by Theorem 2, X is a member of P(m)-D(m). Suppose that $\{V_k : 0 \leq k < m\}$ is a partition of X of power m which satisfies (#). Define $f:X \to P(m)$ by $f(x) = k$ if $x \in V_k$. Condition (#) implies that f is uniformly continuous, so $f:X \to D(m)$ is also uniformly continuous. Therefore, $\{f^{-1}(k) : 0 \leq k < m\} = \{V_k\}$ is a uniform cover of X, so we have established $\neg$ (ii).

(i) $\to$ (ii). The identity mapping $i:P(m) \to D(m)$ is not uniformly continuous, so by (i) there exists a uniformly continuous mapping $f:X \to P(m)$ such that $f:X \to D(m)$ is not uniformly continuous. Then $\{f^{-1}(k) : 0 \leq k < m\}$ is a non-uniform partition of X which satisfies (#), so (ii) holds.

Remark: If X is a non-uniformly discrete proximally discrete space on the natural numbers N, it follows from Corollary 1 that X generates $P(\aleph_0)$. However, $P(\aleph_0)$ need not be a quotient of X. Let $\mathcal{F}$ be a free ultrafilter on N and let $u_{\mathcal{F}}$ be the uniformity on N having the following basis of partitions:

$$\mathcal{U}(F) = \{\{x\}: x \notin F\} \cup \{F\},\ F \in \mathcal{F}.$$

Since $\mathcal{F}$ is a free ultrafilter, $u_{\mathcal{F}}N$ is proximally discrete, but not uniformly discrete. Assume that $Q: u_{\mathcal{F}}N \longrightarrow P(\aleph_0)$ is an onto mapping and define $\mathcal{G} = \{G \subset N: Q^{-1}(G) \in \mathcal{F}\}$. Then $\mathcal{G}$ is an ultrafilter on N and $Q: u_{\mathcal{F}}N \longrightarrow u_{\mathcal{G}}N$ is uniformly continuous, but the identity mapping $i: P(\aleph_0) \longrightarrow u_{\mathcal{G}}N$ is not uniformly continuous since $u_{\mathcal{G}}N$ is not precompact. Hence Q is not a quotient mapping.

<u>Definition</u>: $\mathcal{T} = \cap\{P(m)-D(m): m \geq \aleph_0\}$.

<u>Theorem 3</u>: (i) $\mathcal{T}$ is the largest coreflective subcategory of Unif disjoint from the class $P(m): m \geq \aleph_0\}$.

(ii) $\mathcal{T}$ is the class of zero-dimensionally proximally fine spaces.

Proof: (i) By definition, $\mathcal{T}$ is disjoint from the class $\{P(m): m \geq \aleph_0$. Assume $\{P(m): m \geq \aleph_0\} \cap$ coref $(X) = \emptyset$ *for some* uniform space X. Then by Theorem 2, for each m, $P(m) \notin$ coref (X) implies that X is a member of P(m)-D(m), so X is a member of $\mathcal{T}$.

(ii) Assume that X is zero-dimensionally proximally fine. If $f: X \longrightarrow P(m)$ is uniformly continuous, then $f: X \longrightarrow pD(m)$ is also uniformly continuous, so $f: X \longrightarrow D(m)$ is uniformly continuous. Therefore, X is a member of $\mathcal{T} = \cap\{P(m)-D(m): m \geq \aleph_0$.

Conversely, suppose X is a member of $\mathcal{T}$ and $f: X \longrightarrow pY$ is uniformly continuous, where Y has a basis of uniform partitions. If $f: X \longrightarrow Y$ is not uniformly continuous, choose a uniform partition $\{V_k: 0 \leq k < m\}$ of Y of minimal cardinality m such that $\{f^{-1}(V_k)\}$ is not a uniform cover of X. Define $h: Y \longrightarrow P(m)$ by $h(y) = k$ if $y \in V_k$. Then $h \circ f: X \longrightarrow P(m)$ is uniformly continuous, so by assumption $h \circ f: X \longrightarrow D(m)$ is also uniformly continuous. It follows that $\{(h \circ f)^{-1}(k): 0 \leq k < m\} = \{f^{-1}(V_k)\}$ is a uniform cover of X, which is a contradiction . Therefore, $f: X \longrightarrow Y$ is uniformly continuous, so X is zero-dimensionally proximally fine.

Corollary 2: $\mathcal{F}$ is the largest coreflective subcategory of Unif such that $\mathcal{F} \cap$ Prox Discrete = Unif Discrete.

Proof: It is easy to verify that $\mathcal{F} \cap$ Prox Discrete = Unif Discrete. Suppose that Prox Discrete $\cap$ coref (X) = Unif Discrete for some uniform space X. Then $P(m) \notin$ coref (X) for every $m \geq \aleph_0$, so by Theorem 3(i), X is a member of $\mathcal{F}$, which establishes the result.

Corollary 3: Baire-Fine is the largest coreflective subcategory of Univ disjoint from the class $\{[0,1]\} \cup \{P(m): m \geq \aleph_0\}$.

Proof: A uniform space X is Baire-Fine if and only if it is measurable and proximally fine ($[Ha]_2$, 5.2). Since every precompact measurable space is finite, [0,1] is not Baire-Fine. Since every proximally fine proximally discrete space is uniformly discrete, no P(m) is proximally fine. Hence Baire-Fine is disjoint from the given class. Now suppose that $\{[0,1]\} \cup \{P(m): m \geq \aleph_0\} \cap$ $\cap$ coref (X) = $\emptyset$ for some uniform space X. Since [0,1] is not a member of coref (X), it follows from Theorem 1 that X is finitely measurable and since no P(m) is a member of coref (X), it follows from Theorem 3 that X is a member of $\mathcal{F}$. Let $\{B_k: k \in D\}$ be a completely additive Baire partition of X and define $f: X \longrightarrow D$ by $f(x) =$ = k if $x \in B_k$, where D is a uniformly discrete space. Since X is finitely measurable and B_k is a completely additive Baire family, $f: X \longrightarrow pD$ is uniformly continuous. Since X is a member of $\mathcal{F}$, Theorem 3(ii) implies that $f: X \longrightarrow D$ is uniformly continuous.

Therefore, $\{B_k\} = \{f^{-1}(k):k\in D\}$ is a uniform cover of X. Hence X is Baire-Fine and Corollary 3 is established.

Corollary 4: The following statements are equivalent for a uniform space X:

(i) $\{[0,1],P(\aleph_0)\}\cap$ coref (X) = $\emptyset$.

(ii) Every precompact member of coref (X) is finite.

(iii) Every countable Baire cover of X is uniform.

(X is separably-measurable in the sense of $[\mathrm{Ha}]_1$.)

Proof: Clearly, (ii)$\longrightarrow$(i). The implication (iii)$\longrightarrow$(ii) follows from the facts that (a) the separably measurable spaces are a coreflective subcategory of Unif and (b) the precompact measurable spaces are finite. The implication (i)$\longrightarrow$(iii) is established by an argument analogous to the one presented in the second part of the proof of Corollary 3.

Corollary 5: The separably-measurable spaces are the largest coreflective subcategory of Unif disjoint from $\{[0,1],P(\aleph_0)\}$.

Proof: Corollary 5 follows at once from Corollary 4.

Section three: Atoms

Lemma: Assume X is Baire-Fine and every disjoint family of Baire sets is completely additive. Then X is uniformly discrete.

Proof: Let $\{S_k:0\leq k<m\}$ be a family of Baire sets of power m. Inductively, define $T_0 = S_0$, $T_1 = S_1 - T_0,\ldots, T_j = S_j - \cup\{T_k:0\leq k<j\}$ for $j<m$. Assume that T_j is a Baire set for every $j<n$, where $n<m$. Since $\{T_j:j<m\}$ is a disjoint family, by assumption $\cup\{T_j:j<m\}$ is a Baire set, so $T_n = S_n - \cup\{T_j:j<n\}$ is also a Baire set. Therefore, T_j is a Baire set for every $j<m$, so by

assumption, $\cup\{T_j : j < m\} = \cup\{S_j : j < m\}$ is a Baire set. It follows that Baire(X) is closed under the formation of arbitrary unions and intersections. Since the members of Baire(X) separate distinct pairs of points, every singleton subset is a Baire set, so $\mathcal{U} = \{\{x\} : x \in X\}$ is a completely additive Baire partition. Since X is Baire-Fine, $\mathcal{U}$ is a uniform cover and X is uniformly discrete.

For each regular cardinal $m \geq \aleph_1$, T(m) denotes the uniform structure on the set $[0,m]$ with the basis of uniform covers of the form

$$\{\{x\} : x < \alpha\} \cup \{[\alpha, m]\}, \quad \alpha < m.$$

One can verify that every T(m) is a Baire-Fine space and that

$$\text{Baire}(T(m)) = \{A : (m \notin A \text{ and } |A| < m) \text{ or } (m \in A \text{ and } |A^c| < m)\}.$$

For the remainder of the paper, we will assume that the cardinal m satisfies the preceding restriction when referring to T(m).

Theorem 4: Every non-uniformly discrete Baire-Fine space X generates T(m), for some $m > \aleph_1$.

Proof: Assume that X is a non-uniformly discrete Baire-Fine space. Define $m = \inf\{|\mathcal{S}| : \mathcal{S}$ is a disjoint Baire family and $\cup\mathcal{S} \notin \text{Baire}(X)\}$. By the preceding lemma, m is a well-defined uncountable cardinal and one can verify that m is regular. We claim that X generates T(m).

Assume that $f : T(m) \longrightarrow Y$ is not uniformly continuous. Since T(m) is Baire-Fine, f is not measurable, so there exists $B \in$ Baire(Y) such that $A = f^{-1}(B) \notin \text{Baire}(T(m))$. Without loss of generality, assume that $m \notin A$ and $|A| = m$. Choose a disjoint Baire family $\{B_k : k \in K\}$ such that $|K| = m$ and $\cup\{B_k : k \in K\} \notin$ Baire(X). Let $e : K \to A$ be a bijection and define $h : X \longrightarrow T(m)$ by

$$h(x) = \begin{cases} e(k) & x \in B_k \\ m & x \notin \cup\{B_k : k \in K\}. \end{cases}$$

We claim that h is measurable, and therefore uniformly continuous, since X is Baire-Fine. Suppose $H \in \text{Baire}(T(m))$ and $m \notin H$. Since $|H| < m$, $h^{-1}(H) = \cup\{B_k : e(k) \in H\}$ is the disjoint union of fewer than m Baire sets; hence $h^{-1}(H) \in \text{Baire}(X)$, so h is measurable. On the other hand, $(f \circ h)^{-1}(B) = h^{-1}(A) = \cup\{B_k : k \in K\} \notin \text{Baire}(X)$, so $f \circ h$ is not measurable, and hence is not uniformly continuous. Therefore, X generates T(m).

<u>Remark</u>: The proofs of the Lemma and Theorem 4 show that a Baire-Fine space X generates T(m) if Baire(X) is n-additive for every $n < m$, but not m-additive.

<u>Corollary 6</u>: Every non-uniformly discrete space X generates one of the following spaces:

(i) [0,1]

(ii) P(m), for some $m \geq \aleph_0$.

(iii) T(m), for some $m \geq \aleph_1$.

Proof: Assume that X is a non-uniformly discrete space and define $\mathcal{S} = \{[0,1]\} \cup \{P(m) : m \geq \aleph_0\}$. If $\mathcal{S} \cap \text{coref}(X) \neq \emptyset$, then either (i) or (ii) holds. If $\mathcal{S} \cap \text{coref}(X) = \emptyset$, then by Corollary 3, X is Baire-Fine, so by Theorem 4, X generates T(m), for some $m \geq \aleph_1$.

<u>Definition</u>: A coreflective subcategory $\mathcal{C}$ of Unif is an <u>atom</u> if the only proper coreflective subcategory of $\mathcal{C}$ is Unif Discrete.

<u>Corollary 7</u>: Unif has exactly the following coreflective atoms:

(i) $c\ell$ = coref ([0,1])

(ii) $\mathcal{P}(m)$ = coref (P(m)), $m \geq \aleph_0$

(iii) $\mathcal{T}(m)$ = coref (T(m)), $m \geq \aleph_1$.

Proof: Corollary 6 shows that the subcategories listed above are the only possible coreflective atoms. We will now show that each subcategory is an atom.

(i) Suppose that [0,1] generates the non-uniformly discrete space X. If X is finitely measurable, it has a P-space topology, so every uniformly continuous mapping $[0,1] \longrightarrow X$ is constant. Therefore, X is uniformly discrete. Hence X is not finitely measurable, so by Theorem 1, X generates [0,1].

(ii) Suppose P(m) generates the non-uniformly discrete space X. Since Prox Discrete is a coreflective subcategory, X cannot generate any member of the class $\{[0,1]\} \cup \{T(m): m \geq \aleph_1\}$; hence by Corollary 6, X generates some P(n). Therefore, $P(n) \in$ coref (P(m)), which implies that m = n. (If $n < m$, $P(n) \in$ coref (P(m)) implies that P(n) admits n and hence is uniformly discrete. If $n > m$, the range of every uniformly continuous mapping $P(m) \longrightarrow P(n)$ is uniformly discrete, which also implies that P(n) is uniformly discrete.) Hence X generates P(m).

(iii) Suppose T(m) generates the non-uniformly discrete space X. Then X is Baire-Fine, so Theorem 4 implies that X generates T(n), for some $n \geq \aleph_1$. Therefore, $T(n) \in$ coref(T(m)), which implies that m = n. (If $n < m$, $T(n) \in$ coref(T(m)) implies that T(n) admits n and hence is uniformly discrete. If $n > m$, the range of every uniformly continuous $T(m) \longrightarrow T(n)$ is uniformly discrete, which also implies that T(n) is uniformly discrete.) Hence X generates T(m).

<u>Remark</u>: One can also establish that the intersection of any

two distinct coreflective atoms is exactly the class of uniformly discrete spaces.

Section four: Topological results. Let Tychonov denote the category of completely regular Hausdorff spaces and continuous mappings. In this setting, T(m) will refer to the topological space with the induced uniform topology: for each $\alpha < m$, the singleton set $\{\alpha\}$ is open, and the set V is a neighborhood of m if $|T(m) - V| < m$. (The uniform space T(m) is the fine space associated with this topology.)

Corollary 8: Let X be a non-discrete Tychonov space. Then X generates [0,1] if and only if X is not a P-space.

Proof: Consider X as a fine uniform space. Then X is finitely measurable if and only if X is topologically a P-space, so Corollary 8 follows from Theorem 1.

Corollary 9: Every non-discrete P-space X generates T(m), for some $m \geq \aleph_1$.

Proof: Consider X as a fine uniform space. Since X has a P-space topology, X is Baire-Fine, so by Theorem 4, in Unif X generates some T(m). Since X and T(m) are fine spaces, it follows that X generates T(m) in Tychonov.

Using the analogous definition of a coreflective atom in Tychonov, we can also establish the following result.

Corollary 10: Tychonov has exactly the following coreflective atoms:

(i) coref ([0,1])

(ii) coref (T(m)). $m > \aleph$

Proof: Suppose that [0,1] generates the non-discrete space X. Then X is not a P-space, so by Corollary 8, X generates [0,1].

(ii) Suppose that T(m) generates the non-discrete space X. Then in Unif, T(m) generates the fine uniform space X. Since T(m) is a P-space, X is also a P-space, so it is Baire-Fine. Hence by Theorem 4, X generates some T(n). Therefore, T(m) generates T(n). The argument used in the proof of part (iii) of Corollary 7 shows that m = n, so X generates T(m).

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