Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 3, 499--514

Persistent URL: http://dml.cz/dmlcz/106389

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE
26,3 (1885)

NOTES ON REVEALED CLASSES

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Abstract: We construct a revealed class X such that P(X) is not revealed and furthermore we show that there are two fully revealed classes so that their intersection is no fully revealed class.

Key words: Alternative set theory, revealed and fully revealed class, set-theoretically definable class, real class, similarity, automorphism, revealment.

Classification: Primary 03E70
Secondary 03H15

One of the important notions of the alternative set theory (cf. [V]) is the property "to be revealed"; this notion corresponds in some aspects to the saturation property in the model theory.

A class X is called revealed if for every countable class $Y \subseteq X$ there is a set u with $Y \subseteq u \subseteq X$ (cf. § 5 ch. II [V]); a class X is fully revealed if for every normal formula $\varphi(z,Z)$ (even formal one - element of FL, see below), the class $\{z; \varphi(z,X)\}$ is revealed (cf. § 2 [S-V 1]).

At first we are going to summarize results describing the system of revealed classes - e.g. we describe some properties fulfilled by revealed classes and show under which of the most frequent operations the investigated system of classes is closed. A nontrivial result in this area is that $P(X) = \{x; x \subseteq X\}$

need not be revealed even in the case that X is revealed.

The system of all fully revealed classes is closed under Gödel's operations having one free variable by the definition; e.g. $V-X$, $\mathrm{dom}(X)$, $V\times X$, X^{-1}, $\mathrm{Cnv}_3(X) = \{\langle x,y,z\rangle; \langle y,z,x\rangle \in X\}$ and furthermore $\bigcup X$ and $P(X)$ are fully revealed under the assumption that X is fully revealed. On the other hand, also trivially, this system of classes is not closed under operations working with countably many classes - e.g. for every $n \in FN$, the class $N-n$ is fully revealed, but the class $N-FN = \bigcap\{N-n; n\in FN\}$ is not fully revealed (being revealed) because FN is not revealed.

It is not so trivial to answer the question whether the system of fully revealed classes is closed even under Gödel's operations having two free variables. In this article we are going to show that it is not, unfortunately, the case - we shall see that the intersection of two fully revealed classes need not be fully revealed. Constructing a pair of such classes we shall prove some statements which seem to be interesting themselves.

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Let us note that a class X is revealed iff there is no function f with $FN = \{\alpha \le \mathrm{dom}(f); f''\alpha \subseteq X\}$. In fact, for every countable Y there is a one-one mapping f with $Y = f''FN$ & & $\mathrm{dom}(f) \in N$ by the prolongation axiom; supposing $Y \subseteq X$ and $\neg(\exists u)(Y \subseteq u \subseteq X)$ we get $FN = \{\alpha \le \mathrm{dom}(f); f''\alpha \subseteq X\}$. On the other hand, assuming $FN = \{\alpha \le \mathrm{dom}(f); f''\alpha \subseteq X\}$ and $f''Fn \subseteq u \subseteq$ $\subseteq X$, we obtain $FN = \{\alpha \le \mathrm{dom}(f); f''\alpha \subseteq u\}$ from which Set(FN) follows - contradiction. Thence X is not fully revealed iff there is a normal formula $\varphi(z,Z)$ with set-parameters only (cf. the eighth theorem of § 2 [S-V 1]) so that $FN = \{z; \varphi(z,X)\}$.

Thus the above mentioned result can be reformulated in the following way: there are classes X, Y such that there is no normal formula $\varphi(z,Z)$ with $FN = \{z; \varphi(z,X)\} \vee FN = \{z; \varphi(z,Y)\}$ but there is a normal formula $\psi(z,Z_1,Z_2)$ with $FN = \{z; \psi(z,X,Y)\}$ (φ, ψ with set parameters only). In this formulation our result is not so surprising.

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We use the notation usual in the alternative set theory (cf. [V]); in particular, variables $F,G,f,g,\dots$ run through functions; the symbol $F \circ G$ denotes the composition of F and G.

Let us recall some definitions.

A pair of classes K, S codes a system of classes $\mathfrak{M}$ if $(\forall X)(X \in \mathfrak{M} \equiv (\exists q \in K)\ X = S''\{q\})$; a system of classes is codable if there is a pair coding it.

A formula is normal if no class variable is quantified in it; a formula is called set-formula if there are only set-variables and set-constants in it. We define formal formulae in the alternative set theory as usual and the class of all formal formulae (without constants) which are elements of FN is denoted by the symbol FL. The symbol FL_V denotes the class of formulae of FL in which set-parameters are allowed.

A class X is called set-theoretically definable (element of Sd_0 resp.) if there is a set-formula $\varphi \in FL_V$ ($\varphi \in FL$ resp.) with $X = \{z; \varphi(z)\}$. X is a π-class (σ-class resp.) if it is the intersection (union resp.) of a countable sequence of set-theoretically definable classes.

F is a similarity if for each set-formula $\varphi(z_1,\dots,z_n) \in FL$ and for each $x_1,\dots,x_n \in dom(F)$ we have

$$\varphi(x_1,\dots,x_n) \equiv \varphi(F(x_1),\dots,F(x_n));$$

a similarity whose domain and range is V is called an automorphism (see § 1 ch. V [V]).

A class X is said to be a revealment of a class Y if X is fully revealed and for every normal formula $\varphi(Z) \in FL$ we have $\varphi(X) \equiv \varphi(Y)$ (see § 2 [S-V 2]).

A class is called real if it is a figure in an indiscernibility equivalence (see § 1 [Č-V] and ch. III [V]).

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To obtain a complete picture of revealed classes let us recall results of § 5 ch. II [V]:

(a) If for every set u the intersection $X \cap u$ is a set, then X is revealed; in particular, each set-theoretically definable class is revealed.

(b) If X and Y are revealed, then $X \cap Y$ and $X \cup Y$ are revealed, too.

(c) If $\{X_n; n \in FN\}$ is a descending sequence of revealed classes, then $\bigcap \{X_n; n \in FN\}$ is also revealed and furthermore $dom(\bigcap \{X_n; n \in FN\}) = \bigcap \{dom(X_n); n \in FN\}$.

(d) If $\{X_n; n \in FN\}$ is a descending sequence of nonempty revealed classes, then $\bigcap \{X_n; n \in FN\} \neq 0$.

The most important result from the previously mentioned ones is the last one, it guarantees the importance of the notion of revealness.

The following statement expressing mainly that the system of the revealed classes is closed under union and all Gödel's operations except the complement is formulated for completeness only, its proof is quite trivial. It is necessary to emphasize that the complement of a revealed class need not be revealed; as an example can serve the revealed class N-FN.

<u>Theorem</u>. (a) If X is revealed, then $dom(X), X^{-1}$, $Cnv_3(X)$ and $\bigcup X$ are revealed, too.

(b) If X and Y are revealed classes, then also the class $X \times Y$ is revealed.

(c) If F is a revealed function, then even the class $\bigcup \{ F(x) \times \{x\}; x \in \mathrm{dom}(F)\}$ is revealed.

Proof. (a) If Y is a countable class with $Y \subseteq \mathrm{dom}(X)$ ($Y \subseteq X^{-1}$, $Y \subseteq \mathrm{Cnv}_3(X)$, $Y \subseteq \bigcup X$ resp.), then we are able to choose a countable class $Z \subseteq X$ with $\mathrm{dom}(Z) = Y$ ($Z = Y^{-1}$, $Y = \mathrm{Cnv}_3(Z)$, $(\forall y \in Y)(\exists z \in Z)\, y \in z$ resp.). Assuming X is revealed there is u with $Z \subseteq u \subseteq X$ and thus $Y \subseteq \mathrm{dom}(u) \subseteq \mathrm{dom}(X)$ ($Y \subseteq u^{-1} \subseteq X^{-1}$, $Y \subseteq \mathrm{Cnv}_3(u) \subseteq \mathrm{Cnv}_3(X)$, $Y \subseteq \bigcup u \subseteq \bigcup X$ resp.); $\mathrm{dom}(u)$, u^{-1}, $\mathrm{Cnv}_3(u)$ and $\bigcup u$ are sets by § 1 ch. I [V].

(b) If Z is a countable part of $X \times Y$ then $\mathrm{dom}(Z)$ and $\mathrm{rng}(Z)$ are countable, too, and hence supposing X, Y to be revealed there are u, v with $\mathrm{dom}(Z) \subseteq v \subseteq Y \,\&\, \mathrm{rng}(Z) \subseteq u \subseteq X$ from which $Z \subseteq u \times v \subseteq X \times Y$ follows.

(c) If Y is a countable subclass of $\bigcup \{ F(x) \times \{x\}; x \in \mathrm{dom}(F)\}$ then $\mathrm{dom}(Y) \subseteq \mathrm{dom}(F)$ is also countable and assuming that F is revealed we can choose $f \subseteq F$ with $\mathrm{dom}(Y) \subseteq \mathrm{dom}(f)$ and therefore $Y \subseteq \bigcup \{ f(x) \times \{x\}; x \in \mathrm{dom}(f)\}$ and the lastly mentioned class is a set.

Lemma. If Y is a revealment of FN then there is a revealed mapping F of Y into $N - FN$ with $\bigcap F''Y = FN$ (F being moreover decreasing i.e. $\alpha < \beta \in Y \rightarrow F(\alpha) \geq F(\beta)$).

Proof. Using the same idea as in § 4 [S2] we define by induction a sequence $\{ f_\alpha : \alpha \in \Omega \}$ putting $f_0 = 0$; the property $\mathrm{dom}(f_\alpha) \in Y \,\&\, \mathrm{rng}(f_\alpha) \subseteq N - FN \,\&\, f_\alpha$ is decreasing $\&\, (\forall \beta \in (\alpha \cap \Omega))\, f_\beta \subseteq f_\alpha$ serves as the induction hypothesis.

If f_α ($\alpha \in \Omega$) is constructed then we put $f_{\alpha+1} = f_\alpha \cup \{x\} \times (y - \mathrm{dom}(f_\alpha))$ where y (x resp.) is the smallest element (in a fixed well-ordering which orders V by type Ω) of Y-dom(f_α)($\cap$ rng(f_α)-FN resp.). Our choice is possible since dom(f_α) $\subseteq$ Y and rng(f_α) $\cap$ FN = 0 by the induction hypothesis (and because FN is a proper class and thus Y is also a proper class).

If $\alpha \in \Omega$ is a limit and if $\{f_\beta ; \beta \in (\alpha \cap \Omega)\}$ is constructed, then we choose an increasing sequence $\{\alpha_n ; n \in FN\} \subseteq \subseteq \alpha \cap \Omega$ with $\bigcup\{\alpha_n ; n \in FN\} = \bigcup\{\beta ; \beta \in (\alpha \cap \Omega)\}$ (the countability of $\alpha \cap \Omega$ enables us to make such a choice) and furthermore we fix $\delta \in (\bigcap\{\bigcap \mathrm{rng}(f_\beta) ; \beta \in (\alpha \cap \Omega)\} - FN)$ (this choice is possible since FN is no π-class). By the prolongation axiom there is g with $g(n) = f_{\alpha_n}$ & dom(g) $\in$ N. According to the induction hypothesis we have $n < m \in FN \rightarrow g(n) \subseteq g(m)$ & & g(n) is decreasing & dom(g(n)) $\in$ Y & rng(g(n)) $\subseteq N - \delta$. Thus there is even an infinite natural number α' with the properties in question because Y is fully revealed and therefore, defining $f_\alpha = g(\alpha')$, we obtain dom(f_α) $\in$ Y & rng(f_α) $\subseteq N - \delta$ & & f_α is decreasing & $(\forall n \in FN)\ f_{\alpha_n} \subseteq f_\alpha$ from which the induction hypothesis for α follows because of our choice of the sequence $\{\alpha_n ; n \in FN\}$.

At the end we put $F = \bigcup\{f_\alpha ; \alpha \in \Omega\}$. Evidently F is revealed since if $\{x_n ; n \in FN\} \subseteq$ dom(F), then we can choose $\alpha \in \Omega$ with $\{x_n ; n \in FN\} \subseteq$ dom(f_α); F is a decreasing mapping which is a part of (N-FN) $\times$ Y. Moreover, dom(F) = Y & $\bigcap$ F"Y = FN because of the construction of $f_{\alpha+1}$'s.

<u>Theorem</u>. There is a revealed class X such that P(X) is not revealed.

Proof. Let Y be a revealment of FN (for the existence see § 2 [S-V 2]), F be a revealed mapping of Y into N-FN with

$\bigcap F''Y = FN$ and let $Y \subseteq \alpha \in N$. We define X as $\bigcup\{F(x)\times\{x\};$ $x \in \mathrm{dom}(F)\} \cup \alpha \times (\alpha - Y)$. This class is revealed as the union of two revealed classes ($\alpha - Y$ is revealed since Y is fully revealed). For every $n \in FN$ we have $n \times \alpha \subseteq X$, but according to the choice of F there is no $\beta \in N-FN$ with $\beta \times \alpha \subseteq X$ and thus there is no u with $\{n \times \alpha\ ; n \in FN\} \subseteq u \subseteq P(X)$.

The following trivial lemma is useful.

Lemma. If $\{S_n; n \in FN\}$ is a sequence of revealed classes with $(\forall n \in FN)\ S_{n+1} \subset S_n$ then the class $V - \bigcap\{S_n; n \in FN\}$ is not revealed.

Proof. Let us choose f so that $f(n) \in (S_n - S_{n+1})$ for every $n \in FN$. Evidently $m \geq n \longrightarrow f(m) \in S_n$ and thus assuming the classes S_n are revealed, we can choose a decreasing sequence $\{u_n; n \in FN\}$ with $f''(FN - n) \subseteq u_n \subseteq S_n$. Furthermore $f''FN \subseteq (V - \bigcap\{S_n; n \in FN\})$ and if the lastly mentioned class would be revealed, then there would be a set u such that $f''FN \subseteq u \ \&\ u \cap \bigcap\{S_n; n \in FN\} = 0$. For every $n \in FN$ we would have $f(n+1) \in (u_{n+1} \cap u) \subseteq (u_n \cap u)$ from which

$$0 \neq u \cap \bigcap\{u_n; n \in FN\} \subseteq u \cap \bigcap\{S_n; n \in FN\}$$

would follow by (a) and (d) of the beginning of the paper - contradiction.

Corollary. Each revealed σ-class is set-theoretically definable.

Proof. If $\{S_n; n \in FN\}$ is a sequence of set-theoretically definable classes, then for every $k \in FN$ the class $S'_k = \bigcup\{S_n;\ n \leq k\}$ is set-theoretically definable, too, and thence the class $V - S'_k$ is revealed. If $\bigcup\{S_n; n \in FN\} = V - \bigcap\{V - S'_n; n \in FN\}$ is revealed, then there is $k \in FN$ such that $(\forall n \geq k)(V - S'_n = V - S'_k)$ i.e. $\bigcup\{S_n; n \in FN\} = S'_k$.

Corollary. There is no revealed class which is countable, in particular, the class FN is not revealed.

Corollary. Each fully revealed Π-class is set-theoretically definable.

Proof. Its complement is a revealed σ-class, therefore its complement is set-theoretically definable.

Corollary. Each fully revealed real class is set-theoretically definable.

Proof. In § 1 [Č-V] the authors prove that every revealed real class is a Π-class.

Theorem. (a) A class is set-theoretically definable iff the system of all its revealments is codable.

(b) Every set-theoretically definable class which is no element of Sd_0 has Ω-many revealments (i.e. if a pair K,S codes the system of all its revealments, then K is uncountable) and each element of Sd_0 is its sole revealment.

Proof. According to the second theorem of § 2 [S-V 2], the system $\{Y;\ Y$ is a revealment of $X\}$ equals to the system $\{F''Y_0;$ F is an automorphism$\}$ where Y_0 is an arbitrarily chosen revealment of X (by the first theorem of § 2 [S-V 2] every class has a revealment).

If $X \in Sd_0$, then X is its sole revealment by the second theorem of § 3 [S-V 2].

If X is set-theoretically definable, then there is a set-formula $\varphi(z,z_1)$ of FL and a convenient parameter p_0 (may be an n-tuple) with $X = \{z;\ \varphi(z,p_0)\}$. Let the symbol Mon denote the monad of p_0 in the indiscernibility equivalence $\stackrel{\circ}{=}$ defined in § 1 ch. V [V]. Further let us suppose $X \notin Sd_0$.

Put $A_q = \{p;(\forall z)(\varphi(z,p) \equiv \varphi(z,q))\}$ and let us assume at first that there are $p_1,\dots,p_n \in$ Mon with Mon $\subseteq \bigcup\{A_{p_k};k \leq n\}$.

Since every A_q is set-theoretically definable, the class $\bigcup\{A_{p_k};k\le n\}$ is set-theoretically definable, too. If $p\in\bigcup\{A_{p_k};k\le n\}$ then there is $i\le n$ with $(\forall z)(\varphi(z,p)\equiv\equiv\varphi(z,p_i))$ and moreover for each automorphism F we have $(\forall z)(\varphi(z,F(p))\equiv\varphi(z,F(p_i)))$. Furthermore let us realize that $F(p_i)\overset{\Omega}{=}p_o$ (because $p_i\overset{\Omega}{=}p_o$) and therefore there is $j\le n$ with $(\forall z)(\varphi(z,F(p_i))\equiv\varphi(z,p_j))$ from which $F(p)\in\bigcup\{A_{p_k};k\le n\}$ follows. We have proved that $\bigcup\{A_{p_k};k\le n\}$ is a figure in the indiscernibility equivalence $\overset{\Omega}{=}$ and thence by the nineteenth theorem of § 1 ch. V [V] it is an element of Sd_o. According to the twelfth theorem of the mentioned section there is a definable $p\in\bigcup\{A_{p_k};k\le n\}$ i.e. there is $i\le n$ and a set-formula ψ of FL such that $\{z;\varphi(z,p_i)\}=\{z;\psi(z)\}$. Furthermore there is an automorphism F so that $F(p_o)=p_i$ (because $p\overset{\Omega}{=}p_i$ and because of the sixth theorem of § 1 ch. V [V]) and hence $X=\{z;\varphi(z,p_o)\}=\{z;\varphi(z,F^{-1}(p_i))\}=F^{-1}{}''\{z;\varphi(z,p_i)\}=$ $=F^{-1}{}''\{z;\psi(z)\}=\{z,\psi(z)\}$. This contradicts our assumption $X\notin Sd_o$.

Let $\{p_k;k\in FN\}\subseteq Mon$ be a sequence such that $p_n\notin\bigcup\{A_{p_k};k<n\}$ for every $n\in FN$. The class Mon is a σ-class by the definition and thus it is revealed. Therefore for every $n\in FN$ the class $Mon-\bigcup\{A_{p_k};k<n\}$ is revealed, too, and it is nonempty (p_n being its element). Thence by (d) of the beginning of the paper, even the class $Mon-\bigcup\{A_{p_k};k\in FN\}$ is nonempty.

We suppose that $X=\{z;\varphi(z,p_o)\}\notin Sd_o$. Then X is its revealment and for every $p\in Mon$ there is an automorphism G with $G(p_o)=p$ and furthermore for every automorphism F we have $F(p_o)\in Mon$ and $F''X=F''\{z;\varphi(z,p_o)\}=\{z;\varphi(z,F(p_o))\}$. Thus the system of classes $\{F''X;\ F$ is an automorphism$\}$ is coded by the pair Mon, $\{\langle z,p\rangle ; \varphi(z,p)\}$ and moreover there is no class $Z \subseteq$ Mon which is at most countable so that Mon $\subseteq \bigcup\{A_q; q \in Z\}$. We have proved our statement (b) and one implication of (a).

Let Y_0 be a revealment of a class X. If the system of classes $\{F''Y_0;\ F$ is an automorphism$\}$ is codable, then by the eighth theorem of § 1 [Č-V], the class Y_0 is real. This class is even fully revealed and hence Y_0 is set-theoretically definable by the last Corollary and thus $Y_0 = \{z; \varphi(z,p)\}$ for some $\varphi \in$ FL and a suitable parameter p. Therefore $X = \{z; \varphi(z,p')\}$ for some parameter p' by the definition of revealment. We have proved the second implication of (a) which finishes the proof of our theorem.

Since there are classes which are not set-theoretically definable, the last theorem guarantees that the system of fully revealed classes is not codable i.e. it is very large. On the other hand, the following statement shows that this system is "narrow" - there is rather a small number of types of fully revealed classes if in one type there are classes satisfying the same normal formulae of FL. This is raised by the countability of the class of considered formulae (the system of all subclasses of FL is codable according to the prolongation axiom). (If we would admit in the considered formulae even set-parameters, we would get a quite opposite result, of course.)

The system of all axioms of the form

$$(\forall x)(\exists X)\ \Phi(x,X) \longrightarrow (\exists Y)(\forall x)\ \Phi(x,Y''\{x\})$$

where Φ is an arbitrary formula is called the strong schema of choice; the alternative set theory with the strong schema of choice is consistent relatively to the alternative set theory.

Theorem. If the strong schema of choice is available, then there is a codable system of classes $\mathcal{M}$ such that

$(\forall X)(\exists Y \in \mathcal{M})$ Y is a revealment of X.

Proof. As $\Phi(z,Z)$ we fix the formula

$[(\exists Q)(\forall \varphi \in FL)((\varphi$ is normal $\&$ φ has exactly one free variable$) \rightarrow (\varphi(Q) \equiv \varphi \in z)) \rightarrow (\forall \varphi \in FL)((\varphi$ is normal $\&$ $\&$ φ has exactly one free variable$) \rightarrow (\varphi(Z) \equiv \varphi \in z)] \,\&\, Z$ is fully revealed.

The following result is a slight generalization of the eighth theorem of § 1 [Č-V].

Theorem. If there are x, y so that the system of classes $\{F''X;\ F$ is an automorphism with $F(x) = y\}$ is nonempty and codable then X is a real class.

Proof. Let x, y be sets with the above described property. Since there is an automorphism F with $F(x) = y$, the set $\{\langle y,x\rangle\}$ is a similarity.

To every similarity $\{\langle z,y\rangle\}$ there is an automorphism H with $H(y) = z$ by the sixth theorem of § 1 ch. V [V]. Moreover, if a well-ordering of V of type Ω is chosen, then such an automorphism can be constructed uniquely and we are going to denote it by the symbol H_z.

If a pair of classes K,S codes the system of classes $\{F''X;\ F$ is an automorphism with $F(x) = y\}$ then we put

$\widetilde{K} = \bigcup\{(H_z''K)\times\{z\};\ \{\langle z,y\rangle\}$ is a similarity$\}$

and

$\widetilde{S} = \{\langle H_z(p),\langle H_z(q),z\rangle\rangle;\ \langle p,q\rangle \in S \,\&\, \{\langle z,y\rangle\}$ is a similarity$\}$.

If G is an automorphism with $G(x) = z$ then $\{\langle z,y\rangle\} = \{\langle z,x\rangle\} \circ \{\langle x,y\rangle\}$ is a similarity since the composition of similarities is also a similarity and the converse of a similarity is a similarity, too (see § 1 ch. V [V]). Thus the automorphism H_z had to be chosen and putting $F = H_z^{-1} \circ G$ we have $F(x) = H_z^{-1}(G(x)) = H_z^{-1}(z) = y$ and thus there is $q \in K$ with $F"X = S"\{q\}$. Since $\{\langle z,y\rangle\}$ is a similarity, $\langle H_z(q),z\rangle$ is an element of $\widetilde{K}$ and moreover

$$\widetilde{S}"\{\langle H_z(q),z\rangle\} = \{H_z(p);\langle p,q\rangle \in S\} = H_z"(S"\{q\}) =$$

$$= H_z"(F"X) = H_z"(H_z^{-1}"(G"X)) = G"X.$$

We have proved that the system of classes $\{G"X;\ G$ is an automorphism$\}$ is codable. We finish the proof using the mentioned result of § 1 [Č-V].

<u>Theorem</u>. If F is a similarity which is at most countable, then there are f, g such that $F \cup \{\langle g,f\rangle\}$ is a similarity and such that every automorphism G with $G(f) = g$ is an extension of F.

Proof. We suppose that $\mathrm{dom}(F)$ is at most countable and thus there is f with $f"FN = \mathrm{dom}(F)$. According to the third theorem of § 1 ch. V [V] there is g such that $F \cup \{\langle g,f\rangle\}$ is a similarity. If G is an automorphism with $G(f) = g$, then $G(f(n)) = (G(f))(G(n)) = (G(f))(n) = g(n)$ and $\langle f(n),n\rangle \in f$ implies $(F \cup \{\langle g,f\rangle\})(\langle f(n),n\rangle) \in (F \cup \{\langle g,f\rangle\})(f)$ i.e. $\langle F(f(n)),n\rangle \in g$ from which $F(f(n)) = g(n) = G(f(n))$ follows for every $n \in FN$. We have proved $F \subseteq G$ because $f"FN = \mathrm{dom}(F)$.

<u>Lemma</u>. If F is a similarity which is at most countable, then for every $\alpha \notin FN$ there is $\beta \in \alpha - FN$ such that $F \cup \{\langle\beta,\beta\rangle\}$ is a similarity.

Proof. Let $\mathfrak{M}$ be the system of classes of the form $\{\beta < \alpha;\ m < \beta \ \&\ (\varphi(\beta,x_1,\dots,x_n) \equiv \varphi(\beta,F(x_1),\dots,F(x_n)))\}$ where $m \in FN$, $x_1,\dots,x_n \in \mathrm{dom}(F)$ and φ is a set-formula of FL with exactly n+1 free variables.

For every $m \in FN$ there is a set-formula $\psi(z) \in FL$ so that

$(\exists !\, x)\ \psi(x)\ \&\ \psi(m+1)$.

If $\varphi(z_0, z_1, \dots, z_n)$ is an arbitrary set-formula of FL, then $(\exists x)(\varphi(x, z_1, \dots, z_n)\ \&\ \psi(x))$ is a set-formula of FL, too. Using the fact that F is a similarity, we obtain

$$(\exists x)(\varphi(x, x_1, \dots, x_n)\ \&\ \psi(x)) \equiv (\exists x)(\varphi(x, F(x_1), \dots, F(x_n))\ \&\ \psi(x))$$

i.e.

$$\varphi(m+1, x_1, \dots, x_n) \equiv \varphi(m+1, F(x_1), \dots, F(x_n))$$

for every $x_1, \dots, x_n \in \mathrm{dom}(F)$.

Thus $\mathfrak{M}$ is a countable system of nonempty set-theoretically definable classes such that $X, Y \in \mathfrak{M} \rightarrow X \cap Y \in \mathfrak{M}$. Therefore by (a) and (d) of the beginning of this article we have $\bigcap \{X; X \in \mathfrak{M}\} \neq 0$. According to the definition of $\mathfrak{M}$, every element of $\bigcap \{X; X \in \mathfrak{M}\}$ satisfies our requirements.

Theorem. There are fully revealed classes X and Y such that $X \cap Y$ is not fully revealed.

Proof. Let us define

$\mathrm{Sat} = \{\langle x, \varphi \rangle ; \varphi \in FL\ \&\ \varphi$ is a set-formula with exactly one free variable $\&\ \varphi(x)\}$.

The class Sat determines the satisfaction relation in the model $\langle V, \in \rangle$ and the pair of classes FL, Sat codes the system of classes Sd_0. Furthermore, for every revealment Q of Sat and every $n \in FN$, we have $Q \upharpoonright n = \mathrm{Sat} \upharpoonright n$ since $\mathrm{Sat} \upharpoonright n$ is set-theoretically definable. Since each class has a revealment, we are able to fix Z as a revealment of the class Sat.

Let $\{\alpha_\gamma ; \gamma \in \Omega\}$ be a descending sequence of natural numbers with $FN = \bigcap \{\alpha_\gamma ; \gamma \in \Omega\}$ and let $\{x_\gamma ; \gamma \in \Omega\}$ be an enumeration of the universal class. We are going to construct a sequence of similarities $\{H_\gamma ; \gamma \in \Omega\}$ by induction putting

$H_0 = 0$ and for every limit $\gamma \in \Omega$ we define H_γ as $\bigcup \{H_\beta ;\ \beta \in (\gamma \cap \Omega)\}$.

For every $\gamma \in \Omega$ we construct the similarity $H_{\gamma+1}$ under the assumption that H_γ is a similarity which is at most countable.

By the last lemma we can choose $\beta \in \alpha_\gamma - FN$ such that $H_\gamma \cup \{\langle \beta , \beta \rangle\}$ is a similarity; furthermore according to the last theorem we are able to fix g, f such that $H_\gamma \cup \{\langle \beta , \beta \rangle\} \cup \cup \{\langle g,f \rangle\}$ is a similarity and such that every automorphism F with $F(f) = g$ is an extension of $H_\gamma \cup \{\langle \beta , \beta \rangle\}$.

At first we are going to show that there is an automorphism F with $F(f) = g$ such that $Z \upharpoonright \beta \neq F''(Z \upharpoonright \beta)$. If there would not be such an automorphism then the system of classes

$\{F''(Z \upharpoonright \beta);\ F$ is an automorphism with $F(f) = g\}$

would be codable and nonempty ($\{\langle g,f \rangle\}$ being extendable to an automorphism since it is evidently a similarity) and thus $Z \upharpoonright \beta$ would be real because of the last but one theorem. Since $Z \upharpoonright \beta$ would be fully revealed and real, it would have to be set-theoretically definable by the last Corollary. On the other hand, the pair FL, $Z \upharpoonright \beta$ codes the system of classes Sd_0 and thus we would obtain a contradiction to the first theorem of § 4 [S-V 2]. Our claim is proved.

We have proved that there is an automorphism $F \supseteq H_\gamma \cup \{\langle \beta, \beta \rangle\}$ with $Z \upharpoonright \beta \neq (F''Z) \upharpoonright \beta$ $(= (F''Z) \upharpoonright F(\beta) = F''(Z \upharpoonright \beta))$. Therefore we are able to choose $H_{\gamma+1}$ so that

(1) $H_{\gamma+1}$ is a similarity which is at most countable;

$$H_\gamma \subseteq H_{\gamma+1}$$

(2) $$(\exists \beta < \alpha_\gamma)(\exists x \in Z \upharpoonright \beta)(\exists y \notin Z \upharpoonright \beta)(\langle x,y \rangle \in \\ \in (H_{\gamma+1} \cup H_{\gamma+1}^{-1}))$$

(3) $x_\gamma \in (\mathrm{dom}(H_{\gamma+1}) \cap \mathrm{rng}(H_{\gamma+1}))$.

At the end we put $H = \bigcup\{H_\gamma ; \gamma \in \Omega\}$. Evidently H is an automorphism and for every $\alpha \notin FN$ we have $(H''Z) \restriction \alpha \neq Z \restriction \alpha$, since for every $\gamma \in \Omega$ there is $\beta < \alpha_\gamma$ such that for every one-one mapping F with $F \supseteq H_{\gamma+1}$ we have $(F''Z) \restriction \beta \neq Z \restriction \beta$.

According to the second theorem of § 2 [S-V 2], the class $H''Z$ is a revealment of the class Sat, too, and thence for every $n \in FN$ we get $Z \restriction n = \mathrm{Sat} \restriction n = (H''Z) \restriction n$. Eventually we define $X = Z \times \{0\} \cup V \times \{1\}$ and $Y = V \times \{0\} \cup (H''Z) \times \{1\}$. Both classes X and Y are fully revealed classes and we have $X \cap Y = Z \times \{0\} \cup (H''Z) \times \{1\}$; this class is not fully revealed because

$$\{\alpha ; ((X \cap Y)'' \{0\}) \restriction \alpha = ((X \cap Y)'' \{1\}) \restriction \alpha\} =$$

$$= \{\alpha ; Z \restriction \alpha = (H''Z) \restriction \alpha\} = FN$$

is not revealed by the first Corollary.

Let us note that as a trivial consequence we get that the system of fully revealed classes is closed neither to Cartesian product nor to the pairing operation of classes ($\langle X,Y\rangle^{\sigma} = X \times \{0\} \cup Y \times \{1\}$). On the other hand, according to [S-V 5], to every fully revealed class there is a system of fully revealed classes containing it and closed under all Gödel's operations.

At the end let us note that we have constructed two revealments Z, Z' of the satisfaction class Sat with $FN = \{\alpha ; Z \restriction \alpha = Z' \restriction \alpha\}$, but using the same technique as in the last proof we are able to prove a still stronger result, namely, for every codable system $\mathfrak{M}$ of revealments of the class Sat we can construct a class X with

$$(\forall Y \in \mathfrak{M}) FN = \{\alpha ; X \restriction \alpha = Y \restriction \alpha\}$$

Thus we can construct by induction a system $\{X_\gamma ; \gamma \in \Omega\}$ of revealments of the class Sat so that

$$(\beta, \gamma \in \Omega \ \& \ \beta \neq \gamma) \rightarrow FN = \{\alpha ; X_\beta \upharpoonright \alpha = X_\gamma \upharpoonright \alpha\}$$

i.e. we are able to construct Ω -many revealments of Sat with the property in question.

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(Oblatum 14.1. 1985)