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CONSTRUCTIBILITY AND SHIFTINGS OF VIEW

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Abstract: The axiom of constructibility in the alternative set theory (AST) is introduced and its basic consequences are shown. The corresponding interpretations are interpretations of AST + strong schema of choice in AST such that the class FN is absolute. Using these interpretations we can strengthen the results concerning shiftings of the horizon (cf. [S-V 5]).

Key words: The alternative set theory, constructibility, the class FN, shiftings of view and of the horizon, restriction of view, schema of choice.

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The alternative set theory (AST) can serve as an alternative to Cantor's set theory; it gives us a sufficiently strong framework for a great deal of mathematics (cf. [V]). The axiomatic system of this theory is sketched in the first section; the symbol FN denotes the class of (standard) finite natural numbers (see below).

An interpretation $*$ of T' in T (T', T being stronger than AST) is called a shifting of view (of T' in T) if

$$T \vdash (\forall x)\mathrm{Cls}^*(x) \,\&\, (\forall X^*, Y^*)(X^* \in^* Y^* \equiv X^* \in Y^*).$$

If moreover $T \vdash \mathrm{FN}^* = \mathrm{FN}$, then $*$ is said to be a restriction of view and if $T \vdash \mathrm{FN}^* \neq \mathrm{FN}$, then $*$ is said to be a shifting of the horizon. Of course there is a trivial shifting of view – the identity; other shiftings of view are called nontrivial –

more formally a shifting of view $*$ of T' in T is called nontrivial if $T \vdash (\exists X) \neg Cls^*(X)$. Evidently, there can be shiftings of view of T' in T which are neither restrictions of view nor shiftings of the horizon (in T !). Each shifting of the horizon is a nontrivial shifting of view since FN is no $*$-class.

The importance of shiftings of view lies even on philosophical aspects. In AST we try to describe our understanding of the real world. Sets are considered as formalizations of collections we really meet, classes are formal counterparts of our idealizations and generalizations. Thus shiftings of view describe our different approaches to the real world (the property "to be a set" and membership relation being absolute) - in different approaches we can only change the collection of our idealizations and descriptions (i.e. the system of proper classes).

Collections converging to the horizon of our observation ability (describing unlimited processes) are formalized in AST by countable classes and from formal reasons it is sufficient to restrict ourselves to one countable class - the class of finite natural numbers FN. Hence shiftings of view $*$ with $FN^* \neq FN$ can be considered as a formalization of such approaches which lead to changes of the horizon (shiftings of the horizon).

The schema of axioms of the form

$$(\forall n \in FN)(\exists X)\, \Theta(n,X) \rightarrow (\exists Y)(\forall n \in FN)\, \Theta(n, Y''\{n\})$$

for an arbitrary formula Θ is called the schema of choice; similarly the schema of axioms of the form

$$(\forall x)(\exists X)\, \Theta(x,X) \rightarrow (\exists Y)(\forall x)\, \Theta(x, Y''\{x\})$$

for an arbitrary formula Θ is called the strong schema of choice (cf. the analogous definition of schema of choice in the second order arithmetic).

Inspired with Gödel's constructive process - or with ramified analysis if somebody likes - we shall define for every class Q the system of Q-constructible classes (see § 2). Thus with every constant Q we naturally associate an interpretation $\mathcal{L}(Q)$.

For convenient classes Q we are going to show that $\mathcal{L}(Q)$ is a restriction of view of AST + strong schema of choice in AST (cf. § 3,4).

In [S-V 5] there were constructed shiftings of the horizon in the theory AST + schema of choice. There was a question whether it is possible to construct shiftings of the horizon in AST itself. Using the results of the paper we answer this question affirmatively since the compositions of shiftings of the horizon constructed in [S-V 5] and interpretations $\mathcal{L}(Q)$ for suitable constants Q give us shiftings of the horizon in AST itself (see § 5).

Let us note that using the methods mentioned in [M-S] we are able to construct an interpretation of AST + strong schema of choice in AST, too, (and to demonstrate in this way the consistency of AST + strong schema of choice relatively to AST) but so constructed an interpretation is in no case a shifting of view, hence such an interpretation can hardly be used in the considerations concerning the existence of shiftings of the horizon in AST.

Assuming the axiom of construstibility we are able to define (by a formula) a well-ordering of classes (cf. § 3).

At the end we shall deal with shiftings of view in general setting and we shall see that $\mathcal{L}(Q)$ is in some sense the minimal restriction of view of AST in AST, namely if $*$ is a restriction of view (with parameter Q) of AST in AST, then we cannot prove in AST simultaneously

(a) Q is a $*$-class

(b) there is a $\mathcal{L}(Q)$-class which is no $*$-class.

§ 1. Preliminaries. At first we are going to summarize axioms of AST; further informations concerning the axiomatic system of this theory can be found in [V] or in a more formal way in [S 1]. AST is a theory with the following axioms: extensionality for classes, Morse's schema of classes, $Set(0)$ & & $Set(x \cup \{y\})$, induction for set-formulas (i.e. for every formula φ in which only set-variables and set-constants occur, we accept the axiom

$$V \models [(\varphi(0) \& (\forall x,y)((\varphi(x) \& \varphi(y)) \rightarrow \varphi(x \cup \{y\}))) \rightarrow \rightarrow (\forall x)\varphi(x)],$$

the prolongation axiom i.e.

$$(\forall F)((Fnc(F) \& dom(F) = FN) \rightarrow (\exists f)(F \subseteq f \& Fnc(f))),$$

the axiom of choice in the form that the universal class V can be well-ordered and the axiom of cardinalities i.e. each class can be one-one mapped into FN or onto V.

In AST we admit proper classes which are subclasses of sets - FN is defined as the smallest possible cut of the class of natural numbers N closed under successors (see § 1 ch. II [V]); we have $N-FN \neq 0$.

A class $\preceq$ is called a well-ordering if it is a linear ordering such that every nonempty subclass of $dom(\preceq)$ has the least element.

A well-ordering $\preceq$ is an ordering of type Ω if for each $x \in dom(\preceq)$ the segment $\{y; y \preceq x\}$ can be one-one mapped into FN (is at most countable) while $dom(\preceq)$ can be one-one mapped onto V (is uncountable).

If $\preceq$ is a well-ordering then $0_{\preceq}$ denotes its first element. If $\preceq$ and $\trianglelefteq$ are two well-orderings then $\preceq + \trianglelefteq$ is the well-ordering

$\{\langle\langle a,x\rangle,\langle b,y\rangle\rangle;\ (a = 0 = b \,\&\, x \preceq y) \vee (a = 1 = b \,\&\, x \trianglelefteq y) \vee$
$\vee (a = 0 \,\&\, b = 1 \,\&\, x \in dom(\preceq) \,\&\, y \in dom(\trianglelefteq))\}$;

in particular, $\preceq + 1$ denotes the well-ordering $\preceq + \{\langle 0,0\rangle\}$. We use variables $\preceq$ and $\trianglelefteq$ (sometimes with indexes) for non-empty well-orderings.

If K,S is a pair of classes, then Dc(K,S) denotes the system of classes $\{X;\ (\exists q \in K)\ X = S''\{q\}\}$. A system of classes $\mathfrak{M}$ is called codable if there is a pair K,S with $\mathfrak{M} = Dc(K,S)$; we are going to write Dc(S) instead of $Dc(dom(S),S) \cup \{0\}$.

A formula is said to be normal if it contains no quantifier binding a proper class. Metamathematical formulas are denoted by symbols $\Phi, \Psi, \Theta, \ldots$; we can define formal formulas in AST as usual (cf. [S 1]) and the class of formal formulae without parameters which are elements of FN is denoted by the symbol FL; variables $\varphi, \psi, \vartheta, \ldots$ run through elements of FL.

For normal formulae which are elements of FL and all classes $X_1,\ldots,X_n$, the satisfaction relation in the model$(V,\in,X_1,\ldots,X_n)$ can be defined (see § 3 [S 1]) and we shall write $\vartheta(X_1,\ldots,X_n)$ instead of $(V,\in,X_1,\ldots,X_n) \models \vartheta(X_1,\ldots,X_n)$.

For every $\varphi \in$ FL and every codable class $\mathfrak{M}$, the symbol $\varphi^{(\mathfrak{M})}$ denotes the formula resulting from φ by restriction of all quantifiers binding class variables to elements of $\mathfrak{M}$ (quantifiers binding set-variables are left without change); similarly for metamathematical formulae, but in this case the codability of $\mathfrak{M}$ is not required. Thus e.g. the symbol $((\exists X)(\forall y)\ y \in X)^{(\mathfrak{M})}$ denotes the formula $(\exists X \in \mathfrak{M})(\forall y)\ y \in X$.

The formula $\varphi^{(\mathfrak{M})}$ expresses the validity of φ in the model determined by the system of classes $\mathfrak{M}$ (and all sets) and the usual membership relation. Moreover, under the assumption that $\mathfrak{M}$ is codable, the formula $\varphi^{(\mathfrak{M})}$ is (equivalent to) a normal one - this is the reason why we have defined $\varphi^{(\mathfrak{M})}$ for codable $\mathfrak{M}$ only.

If $\mathcal{T}$ is a formal theory (i.e. a subclass of FL) and if $\mathfrak{M}$ is a codable system of classes then $\mathcal{T}^{(\mathfrak{M})}$ means $(\forall \varphi \in \mathcal{T})\, \varphi^{(\mathfrak{M})}$; similarly for metamathematical theories. $\mathcal{AST}$ denotes the class of these elements of FL which are formal axioms of AST.

If $*$ is an interpretation of T′ in T then Θ is called absolute iff the formula

$$(\forall X_1^*,\dots,X_n^*)(\, \Theta^*(X_1^*,\dots,X_n^*) \equiv \Theta\, (X_1^*,\dots,X_n^*))$$

is provable in T.

§ 2. <u>The axiom of constructibility</u>. In this section for every class Q we define the system of Q-constructible classes; for this purpose the auxiliary property Φ is useful.

The symbol $\Phi(\preccurlyeq, S, Q)$ denotes the formula

$$\mathrm{dom}(S) = \mathrm{dom}(\preccurlyeq) \,\&\, S''\{0_{\preccurlyeq}\} = Q \times \{\langle 0, 0_{\preccurlyeq}\rangle\} \cup$$
$$\cup \{\langle y, \langle z, 1\rangle, 0_{\preccurlyeq}\rangle ; y \in z \in V\} \,\&\, (\forall x \in (\mathrm{dom}(\preccurlyeq) - \{0_{\preccurlyeq}\}))\, S''\{x\} =$$
$$= \{\langle\langle y_1,\dots,y_k\rangle, \langle \varphi, q_1,\dots,q_n\rangle, x\rangle ; \varphi \in \mathrm{FL} \,\&\, q_1,\dots,q_n \in$$
$$\in \mathrm{dom}(S''\{y; y < x\}) \,\&\, \varphi \text{ has exactly } k{+}n \text{ free variables } \&$$
$$\& \; \varphi^{(\mathrm{Dc}(S''\{y;y<x\}))}(y_1,\dots,y_k, \mathrm{rng}(S)''\{q_1\},\dots,\mathrm{rng}(S)''\{q_n\})\}.$$

At first let us realize that the system of classes $\mathrm{Dc}(S''\{y;y<x\})$ is codable and hence the symbol

$$\varphi^{(\mathrm{Dc}(S''\{y;y<x\}))}(y_1,\dots,y_k, \mathrm{rng}(S)''\{q_1\},\dots,\mathrm{rng}(S)''\{q_n\})$$

is meaningful for $q_1,\dots,q_n \in \mathrm{dom}(S''\{y;y<x\})$ and expresses the validity of the formula

$$\varphi(y_1,\dots,y_k, (S''\{y;y<x\})''\{q_1\},\dots,(S''\{y;y<x\})''\{q_n\})$$

in the model determined by the system of classes $Do(S''\{y; y < x\})$ i.e. by the system of classes constructed up to the stage x. The formula in question is equivalent to a normal one having $S''\{y; y < x\}$ as the only additional parameter, thus even the whole formula Φ is (equivalent to) a normal formula.

Let $\Phi(\preceq, S, Q)$ and $\Phi(\trianglelefteq, \tilde{S}, Q)$ and let G be an isomorphism of $\preceq$ onto $\trianglelefteq$. Then we can define a mapping ($\tilde{G}$, say) of $dom(rng(S))$ onto $dom(rng(\tilde{S}))$ so that

$$\tilde{G}(\langle\langle \varphi, q_1, \ldots, q_n\rangle, x\rangle) = \langle\langle \varphi, \tilde{G}(q_1), \ldots, \tilde{G}(q_n)\rangle, G(x)\rangle$$

and

$$\tilde{G}(\langle 0, 0_{\preceq}\rangle) = \langle 0, 0_{\trianglelefteq}\rangle \;\&\; \tilde{G}(\langle\langle z, 1\rangle, 0_{\preceq}\rangle) = \langle\langle z, 1\rangle, 0_{\trianglelefteq}\rangle .$$

Such a mapping is determined uniquely and moreover by induction for every $x \in dom(\preceq)$ and every $q \in dom(S''\{x\})$ we can prove the equality $(S''\{x\})''\{q\} = (\tilde{S}''\{G(x)\})''\{\tilde{G}(q)\}$. In particular, to each $\preceq$ and Q there is at most one S with $\Phi(\preceq, S, Q)$.

On the other hand, for every $\preceq$ and Q there is S with $\Phi(\preceq, S, Q)$. This can be proved by induction using Morse's schema (the definition in question is correct since in the definition of $S''\{x\}$ only the class $S \upharpoonright \{y; y < x\}$ is used).

Further we put

$$\mathcal{L}(\preceq, Q) = \{X; (\exists S)(\Phi(\preceq, S, Q) \;\&\; X \in Do(rng(S)))\}.$$

This definition is in harmony with ramified analysis because $L(\preceq + 1, Q)$ is just the system of classes parametrically definable in the model determined by $\mathcal{L}(\preceq, Q)$; if $\preceq$ has no last element, then $\mathcal{L}(\preceq, Q) = \bigcup\{\mathcal{L}(\preceq \upharpoonright \{y; y < x\}, Q); x \in dom(\preceq)\}$ and furthermore

$$\mathcal{L}(\{\langle 0,0\rangle\}, Q) = V \cup \{Q\}.$$

Let us note that the equality

$$\mathcal{L}(\preceq, Q) = \{X; (\forall S)(\Phi(\preceq, S, Q) \rightarrow X \in Do(rng(S)))\}$$

holds because of $(\forall \leq)(\forall Q)(\exists ! S)\, \Phi(\leq, S, Q)$; moreover, if $\leq$ and $\trianglelefteq$ are isomorphic, then $(\mathcal{L}(\leq, Q) = \mathcal{L}(\trianglelefteq, Q)$.

Now we are able to define the system of Q-constructible classes (in symbols $\mathcal{L}(Q)$); to obtain this system as small as possible we shall use the idea due to R. Gandy and consider the following two cases:

(A) There is a well-ordering $\leq$ such that $\mathcal{L}(\leq + 1, Q)$ does not contain a well-ordering of type greater or equal to $\leq$. In this case let us fix $\leq_0$ as a well-ordering of the smallest possible ordinal type having the property in question. Further we define

$$\mathcal{L}(Q) = \mathcal{L}(\leq_0, Q).$$

(B) There is no well-ordering with the property described in the case (A). In this case we put

$$\mathcal{L}(Q) = \bigcup \{\mathcal{L}(\leq, Q); \leq \text{ is a well-ordering}\}.$$

The statement

$$(\forall X)\; X \in \mathcal{L}(Q)$$

is called the axiom of Q-constructibility and the formula

$$(\exists Q)(\forall X)\; X \in \mathcal{L}(Q)$$

is said to be the axiom of constructibility.

Note. We have restricted the system of classes - to a constant Q we constructed the system of Q-constructible classes. However, the original Gödel's purpose was to restrict the collection of sets (in Gödel-Bernays set theory and so achieve the validity of the Continuum Hypothesis, cf. [G]). In AST we are able to restrict the collection of sets by many ways - see e.g. endomorphic universes [S-V 1]. On the other hand we are not able to restrict the universal class suitably - more precisely we cannot choose sets using a set-formula so that the class of chosen sets has properties analogical to the class of Gödel's constructible sets. In fact, if $\Theta(z)$ is a set-formula and if the class $\{z, \Theta(z)\}$ is closed under the sole conveniently chosen operation i.e. if we have $(\Theta(x) \& \Theta(y)) \longrightarrow \Theta(x \cup \{y\})$ then $V = \{z;\ \Theta(z)\}$ by the axiom of induction (and an analogue of the class of Gödel's constructible sets has to be closed under $x \cup \{y\}$ evidently). The basis of this impossibility of the construction of a set-theoretically definable class different from V and closed under the operation $x \cup \{y\}$ lies in the fact that all sets in AST are finite from the point of view of Cantor's set theory (they satisfy all ZF-axioms if the axiom of infinity is replaced by its negation). Thus it seems to be hardly possible to use Gödel's method to restrict the universal class without essential changes.

§ 3. <u>Some consequences of the axiom of constructibility.</u> In this section we introduce a formula $\Psi(X,Y,Q)$ which represents a well-ordering of Q-constructible classes and using it we are going to show that the strong schema of choice is a consequence of the axiom of constructibility. Furthermore, we shall specify what we mean by the minimality of the system of classes $\mathcal{L}(Q)$ and supposing the axiom of Q-constructibility we shall see that $\neg(\exists \leq)\ \mathcal{AST}^{(\mathcal{L}(\leq,Q))}$, this result will be used in the last section.

In AST there are well-orderings of the universal class, let us fix one of them, say $\leq_1$. To every well-ordering $\leq$ we define the well-ordering $\tilde{\leq}$ putting

$\langle a,x\rangle \tilde{\leq} \langle b,y\rangle \equiv (x < y \vee [x = 0_{\leq} = y \& (a = 0 \vee (a \neq 0 \neq b \& \& a \leq_1 b))] \vee [x = y \neq 0_{\leq} \& x \in dom(\leq) \& (\exists \varphi, q_1, \ldots, q_n, \psi, q_1', \ldots, q_m')(a = \langle \varphi, q_1, \ldots, q_n\rangle \& b = \langle \psi, q_1', \ldots, q_m'\rangle \& (\varphi \leq_1 \psi \vee$

$\vee(\varphi = \psi \;\&\; n \in m) \vee (\varphi = \psi \;\&\; n = m \;\&\; (\exists i \leq n)\,[(\forall j \leq i)(q_j = q_j') \;\&$
$\&\; (q_{i+1} \precsim q'_{i+1} \vee i = m)])))])$.

If $\Phi(\preccurlyeq, S, Q)$, then we have $\mathrm{dom}(\mathrm{rng}(S)) \subseteq \mathrm{dom}(\precsim)$ and furthermore it is $q \in (\mathrm{dom}(\precsim) - \mathrm{dom}(\mathrm{rng}(S))) \rightarrow \mathrm{rng}(S)''\{q\} = 0$.

We define further

$\Psi(X,Y,Q) \equiv (\exists \preccurlyeq)(\exists S)(\Phi(\preccurlyeq, S, Q) \;\&\; (\exists q \in \mathrm{dom}(\precsim))\,(X =$
$= \mathrm{rng}(S)''\{q\} \;\&\; (\forall q' \precsim q)(Y \neq \mathrm{rng}(S)''\{q'\})))$.

If G is an isomorphism of $\preccurlyeq$ onto $\trianglelefteq$ then for the mapping $\tilde{G}$ defined in the last section we can prove by induction

$(\forall q_1, q_2 \in \mathrm{dom}(\tilde{G}))(q_1 \precsim q_2 \equiv \tilde{G}(q_1) \trianglelefteq \tilde{G}(q_2))$.

According to the second section we have

$(\Psi(X,Y,Q) \;\&\; \Psi(Y,X,Q)) \rightarrow X = Y$

and for every $X, Y \in \mathcal{C}(Q)$ the disjunction

$\Psi(X,Y,Q) \vee \Psi(Y,X,Q)$

holds since for every such class there are $\preccurlyeq$, S and $q_1, q_2 \in$ $\in \mathrm{dom}(\precsim)$ with

$\Phi(\preccurlyeq, S, Q) \;\&\; X = \mathrm{rng}(S)''\{q_1\} \;\&\; Y = \mathrm{rng}(S)''\{q_2\}$.

Moreover, let us realize that the transitivity i.e.

$(\Psi(X,Y,Q) \;\&\; \Psi(Y,Z,Q)) \rightarrow \Psi(X,Z,Q)$

is trivial.

Let us note that for each Y, the system $\{X;\ \Psi(X,Y,Q)\}$ is codable.

<u>Metatheorem</u>. To every formula $\Theta(Z_1, Z_2)$ there is a formula $\tilde{\Theta}(Z_1, Z_2)$ such that in AST + axiom of Q-constructibility we can prove

(a) $(\forall X, Y)(\tilde{\Theta}(X,Y) \rightarrow \Theta(X,Y))$

(b) $(\forall X)((\exists Y)\, \Theta(X,Y) \rightarrow (\exists !\, Y)\, \tilde{\Theta}(X,Y))$.

Demonstration. Putting

$\tilde{\Theta}(Z_1, Z_2)$ iff $\Theta(Z_1, Z_2) \;\&\; \neg\, (\exists Z)(\Psi(Z, Z_2, Q) \;\&$
$\&\; \Theta(Z_1, Z) \;\&\; Z \neq Z_1)$

we get the implication (a) trivially. Let us proceed in AST + + $(\forall X)\ X \in \mathcal{L}(Q)$. Assuming

$$\Phi(\preccurlyeq, S, Q) \,\&\, (\exists Y)(\Theta(X,Y) \,\&\, Y \in \mathcal{L}(\preccurlyeq, Q))$$

we choose q as the least element in the ordering $\succcurlyeq$ so that $\Theta(X, \mathrm{rng}(S)''\{q\})$ and we obtain $\tilde{\Theta}(X, \mathrm{rng}(S)''\{q\})$ according to the definition of Ψ.

Corollary. The axiom of constructibility implies (in AST)

(a) strong schema of choice

(b) schema of dependent choices i.e. the system of axioms of the form

$$(\forall Z_1)(\exists Z_2)\,\Theta(Z_1,Z_2) \longrightarrow (\forall X)(\exists Y)(\mathrm{dom}(Y) = FN \,\&\, Y''\{0\} = X \,\&\, (\forall n \in FN)\,\Theta(Y''\{n\}, Y''\{n+1\}))$$

where Θ is an arbitrary formula.

Demonstration. If a formula Θ is given then $\tilde{\Theta}$ denotes the formula constructed in the last Metatheorem. Let us proceed in AST + $(\forall X)\ X \in \mathcal{L}(Q)$:

(a) we put $Y = \{\langle y,x\rangle ; (\exists Z)(\tilde{\Theta}(x,Z) \,\&\, y \in Z)\}$.

(b) Assuming $(\forall Z_1)(\exists Z_2)\,\Theta(Z_1,Z_2)$, we define for every X the class Y by induction putting $Y''\{0\} = X$ and choosing $Y''\{n+1\}$ so that $\tilde{\Theta}(Y''\{n\}, Y''\{n+1\})$.

Metalemma. If $\mathfrak{M}$ is a system of classes containing all sets with $AST^{(\mathfrak{M})}$, then for every Q and $\preccurlyeq$ elements of $\mathfrak{M}$ we have

$$\mathcal{L}(\preccurlyeq, Q) \subseteq \mathfrak{M} \,\&\, (\exists S \in \mathfrak{M})\ \Phi(\preccurlyeq, S, Q).$$

Demonstration. According to the second section we have $[(\exists S)\ \Phi(\preccurlyeq, S, Q)]^{(\mathfrak{M})}$ because of $AST^{(\mathfrak{M})}$ and because $\preccurlyeq$ is an $\mathfrak{M}$-well-ordering, too. Thus using the absoluteness of Φ (in the interpretation determined by $\mathfrak{M}$) we get $(\exists S \in \mathfrak{M})\Phi(\preccurlyeq, S, Q)$, from which $(\forall q)\mathrm{rng}(S)''\{q\} \in \mathfrak{M}$ follows.

__Theorem.__ If $\mathfrak{M}$ is a codable system of classes with $V \cup \{Q, FN\} \subseteq \mathfrak{M}$ & $\mathcal{AST}^{(\mathfrak{M})}$ then $\mathcal{L}(Q) \subseteq \mathfrak{M}$.

Proof. The property "to be a linear ordering" is absolute (in the interpretation determined by $\mathfrak{M}$) trivially; moreover, $\mathfrak{M}$ contains all countable classes because this system contains FN and all sets, thence even the property "to be a well-ordering" is absolute.

If there is $X \in (\mathcal{L}(Q) - \mathfrak{M})$, then there is a well-ordering such that no element of $\mathfrak{M}$ is isomorphic to it by the last result. Let us fix a constant $\trianglelefteq$ as a well-ordering of the smallest possible ordinal type having this property. By the definition of $\trianglelefteq$ to each $x \in \mathrm{dom}(\trianglelefteq)$ there is an $\mathfrak{M}$-well-ordering isomorphic to $\trianglelefteq \restriction \{y; y \triangleleft x\}$ and conversely to each $\mathfrak{M}$-well-ordering there is $x \in \mathrm{dom}(\trianglelefteq)$ so that $\trianglelefteq \restriction \{y; y \triangleleft x\}$ is isomorphic to the $\mathfrak{M}$-well-ordering in question. The well-ordering $\trianglelefteq$ cannot have the last element because of $\mathcal{AST}^{(\mathfrak{M})}$ and thus by the last Metalemma we obtain

$$X \in \mathcal{L}(\trianglelefteq, Q) \equiv [(\exists \leq)(\exists S)(\Phi(\leq, S, Q) \,\&\, X \in \mathrm{Dc}(\mathrm{rng}(S)))]^{(\mathfrak{M})}.$$

Therefore every class parametrically definable in the model determined by the system of classes $\mathcal{L}(\trianglelefteq, Q)$ (i.e. every element of $\mathcal{L}(\trianglelefteq + 1, Q)$) is parametrically definable, too, in the model determined by $\mathfrak{M}$, hence it is an element of $\mathfrak{M}$ because $\mathcal{AST}^{(\mathfrak{M})}$ is assumed. Since $\trianglelefteq \notin \mathfrak{M}$ we have $\trianglelefteq \notin \mathcal{L}(\trianglelefteq + 1, Q)$ from which $\mathcal{L}(Q) \subseteq \mathcal{L}(\trianglelefteq, Q) \subseteq \mathfrak{M}$ follows.

We have proved, moreover, that if there is a codable system of classes $\mathfrak{M}$ with $V \cup \{Q, FN\} \subseteq \mathfrak{M}$ & $\mathcal{AST}^{(\mathfrak{M})}$ then $(\exists \trianglelefteq)\, \mathcal{L}(Q) = \mathcal{L}(\trianglelefteq, Q)$ (i.e. case (A)). Let us note that in the next section we shall show (for convenient constants Q's) even the converse implication (we have, moreover, $\mathcal{AST}^{\mathcal{L}(Q)}$ in this case) and thus we shall be able to conclude that if there is a codable system of classes $\mathfrak{M}$ with $V \cup \{Q, FN\} \subseteq \mathfrak{M}$ & $\mathcal{AST}^{(\mathfrak{M})}$, then the system of classes $\mathcal{L}(Q)$ is the minimal one with the mentioned property. Consequently, using the following statement, the existence of a codable system of classes $\mathfrak{M}$ with $V \cup \{Q, Fn\} \subseteq \mathfrak{M}$ & $\mathcal{AST}^{(\mathfrak{M})}$ will be excluded in AST + axiom of Q-constructibility.

Theorem. If $(\forall X)\, X \in \mathcal{L}(Q)$, then there is no well-ordering $\trianglelefteq$ so that $\mathcal{AST}^{(\mathcal{L}(\trianglelefteq, Q))}$.

Proof. If $\mathcal{AST}^{(\mathcal{L}(\trianglelefteq, Q))}$, then $\mathcal{L}(Q) \subseteq \mathcal{L}(\trianglelefteq, Q)$ by the last theorem and hence the system of all classes would be codable - this would contradict the second theorem of § 5 ch.I [V.].

Let us note that in the last results we can assume (Morse's schema)$^{(\mathfrak{M})}$ and (*Morse's schema*)$^{(\mathfrak{M})}$ instead of AST$^{(\mathfrak{M})}$ and $\mathcal{AST}^{(\mathfrak{M})}$ respectively.

§ 4. The interpretation $\mathfrak{L}(Q)$. The system of Q-constructible classes determines naturally an interpretation which will be denoted $\mathfrak{L}(Q)$; formally

$$Cls^{\mathfrak{L}(Q)}(X) \equiv X \in \mathcal{L}(Q) \text{ and}$$

$$X^{\mathfrak{L}(Q)} \in^{\mathfrak{L}(Q)} Y^{\mathfrak{L}(Q)} \equiv X^{\mathfrak{L}(Q)} \in Y^{\mathfrak{L}(Q)}.$$

In this section we are going to show - for convenient constants Q - that the interpretation $\mathfrak{L}(Q)$ is a restriction of view of AST + axiom of Q-constructibility in AST. For this purpose the following Lemma is useful.

Lemma. If $\leq$ is an element of $\mathcal{L}(\trianglelefteq, Q)$ and if FN is definable from Q then there is $S \in \mathcal{L}(\trianglelefteq + \leq + 1, Q)$ with $\Phi(\leq, S, Q)$.

Proof. There is S with $\Phi(\leq, S, Q)$ by § 2 and we have to show $S \in \mathcal{L}(\trianglelefteq + \leq + 1, Q)$. To obtain a contradiction let us suppose that x is the smallest element in the ordering $\preceq$ so that

$$S \restriction \{z; z < x\} \notin \mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}) + 1, Q).$$

At first let us assume that x is the successor of an element y in the well-ordering $\preceq$. According to our choice of x we have

$$S \restriction \{z; z < y\} \in \mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}), Q)$$

because $\trianglelefteq + (\preceq \restriction \{z; z < y\}) + 1$ is isomorphic to $\trianglelefteq + (\preceq \restriction \{z; z < x\})$.

For every $q \in \mathrm{dom}(S''\{y\})$ the class $(S''\{y\}) \restriction \{q'; q' \mathcal{Z} q\}$ is definable by a normal formula with parameters q and $S \restriction \{z; z < y\}$ (or Q if $y = 0_{\preceq}$) only, therefore this class is an element of the system of classes $\mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}), Q)$. The class $\mathrm{dom}(S''\{y\})$ is definable, too, using a normal formula with parameters Q (or FL if somebody prefers) and $\mathrm{dom}(S''\{z; z < y\})$ only and hence the class

$$S \restriction \{z; z < x\} = S \restriction \{z; z < y\} \cup \bigcup \{S''\{y\} \restriction \{q'; q' \mathcal{Z} q\}; q \in \mathrm{dom}(S''\{y\})\}$$

is definable in the model determined by the system of classes $\mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}), Q)$ and thence it is an element of $\mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}) + 1, Q)$.

If $x \neq 0_{\preceq}$ is limit then for every $y < x$ we have

$$S \restriction \{z; z < y\} \in \mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < y\}) + 1, Q) \subseteq \subseteq \mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}), Q)$$

and therefore using the uniqueness mentioned in the second section we obtain

$$S \restriction \{z; z < x\} = \bigcup \{S \restriction \{z; z < y\}; y < x\} = \bigcup \{\tilde{S} \in \mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}), Q); (\exists y < x)\ \Phi(\preceq \restriction \{z; z < y\}, \tilde{S}, Q)\}.$$

Since Φ is normal, the formula $\Phi^{(\mathcal{L}(\trianglelefteq + (\preceq \restriction \{z; z < x\}), Q))}$

is the same as the formula Φ and thus

$$S \restriction \{z; z < x\} \in \mathcal{L}(\preceq + (\preceq \restriction \{z; z < x\}) + 1, Q)$$

as a class definable in the model determined by $\mathcal{L}(\preceq + (\preceq \restriction \{z; z < x\}), Q)$. Since $S \restriction \{z; z < 0_{\preceq}\} = 0$ is evidently an element of $\mathcal{L}(\preceq + 1, Q)$, we are done.

In the following we shall assume that $Q''\{0\}$ is a well-ordering of V of type Ω. Under this assumption we are able to show that $\mathcal{L}(Q)$ is an interpretation of AST + axiom of Q-constructibility in AST and if moreover the alternative (A) holds then $\mathcal{L}(Q)$ determines even a model of $\mathcal{AST}$. Of course, it would be sufficient to suppose that a well-ordering with the desired properties is defined by whatever combination of Gödel's operations from Q, but in the general case $\mathcal{L}(Q)$ need not be an interpretation of AST - at the end of the paper we shall see that $\mathcal{L}(0)$ cannot be an interpretation of the axiom of choice in AST.

<u>Metatheorem</u>. The interpretation $\mathcal{L}(Q)$ is a restriction of view of AST + $(\forall X)\, X \in \mathcal{L}(Q)$ in AST +"$Q''\{0\}$ is a well-ordering of V of type Ω" . Moreover in the lastly mentioned theory we can prove

$$(\exists \preceq)(\mathcal{L}(Q) = \mathcal{L}(\preceq, Q)) \rightarrow [\mathcal{AST} + (\forall X)\, X \in \mathcal{L}(Q)]^{(\mathcal{L}(Q))}.$$

Demonstration. We write $\mathcal{L}$ instead of $\mathcal{L}(Q)$. Directly from the definition of $\mathcal{L}(\preceq, Q)$ we see that these systems are closed under Gödel's operations (except $\mathcal{L}(\{\langle 0,0\rangle\}, Q)$, may be) and hence the same is true for the system of all $\mathcal{L}$-classes. Evidently each set is an $\mathcal{L}$-set; formulae Φ , Ψ are normal and thence they are absolute.

Furthermore $FN^{\mathcal{L}} = FN$ since the class FN is describable from $Q''\{0\}$ (e.g. $FN = \{\alpha \in N; (\exists f)(\exists x \in dom(Q''\{0\}))($ f is an isomorphism of $\in \cap \alpha^2$ onto $Q''\{0\} \upharpoonright \{y; \langle y,x\rangle \in Q''\{0\}\}\})$. Every countable class is an $\mathcal{L}$-class because every countable class is of the form f"FN according to the prolongation axiom. Thus the property "to be a well-ordering" is absolute because the property "to be a linear ordering" is absolute trivially.

In the case (A), for each $\leq$ smaller than $\leq_0$ there is a well-ordering $\leq^{\mathcal{L}}$ isomorphic to $\leq$ since if there is no such $\leq^{\mathcal{L}}$, in $\mathcal{L}(\leq + 1,Q)$ cannot be a well-ordering of type greater or equal to $\leq$ and this contradicts our choice of $\leq_0$.

In the case (B) for every $\leq$ there is $\leq^{\mathcal{L}}$ isomorphic to $\leq$ because in $\mathcal{L}(\leq + 1,Q)$ is a well-ordering of type greater or equal to $\leq$ according to the assumption (B) and because if $\trianglelefteq$ is an $\mathcal{L}$-well-ordering then $\trianglelefteq \upharpoonright \{y; y \triangleleft x\}$ is an $\mathcal{L}$-well-ordering, too, for each $x \in \mathrm{dom}(\trianglelefteq)$.

Thus we get as a trivial consequence

$$(\forall X^{\mathcal{L}})(\exists \leq^{\mathcal{L}})\, X^{\mathcal{L}} \in \mathcal{L}(\leq^{\mathcal{L}},Q).$$

Moreover, by the previous Lemma we have

$$\mathcal{L}(\leq^{\mathcal{L}},Q) = \mathcal{L}^{\mathcal{L}}(\leq^{\mathcal{L}},Q)$$

(because $\trianglelefteq^{\mathcal{L}} + \leq^{\mathcal{L}}$ is an $\mathcal{L}$-well-ordering) and therefore we have proved the $\mathcal{L}$-axiom of Q-constructibility.

We are going to show that for every formula Θ and every well-ordering $\leq^{\mathcal{L}}$ there is a well-ordering $\trianglelefteq^{\mathcal{L}}$ greater than $\leq^{\mathcal{L}}$ so that

$$(\forall X_1,\dots,X_n \in \mathcal{L}(\leq^{\mathcal{L}},Q))(\Theta^{\mathcal{L}}(X_1,\dots,X_n) \equiv$$
$$\equiv \Theta^{(\mathcal{L}(\trianglelefteq,Q))}(X_1,\dots,X_n)).$$

According to the last paragraph and to the properties of the formula Ψ, to every $X^{\mathcal{L}}$ we can choose one well-ordering $\leq^{\mathcal{L}}_X$ with $X^{\mathcal{L}} \in \mathcal{L}(\leq^{\mathcal{L}}_X,Q)$. Furthermore since every $\mathcal{L}(\leq,Q)$ is codable we are able to choose to every $\leq^{\mathcal{L}}$ a well-ordering

with the desired property using properties of the formula Ψ, the usual construction based on Skolem's functions given by Θ and the fact that for each codable system of well-orderings there is a well-ordering greater than each element of the system in question. Thus in the case (B) we are done because to $\preceq$ there is a well-ordering $\preceq^{\mathcal{L}}$ isomorphic to $\preceq$, but in the case (A) we have to prove moreover that $\preceq$ is of type smaller than $\leq_0$ (and then again there is $\preceq^{\mathcal{L}}$ isomorphic to $\preceq$).
We have

$$\Psi(X^{\mathcal{L}},Y^{\mathcal{L}}) \equiv \Psi^{\mathcal{L}}(X^{\mathcal{L}},Y^{\mathcal{L}}) \equiv \Psi^{(\mathcal{L}(\leq_0,Q))}(X^{\mathcal{L}},Y^{\mathcal{L}})$$

and

$$X^{\mathcal{L}}\in \mathcal{L}(\preceq_1^{\mathcal{L}},Q) \equiv X^{\mathcal{L}}\in \mathcal{L}^{\mathcal{L}}(\preceq_1^{\mathcal{L}},Q) \equiv X^{\mathcal{L}}\in \mathcal{L}^{(\mathcal{L}(\leq_0,Q))}(\preceq_1^{\mathcal{L}},Q)$$

for every $\preceq_1^{\mathcal{L}}$ and therefore $\preceq$ is definable in the model determined by $\mathcal{L}(\leq_0,Q)$, thence it is an element of $\mathcal{L}(\leq_0+1,Q)$ and thus our statement follows from the choice of $\leq_0$.

In particular, for every Θ and every $X_1^{\mathcal{L}},\dots,X_n^{\mathcal{L}}$ there is $\preceq^{\mathcal{L}}$ with $X_1^{\mathcal{L}},\dots,X_n^{\mathcal{L}} \in \mathcal{L}(\preceq^{\mathcal{L}},Q)$ such that

$$(\forall x)(\Theta^{\mathcal{L}}(x,X_1^{\mathcal{L}},\dots,X_n^{\mathcal{L}}) \equiv \Theta^{(\mathcal{L}(\preceq^{\mathcal{L}},Q))}(x,X_1^{\mathcal{L}},\dots,X_n^{\mathcal{L}})$$

and hence we have proved the $\mathcal{L}$-Morse's schema because

$$\{x,\ \Theta^{\mathcal{L}}(x,X_1^{\mathcal{L}},\dots,X_n^{\mathcal{L}})\}\in \mathcal{L}(\preceq^{\mathcal{L}}+1,Q).$$

If $\mathcal{L}(Q) = \mathcal{L}(\preceq,Q)$ for some $\preceq$, the previous considerations are true even for formal formulae.

The $\mathcal{L}$-axiom of extensionality and all $\mathcal{L}$-axioms concerning sets are trivial (the property $V \models \varphi$ being absolute); the $\mathcal{L}$-axiom of choice and the $\mathcal{L}$-axiom of cardinalities hold according to the definition of $\mathcal{L}(\{\langle 0,0\rangle\},Q)$ and to the requirement put on the constant Q. At the end let us consider that the $\mathcal{L}$-prolongation axiom is an easy consequence of the prolongation axiom, absoluteness of the class FN and the fact that each set is an $\mathcal{L}$-set.

Corollary. If $Q"\{0\}$ is a well-ordering of V of type Ω then the following is equivalent:

(a) $(\exists \preceq)\ \mathcal{L}(Q) = \mathcal{L}(\preceq, Q)$ (case (A))

(b) there is a codable system of classes $\mathfrak{M}$ with $Q, FN \in \mathfrak{M}$ & $\mathcal{AST}^{(\mathfrak{M})}$.

§ 5. Shiftings of view. Let us recall some definitions from [S-V 1] and [S-V 2]. A class X is said to be revealed if for every countable $Y \subseteq X$ there is a set u with $Y \subseteq u \subseteq X$ and X is called fully revealed if every class of the form $\{x; \varphi(x,X)\}$ is revealed under the assumption that $\varphi \in FL$ is a normal formula; X is a revealment of Y iff X is fully revealed and for every normal formula $\varphi \in FL$ we have $\varphi(X) \equiv \varphi(Y)$ (in other words X is fully revealed iff there is no normal formula $\varphi \in FL$ describing FN using the parameter X and set-parameters only).

Normal formulae (even elements of FL) are absolute in each shifting of view.

Let $*$ be a restriction of view. Then the properties "to be revealed" and "to be fully revealed" are absolute since $FN^* = FN$ and since every countable class is of the form f"FN. Thus even the property "X is a revealment of Y" is absolute. Furthermore let us realize that the property $X \in \mathcal{L}(\preceq, Q)$ is absolute since it is equivalent both to $(\forall S)(\Phi(\preceq, S, Q) \rightarrow X \in Dc(rng(S)))$ and to $(\exists S)(\Phi(\preceq, S, Q)\ \&\ X \in Dc(rng(S)))$ and since the formulae $\Phi(\preceq, S, Q)$ and $X \in Dc(rng(S))$ are absolute. At the end let us appreciate that for every $\preceq^*$ and Q^* we have $(\forall X)(X \in \mathcal{L}(\preceq^*, Q^*) \rightarrow Cls^*(X))$ by Metalemma of § 3. Thus, in particular, if for each well-ordering there is a $*$-well-ordering isomorphic to the given one then $(\forall X \in \mathcal{L}(Q^*))\ Cls^*(X)$.

In [S-V 5] we have constructed shiftings of the horizon in AST + schema of choice, the following statement shows the existence of shiftings of the horizon in AST itself.

<u>Metatheorem</u>. There are shiftings of the horizon in AST, moreover in AST + "B is a revealment of FN" we can construct a shifting of the horizon $*$ with $FN^* = B$.

Demonstration. According to § 1 [S-V 5] there is a shifting of the horizon $\#$ in the theory AST + "B is a revealment of FN" + schema of choice and moreover $\#$ fulfils the requirement $FN^{\#} = B$. Let us fix $\leq$ so that $\leq$ is a well-ordering of V of type Ω (the existence follows from the axiom of cardinalities and from the construction of Ω in § 3 ch. II [V]) and let us put $Q = \leq \times \{0\} \cup B \times \{1\}$. The interpretation $*$ is defined as the composition of $\mathcal{L}(Q)$ and $\#$. Now, it is sufficient to realize that $\mathcal{L}(Q)$ is a restriction of view of AST + "B is a revealment of FN" + schema of choice in AST + "B is a revealment of FN" by § 3,4 and absoluteness mentioned at the beginning of this section. At the end let us consider that in [S-V 2] the existence of revealments of FN was proved.

Thus the question whether there are (nontrivial) shiftings of the horizon in AST was solved positively. However, let us mention an open problem in this area: For every so far constructed shifting of the horizon $*$ in T we have $T \vdash *$-schema of choice and thus if $T \vdash \neg$ schema of choice then there are statements which are not absolute. Question is if we are able to construct shiftings of the horizon in AST in such a way that all statements are absolute (writing AST + schema of choice instead of AST, the problem is solved confirmatively, see [S-V 5]).

Dealing with the existence of nontrivial restrictions of view we shall obtain a partial answer since we shall see that in AST + axiom of Q-constructibility we are not able to construct a nontrivial restriction of view $*$ such that Q is a $*$-class.

<u>Metatheorem</u>. If $*$ is a restriction of view of AST + + $(\forall X)\ X \in \mathcal{L}(Q)$ in a theory T such that $T \vdash (\exists X \in \mathcal{L}(Q)) \neg$ $\neg \mathrm{Cls}^*(X)$ then we can fix $\preceq$ so that the formula

$$\mathrm{Cls}^*(X) \equiv X \in \mathcal{L}(\preceq, Q)$$

is provable in T (with the constant $\preceq$).

Demonstration. Let us proceed in T. According to $(\exists X \in \mathcal{L}(Q)) \neg \mathrm{Cls}^*(X)$, § 2 and to Metalemma of the third section, there is a well-ordering such that no $*$-well-ordering is isomorphic to it. Let us fix $\preceq$ as a well-ordering of the smallest possible ordinal type with this property. The formula $(\forall X \in \mathcal{L}(\preceq, Q))\ \mathrm{Cls}^*(X)$ follows trivially from the mentioned Metallemma and from the definition of $\preceq$ since $\preceq$ cannot have the last element.

The interpretation $*$ is furthermore supposed to be an interpretation of the axiom of Q-constructibility in T and thus to every X^* there is $\leq^*$ with $X^* \in^* \mathcal{L}^*(\leq^*, Q)$; however, the last formula is equivalent to the formula $X^* \in \mathcal{L}(\leq^*, Q)$ and thence by the definition of $\preceq$, for each X^* there is $x \in \mathrm{dom}(\preceq)$ with $X \in \mathcal{L}(\preceq \restriction \{y; y \prec x\}, Q)$ and therefore $X^* \in \mathcal{L}(\preceq, Q)$. We have proved $(\forall X^*) X^* \in \mathcal{L}(\preceq, Q)$ and we are done.

<u>Corollary</u>. If $*$ is a restriction of view in a theory T such that $T \vdash \mathrm{Cls}^*(Q) \,\&\, (\exists X \in \mathcal{L}(Q)) \neg \mathrm{Cls}^*(X) \,\&$ "Q"$\{0\}$ is a well-ordering of V of type Ω " then we can fix $\preceq$ so that $\mathrm{Cls}^{\#}(X)$ defined by $X \in \mathcal{L}(\preceq, Q)$ determines a nontrivial restriction of view in T (with the constant $\preceq$).

Demonstration. The composition of $*$ and $\mathcal{L}(Q)$ is a restriction of view of AST + $(\forall X)\, X \in \mathcal{L}(Q)$ in T, therefore our statement is a trivial consequence of the previous result.

Let us continue in a sufficiently strong metamathematics (e.g. ZF is strong enough, cf. § 9 [S 3]). Furthermore, let us assume that $*$ is a nontrivial restriction of view (with Q as a parameter) of AST in AST + $(\forall X)\, X \in \mathcal{L}(Q)$ such that

AST + $(\forall X)\, X \in \mathcal{L}(Q) \vdash \mathrm{Cls}^{*}(Q)$.

According to § 9 [S 3] and to the fourth section of this article there is a model $\mathcal{O}\!\mathcal{l}$ and its class Q so that

$\mathcal{O}\!\mathcal{l} \models$ AST + $(\forall X)\, X \in \mathcal{L}(Q)$ + "Q"$\{0\}$ is a well-ordering of V of type Ω " and $\mathrm{FN}^{\mathcal{O}\!\mathcal{l}} = \omega$ (ω denotes the set of metamathematical natural numbers). By the last Corollary we can fix an $\mathcal{O}\!\mathcal{l}$-well-ordering $\trianglelefteq$ so that $\mathcal{O}\!\mathcal{l} \models \mathrm{AST}^{(\mathcal{L}(\trianglelefteq, Q))}$. However, $\mathrm{FN}^{\mathcal{O}\!\mathcal{l}} = \omega$ and hence we have even $\mathcal{O}\!\mathcal{l} \models \mathcal{AST}^{(\mathcal{L}(\trianglelefteq, Q))}$ which contradicts the last statement of the third section.

We have proved that there is no nontrivial restriction of view $*$ of AST in AST + $(\forall X)\, X \in \mathcal{L}(Q)$ with Q as a parameter such that Q is a $*$-class in all cases.

Open problem. Is there a nontrivial restriction of view in AST (i.e. is it provable in AST that there is a well-ordering $\leq$ of V of type Ω with $(\exists X)\, X \notin \mathcal{L}(\leq \times \{0\})$)?

At the end we are going to show that $\mathcal{L}(0)$ need not be an interpretation of AST. Let $\mathcal{R}$ denote the system of real classes defined in [Č-V]; by the sixth theorem of § 1 of the cited paper $\mathcal{R}$ determines an interpretation of Morse's schema. Assuming $\mathcal{O}\!\mathcal{l} \models$ AST and $\mathrm{FN}^{\mathcal{O}\!\mathcal{l}} = \omega$ we have $\mathcal{O}\!\mathcal{l} \models (\textit{Morse's schema})^{(\mathcal{R})}$ and thus according to the third section (supposing coincidence of the classes of metamathematical and finite natural numbers we need not require the codability of the system of classes in question) we obtain $\mathfrak{A} \models \mathcal{L}(0) \subseteq \mathcal{N}$ and therefore $\mathfrak{A} \models$ "there is no well-ordering of V in $\mathcal{L}(0)$" by § 1 [Č-V]. We have proved that $\mathcal{L}(0)$ is no interpretation of the axiom of choice in AST.

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