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ON THE COMPLEXITY OF THE SUBGRAPH PROBLEM

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Abstract: The complexity of the problem "Does a given graph contain a complete subgraph with k vertices?" is $O(n^k)$.

Key words: Complexity of the subgraph problem, complete subgraph.

Classification: 05C99

This note is motivated by the complexity of the following decision problem:

Given a graph G and a positive integer k, does there exist a subgraph of G isomorphic to K_k (= the complete graph with k vertices)?

The following particular question was considered independently by L. Lovász and one of us:

Is the complexity of the above problem $O(n^k)$?

In this note we give a positive answer to this question in a slightly more general form. Let us note that we have been informed by L. Lovász that F.K. Chung and R. Karp obtained independently also a solution to the above problem.

Let us stress that all the solutions are based on the fast matrix multiplication and that it is not clear whether one could devise a purely combinatorial algorithm.

1: Fast recognition of complete subgraphs

1.1: First we show how to detect a triangle:

Let G be a graph with vertices $x_1,\dots,x_n$ and let A be the adjacency matrix of G (i.e. $a_{ij} = 1$ if x_i, x_j form an edge of G, $a_{ij} = 0$ otherwise). Compute the matrix $B := A^2$. Then the graph G does not contain a triangle if and only if $\min(a_{ij},b_{ij}) = 0$ for all i, j (as b_{ij} is the number of paths of length 2 between x_i and x_j). The complexity of this procedure is $O(n^{\alpha})$ providing we use an $O(n^{\alpha})$ algorithm for the matrix multiplication. It is well known that one may achieve $\alpha < 3$ (see Concluding remarks). If $0 < a_{ij} \leq b_{ij}$ for some i, j then G contains a triangle of the form $\{x_i,x_j,x_k\}$. The third vertex x_k can be found in O(n) steps by checking all the remaining vertices.

1.2: This procedure may be used for detection of complete subgraphs of size 3ℓ in $O(n^{\ell\cdot\alpha})$ steps as follows:

For a given graph G of size n we construct an auxiliary graph H of size $O(n^{\ell})$ with the following property: H contains a triangle iff G contains a complete subgraph of size 3ℓ.

Thus the detection of triangles in H yields an $O(n^{\ell\cdot\alpha})$ algorithm for the detection of a complete subgraph of size 3ℓ.

The graph H may be defined as follows:

$V(H) = \{Y \subseteq V(G);\ |Y| = \ell$ and Y forms a complete subgraph in G$\}$

$E(H) = \{\{Y,Y'\};\ Y \neq Y'$ and $Y \cup Y'$ forms a complete subgraph in G$\}$.

1.3: Let us also remark that the vertex sizes which are not divisible by 3 do not present a difficulty by the following:

For a subset Y of vertices of graph G = (V,E) put N(Y) = $= \{v \in V \mid \{y,v\} \in E$ for every $y \in Y\}$.

$N(Y)$ is the set of all common neighbors of the set Y.

Consider all graphs $G_1, \dots, G_n$ which are induced by the sets $N(\{x\})$, $x \in V$. Then a graph G_i contains a complete subgraph of size 3ℓ if and only if G contains a complete subgraph of size $3\ell + 1$.

Similarly if we consider all graphs which are induced by the sets $N(\{x,y\})$, $\{x,y\} \in E$ we can detect a complete subgraph of size $3\ell + 2$.

Thus, using the previous $O(n^{\ell \cdot \alpha})$ for a 3ℓ-complete subgraph, we can detect a $(3\ell + i)$-complete subgraph of a graph with n vertices in $O(n^{i + \ell \cdot \alpha})$ steps, $i = 0,1,2$,

2: Fast recognition of arbitrary subgraphs

2.1: Here we prove

<u>Proposition</u>. Let F be a fixed graph with k vertices. Let there exist an $O(n^{\alpha(k)})$ algorithm for finding a K_k in a graph with n vertices. Then the following two problems can be solved in $O(n^{\alpha(k)})$ steps for arbitrary graph G with n vertices:

(1) Does G contain F as an induced subgraph?

(2) Does G contain F as a (not necessarily induced) subgraph?

We give two proofs.

2.2: <u>Proof I</u>: For a given instance F,G of the problem ((1) or (2)) we construct auxiliary graphs H_1 and H_2 of size $k \times n$ with the property that H_i contains a complete graph of size k iff the answer to the problem (i) is positive, $i = 1,2$.

Put $V(H_1) = V(H_2) = V(F) \times V(G)$. Denote by E_1, E_2 and E_3 the following three sets:

$\{(f_1,g_1),(f_2,g_2)\} \in E_1$ iff $f_i \in V(F)$, $g_i \in V(G)$, $f_1 \neq f_2$ and $g_1 \neq g_2$.

$\{(f_1,g_1),(f_2,g_2)\} \in E_2$ iff $f_1 \in V(F)$, $g_1 \in V(G)$, $\{f_1,f_2\} \in E(F)$ and $\{g_1,g_2\} \in E(G)$.

$\{(f_1,g_1),(f_2,g_2)\} \in E_3$ iff $f_1 \in V(F)$, $g_1 \in V(G)$, $\{f_1,f_2\} \notin E(F)$ or $\{g_1,g_2\} \in E(G)$.

Put $E(H_1) = E_1 \cap E_2$, and $E(H_2) = E_1 \cap E_3$.

It is easy to see that these graphs have the desired properties.

2.3: Proof II: We consider only the case $k = 3\ell$.

We construct an auxiliary graph H as follows:

Let $X_1 \cup X_2 \cup X_3$ be a partition of the set V(F) into parts of size ℓ. Denote by F_i the subgraph of F induced by the set X_i and denote by $F_{i,j}$ the subgraph of F induced by the set $X_i \cup X_j$, $i, j = 1,2,3$, $i \neq j$.

Denote by V_i the set of all embeddings of F_i into G (explicitly: $f \in V_i$ iff $f: X_i \longrightarrow V(G)$ is one-to-one and $\{f(x),f(y)\} \in E(G) \Longleftrightarrow \{x,y\} \in E(F_i)$).

Put $V(H) = V_1 \cup V_2 \cup V_3$ and $\{f,f'\} \in E(H)$ iff $f \in V_i$, $f' \in V_j$ ($i \neq j$) and the mapping $f \cup f'$ is an embedding of $F_{i,j}$ into G.

Clearly H contains a triangle if and only if G contains an induced subgraph isomorphic to F.

3: Concluding remarks

3.1: Instead of Strassen algorithm [2] we could use any of its refinements.

The current best performance $n^{2,495364}$ is due to Coppersmith and Winograd, see [1].

3.2: Apart from the problem of finding a combinatorial algorithm (see the introduction) the following question may be of interest:

Does there exist a graph F with k vertices for which the decision problem

"does G contain F as an induced subgraph"

is easier than the corresponding problem for the complete graph K_k?

Of course the non-induced subgraph problem is easier (e.g. for forests).

R e f e r e n c e s :

[1] D. COPPERSMITH, S. WINOGRAD: On the asymptotic complexity of matrix multiplication, in: Proceedings 22nd Symposium on Foundations of Comp. Sci, 1981, p. 82-90.

[2] V. STRASSEN: Gaussian elimination is not optimal, Num. Math. 13(1969), 354-356.

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