Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 26 (1985), No. 1, 3--21

Persistent URL: http://dml.cz/dmlcz/106345

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

26,1 (1985)

SEMILINEAR PARABOLIC SYSTEMS

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Abstract: We describe a general local existence and regularity result for semilinear parabolic systems of even order. In particular we obtain classical solutions without compatibility conditions for the nonlinearity. Moreover, we describe a simple method for obtaining global existence by means of a generalization of the Gagliardo-Nirenberg inequality to fractional orders of the derivatives.

Key words: Local and global existence, regularity, parabolic systems, time-dependent boundary conditions.

Classification: 35K60, 35B65

In these lectures we review some recent results of the author concerning local and global existence and regularity for semilinear parabolic systems of arbitrary even order. It is one of the main features of our approach to prove first of all a very general existence and regularity theorem, which guarantees the existence of classical solutions on a maximal time interval. In possession of this general theorem one can then treat the question of global existence separately by establishing appropriate a priori bounds in some weak norm without worrying any more about existence questions.

This paper was presented on the International Spring School on Evolution Equations, Dobřichovice by Prague, May 21-25, 1984 (invited paper).

1. <u>Regular Parabolic Systems</u>. Throughout this paper ℓ, m, n and N are fixed positive integers and k is an integer satisfying $0 \leq k \leq k+1 \leq \ell \leq 2m$, T is a fixed positive real number, and Ω is a bounded domain in $\mathbb{R}^n$ of class $C^{2m+\ell}$. Moreover $\mathbb{\Gamma}$ denotes a (necessarily finite) set of nonempty open and closed subsets Γ of $\partial\Omega$ which are pairwise disjoint and whose union equals $\partial\Omega$.

We denote by $\mathcal{A}(t)$ for each $t \in [0,T]$ a linear differential operator of order 2m acting on N-vector-valued functions $u: \Omega \to \mathbb{C}^N$ of the form

$$\mathcal{A}(t)u := (-1)^m \sum_{|\alpha| \leq 2m} a_\alpha(\cdot,t) D^\alpha u,$$

where

$$[t \mapsto a_\alpha(\cdot,t)] \in C^{2-}([0,T], C^\ell(\overline{\Omega}, \mathcal{L}(\mathbb{C}^N)))$$

for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2m$ (where C^{2-} means that the functions have (locally) Lipschitz continuous first derivatives). Moreover:

$$\mathcal{B}(t) := \{\mathcal{B}_\Gamma(t) \mid \Gamma \in \mathbb{\Gamma}\}$$

denotes for each $t \in [0,T]$ a system of boundary operators on $\partial\Omega$, where

$$\mathcal{B}_\Gamma(t) := (\mathcal{B}_\Gamma^1(t), \ldots, \mathcal{B}_\Gamma^{mN}(t))$$

and

$$\mathcal{B}_\Gamma^\rho(t)u := \sum_{|\alpha| \leq m_{\rho,\Gamma}} b_{\alpha,\Gamma}^\rho(\cdot,t) D^\alpha u$$

with $0 \leq m_{\rho,\Gamma} < 2m$ and

$$[t \mapsto b_{\alpha,\Gamma}^\rho(\cdot,t)] \in C^{2-}([0,T], C^{2m+\ell-m_{\rho,\Gamma}}(\Gamma, \mathcal{L}(\mathbb{C}^N, \mathbb{C})))$$

for $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m_{\rho,\Gamma}$, $1 \leq \rho \leq mN$, and $\Gamma \in \mathbb{\Gamma}$.

We let

$$a(x,t,\xi) := \sum_{|\alpha|=2m} a_\alpha(x,t)\,\xi^\alpha \in \mathcal{L}(\mathbb{C}^N)$$

and

$$a_\vartheta(x,t,\xi,\tau) := a(x,t,\xi) + e^{i\vartheta}\tau^{2m} I_N \in \mathcal{L}(\mathbb{C}^N)$$

for $(x,t,\xi) \in \overline{\Omega}\times[0,T]\times\mathbb{R}^n$, $\vartheta \in [-\pi,\pi]$, and $\tau \in \mathbb{R}$, where I_N denotes the identity in $\mathcal{L}(\mathbb{C}^N)$. Similarly,

$$b_\Gamma^\rho(x,\xi) := \sum_{|\alpha|=m_{\rho,\Gamma}} b_{\alpha,\Gamma}^\rho(x)\,\xi^\alpha$$

for $|\alpha| \leq m_{\rho,\Gamma}$, $1 \leq \rho \leq mN$, and $\Gamma \in \mathbf{\Gamma}$, and $b(x,\xi)$ denotes the $(mN\times N)$-matrix with rows $b_\Gamma^\rho(x,\xi)$ for all $x \in \Gamma$, $\Gamma \in \mathbf{\Gamma}$ and $\xi \in \mathbb{R}^n$.

For $1 < p < \infty$ and $s \in \mathbb{R}^+$ we let $W_p^s := W_p^s(\Omega, \mathbb{C}^N)$ and, if $2m \leq s \leq 2m + \ell$,

$$\mathcal{W}_p^{s-1/p} := \prod_{\Gamma\in\mathbf{\Gamma}} \prod_{\rho=1}^{mN} W_p^{s-1/p-m_{\rho,\Gamma}}(\Gamma, \mathbb{C}^N).$$

Then $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbf{\Gamma})$, $0 \leq t \leq T$, is said to be a <u>regular parabolic initial boundary value problem</u> (IBVP) <u>of</u> (class C^ℓ and) <u>order</u> 2m provided the following additional conditions (R), (C) and (S) are satisfied:

(R) There exists a number $\alpha \in (\pi/2, \pi)$ such that

$$\det a_\vartheta(x,t,\xi,\tau) \neq 0$$

and the polynomial of one complex variable

$$\lambda \mapsto \det a_\vartheta(x,t,\xi + \lambda\nu(x),\tau)$$

has precisely mN roots $\lambda_j^+(\vartheta,x,t,\xi,\tau)$, $1 \leq j \leq mN$, with positive imaginary parts for each

$$(1) \qquad \begin{array}{l}(\vartheta,x,t,\xi,\tau) \in [-\alpha,\alpha]\times\partial\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R} \\ \text{with } (\xi\,|\,\nu(x)) = 0 \text{ and } (\xi,\tau) \neq (0,0),\end{array}$$

where ν is the outer normal on $\partial\Omega$ and $(\cdot\,|\,\cdot)$ the euclidean inner product.

(C) For each $(\vartheta, x, t, \xi, \tau)$ satisfying (1) the rows of the matrix-valued function of one complex variable

$$\lambda \mapsto b(x,t,\xi+\lambda\nu(x))\tilde{a}_{\vartheta}(x,t,\xi+\lambda\nu(x),\tau)$$

are linearly independent modulo $\prod_{j=1}^{mN}(\lambda-\lambda_j^+(\vartheta,x,t,\xi,\tau))$ (as polynomials in λ), where $\tilde{a}_{\vartheta}(x,t,\eta,\tau)$ is the matrix whose elements are the cofactors of the elements of the transposed matrix of $a_{\vartheta}(x,t,\eta,\tau)$. If $N = 1$ we put $\tilde{a}_{\vartheta}(x,t,\eta,\tau) := 1$.

(S) For each $t \in [0,T]$ there exists a number $\lambda \in \mathbb{C}$ such that the linear operator

$$(\lambda + \mathcal{A}(t), \mathcal{B}(t)) : W_2^{2m} \to L_2 \times \mathcal{W}_2^{2m-1/2}$$

is surjective.

In the remainder of this section we give some important examples of regular parabolic IBVPs. For this purpose we recall that $(\mathcal{A}(t), \Omega)$, $0 \leq t \leq T$, is said to be a <u>strongly parabolic system</u> if

$$\mathrm{Re}(a(x,t,\xi)\eta \mid \eta) > 0$$

for all $(x,t,\xi,\eta) \in \bar{\Omega} \times [0,T] \times \mathbb{R}^n \times \mathbb{C}^N$ with $\xi \neq 0$ and $\eta \neq 0$ (where now $(\cdot \mid \cdot)$ denotes the "euclidean" inner product in $\mathbb{C}^N$, which is linear in the first and antilinear in the second variable).

(1.1) <u>Examples</u>: (a) Suppose that $N = 1$ (the case of "one equation"), that $(\mathcal{A}(t), \Omega)$, $0 \leq t \leq T$, is strongly parabolic and that $\mathcal{B}(t)$ is a system of m boundary operators covering $\mathcal{A}(t)$ (that is, satisfying the complementing conditions; e.g. [11]) in the usual sense. <u>Then</u> $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \Gamma)$, $0 \leq t \leq T$, <u>is a regular parabolic</u> IBVP <u>of order</u> 2m.

(b) Suppose that $(\mathcal{A}(t), \Omega)$, $0 \leq t \leq T$, is a strongly parabolic system. Moreover, suppose that for each $\Gamma \in \mathbf{\Gamma}$ and $t \in [0,T]$

there are given m vector fields $\beta_{j,\Gamma}(\cdot,t)$ on Γ such that

$$[t \mapsto \beta_{j,\Gamma}(\cdot,t)] \in C^{2-}([0,T], C^{2m+\ell-1}(\Gamma, \mathbb{R}^n))$$

and $(\beta_{j,\Gamma}(x,t) \mid \nu(x)) > 0$ for $j = 1,\dots,m$, $x \in \Gamma$ and $t \in [0,T]$. Then define $(N \times N)$-matrix-valued boundary operators $\hat{\mathcal{B}}_{j,\Gamma}(t)$ by

$$\hat{\mathcal{B}}_{j,\Gamma}(t)u := \frac{\partial^{k+j-1}u}{\partial \beta_{j,\Gamma}(\cdot,t)^{k+j-1}} + \text{lower order terms},$$

where $k := k_\Gamma$ is a fixed integer on Γ with $0 \leq k_\Gamma \leq m$, $1 \leq j \leq m$, $\Gamma \in \mathbb{\Gamma}$, and $t \in [0,T]$. Finally let $\mathcal{B}_\Gamma(t) := (\hat{\mathcal{B}}_{1,\Gamma}(t),\dots$ $\dots,\hat{\mathcal{B}}_{m,\Gamma}(t))$. Then $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbb{\Gamma})$, $0 \leq t \leq T$, <u>is a regular parabolic</u> IBVP <u>of order</u> 2m. Observe that this example covers in particular the case of Dirichlet boundary conditions, where $\beta_{j,\Gamma}(\cdot,t) = \nu$ for $j = 1,\dots,m$, $t \in [0,T]$ and $\Gamma = \partial\Omega$.

(c) <u>Second order strongly parabolic systems</u>: We suppose that m = 1 and use the summation convention, where j, k run from 1 to n. Then we write $\mathcal{A}(t)$ in the form

$$\mathcal{A}(t)u = -a_{jk}(\cdot,t)D_jD_ku + a_j(\cdot,t)D_ju + a_o(\cdot,t)u$$

and consider a boundary operator of the form

$$\mathcal{B}(t)u = \sigma a_{jk}(\cdot,t)\,\nu^j D_k u + (I_N - \sigma)u + \sigma b(\cdot,t)$$

where $\sigma := \operatorname{diag}(\sigma_1,\dots,\sigma_N)$ is a diagonal matrix such that $\sigma_j \in C(\partial\Omega, \{0,1\})$. Thus each σ_j equals either 0 or 1 and is constant on each component of $\partial\Omega$. If $\sigma_j = 0$ then the j-th equation of $\mathcal{B}(t)u = 0$ is simply the Dirichlet condition $u^j = 0$ on the corresponding part Γ of $\partial\Omega$. Of course, $u = (u^1,\dots$ $\dots,u^N)$ and $\nu = (\nu^1,\dots,\nu^n)$. Observe that the function $\sigma(\cdot)$ defines implicitly a boundary decomposition $\mathbb{\Gamma}$. <u>Then</u> $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \mathbb{\Gamma})$, $0 \leq t \leq T$, <u>is a regular parabolic second order</u> IBVP <u>provided</u> $(\mathcal{A}(t), \Omega)$, $0 \leq t \leq T$, <u>is a strongly parabo-</u>

lic system.

(d) Block-triangular second order parabolic systems: We suppose again that $m = 1$ and that we can write $\mathcal{A}(t)$ and $\mathcal{B}(t)$ as upper triangular block-matrix differential operators:

$$\mathcal{A}(t) = [\mathcal{A}^{\rho\sigma}(t)]_{1\leq\rho,\sigma\leq r}, \quad \mathcal{B}(t) = [\mathcal{B}^{\rho\sigma}(t)]_{1\leq\rho,\sigma\leq r},$$

where $\mathcal{A}^{\rho\sigma} = \mathcal{B}^{\rho\sigma} = 0$ for $\rho > \sigma$ and where $(\mathcal{A}^{\rho\rho}(t), \mathcal{B}^{\rho\rho}(t), \Omega, \Gamma)$, $0 \leq t \leq T$, is for each $\rho = 1,\ldots,r$ a second order regular parabolic IBVP acting on N_ρ-vector-valued functions with $N_\rho \in \mathbb{N}^*$ and $N_1 + \ldots + N_r = N$. Then $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \Gamma)$, $0 \leq t \leq T$, is a regular parabolic second order IBVP. Observe that a block-triangular parabolic system need not be strongly parabolic. Finally it is clear how this example can be generalized to block-triangular parabolic systems of order 2m. □

The proof that the above examples define regular parabolic IBVPs is not quite trivial and will be given in [4].

2. Existence and Regularity. Throughout the remainder of this paper we presuppose that $(\mathcal{A}(t), \mathcal{B}(t), \Omega, \Gamma)$, $0 \leq t \leq T$, is a regular parabolic IBVP of order 2m. We put $M := N \sum_{|\alpha| \leq k} 1$, where $\alpha \in \mathbb{N}^n$, and we suppose that

$$f \in C^{2-}([0,T] \times \bar{\Omega} \times \mathbb{C}^M, \mathbb{C}^N). \tag{1}$$

This means that f is continuously real differentiable with respect to all variables and that these derivatives are locally Lipschitz continuous.

For $1 < p < \infty$ and $0 \leq s \leq 2m$ we let

$$W^s_{p,\mathcal{B}(t)} := \{u \in W^s_p \mid \mathcal{B}^\rho_\Gamma(t)u = 0 \text{ for } m_{\rho,\Gamma} < s - 1/p\}.$$

Thus $W^s_{p,\mathcal{B}(t)}$ is for each $t \in [0,T]$ a closed linear subspace of

W_p^s and $W_{p,\mathcal{B}(t)}^s = W_p^s$ for $0 \leq s \neq \hat{m}_{\partial\Omega} + 1/p$, where $\hat{m}_{\partial\Omega} := \min\{m_{\varrho,\Gamma} \mid 1 \leq \varrho \leq mN, \Gamma \in \boldsymbol{\Gamma}\}$.

After these preparations we can formulate our basic *existence, uniqueness, continuity and regularity*

(2.1) *Theorem:* *Suppose that* $n < p < \infty$, *that* $0 \leq s < \min\{1, m/2\}$, *that* $\max\{2s, s + k + n/p\} < \sigma < 2m$ *with* $\sigma \leq \ell$, *and that* $s, \sigma \notin \mathbb{N} + 1/p$. *Then the* IBVP

$$(P)_{(t_0,u_0)} \qquad \begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t)u &= f(t,x,u,Du,\dots,D^k u) && \text{in } \Omega \times (t_0,T] \\ \mathcal{B}(t)u &= 0 && \text{on } \partial\Omega \times (t_0,T] \\ u(\cdot,t_0) &= u_0 && \text{on } \Omega \end{aligned}$$

has for each $(t_0,u_0) \in [0,T) \times W_{p,\mathcal{B}(t_0)}^\sigma$ *a unique maximal solution*

$$u(\cdot,t_0,u_0) \in C^1(\dot{J}, W_p^s) \cap C(\dot{J}, W_p^{2m+s}) \cap C^{(\sigma-\varrho)/2m}(J, W_p^\varrho)$$

for every $\varrho \in [0,\sigma]$, *where* $J := J(t_0,u_0)$ *is the maximal interval of existence and* $\dot{J} := J \setminus \{t_0\}$. *Moreover,* $J(t_0,u_0)$ *is right open in* $[t_0,T]$,

$$\mathfrak{D}_{p,\mathcal{B}(t_0)}^\sigma := \{(t,v) \in [t_0,T] \times W_{p,\mathcal{B}(t_0)}^\sigma \mid t \in J(t,v)\}$$

is open in $[t_0,T] \times W_{p,\mathcal{B}(t_0)}^\sigma$, *and*

$$(2) \qquad u(\cdot,t_0,\cdot) \in C^{0,1-}(\mathfrak{D}_{p,\mathcal{B}(t_0)}^\sigma, W_p^\sigma)$$

[*that is,* $(t,v) \mapsto u(t,t_0,v)$ *is continuous in* t *and locally Lipschitz continuous in* v *on* $\mathfrak{D}_{p,\mathcal{B}(t_0)}^\sigma$].

Corollary 1: *Suppose in addition that* $s > n/p$. *Then* $u(\cdot,t_0,u_0)$ *is the unique classical solution of* $(P)_{(t_0,u_0)}$ *and*

$$u(\cdot,t_0,u_0) \in C^1(\dot{J}, C^\mu(\bar{\Omega}, \mathbb{C}^N)) \cap C(\dot{J}, C^{2m+\mu}(\bar{\Omega}, \mathbb{C}^N)),$$

where $\mu := s - n/p$.

In the important autonomous case the uniqueness, the openness of $\mathfrak{D}^{\sigma}_{p,\mathcal{B}}$ and the continuity assertion (1) imply that $u(\cdot,0,\cdot)$ defines a (local) semiflow on the Banach space $W^{\sigma}_{p,\mathcal{B}}$. More precisely we have the following

Corollary 2: Let the hypotheses of Theorem (2.1) be satisfied and assume in addition that $\mathcal{A}$, $\mathcal{B}$ and f are independent of t. Then $\varphi := u(\cdot,0,\cdot)$ is a semiflow on $W^{\sigma}_{p,\mathcal{B}}$ such that $\varphi \in C^{0,1-}(\mathfrak{D}^{\sigma}_{p,\mathcal{B}}, W^{\sigma}_{p})$ and such that bounded semiorbits are relatively compact.

(2.2) Remarks: (a) The solution $u(\cdot,t_o,u_o)$ of $(P)_{(t_o,u_o)}$ is independent of $p \in (n,\infty)$ and of s for $t > t_o$. Thus, in particular, the maximal interval of existence $J(t_o,u_o)$ does neither depend on p nor on s.

(b) $u(\cdot,t_o,u_o)$ is a global solution of $(P)_{(t_o,u_o)}$, that is, $J(t_o,u_o) = [t_o,T]$, provided $F(\text{graph } u(\cdot,t_o,u_o))$ is bounded in L_p for some $p \in (n,\infty)$, where $F(t,u)(x) := f(t,x,u(x),Du(x),\dots \dots,D^k u(x))$.

(c) Let the assumptions of Theorem (2.1) be satisfied and suppose in addition that

$$W^r_{p,\mathcal{B}(t)} = W^r_{p,\mathcal{B}(0)} \qquad \forall t \in [0,T] \, ,$$

where $0 < r < \sigma$ and $r \notin \mathbb{N} + 1/p$. Moreover, let the following compatibility conditions be satisfied: $F(t,v) \in W^r_{p,\mathcal{B}(0)}$ for all $v \in W^{2m}_{p,\mathcal{B}(t)}$ and $t \in [0,T]$, and $u_o \in W^{2m}_{p,\mathcal{B}(t_o)}$ with $\mathcal{A}(t_o)u_o \in W^r_{p,\mathcal{B}(0)}$. Then

$$u(\cdot,t_o,u_o) \in C^1(J(t_o,u_o),W^{\rho}_p) \cap C(J(t_o,u_o),W^{2m+\rho}_p)$$

for every $\rho \in /(0,r)$ with $\rho \notin \mathbb{N} + 1 \, p$, that is, we obtain "regularity up to $t = t_o$".

(d) It is not necessary that f be defined on all of $\mathbb{C}^M$. In fact, $\mathbb{C}^M$ in (1) can be replaced by an arbitrary nonempty open subset of $\mathbb{C}^M$. Moreover F need not be a local operator.

(e) Theorem (2.1) remains true if Ω is an unbounded domain which is uniformly regular of class $2m + \ell$ in the sense of Browder [5], provided one imposes additional mild regularity conditions upon the coefficients of $\mathcal{A}$, $\mathcal{B}$ and f "near infinity".

(f) It should be noted that the integer ℓ measures the continuity properties of the data. If one is willing to put $\ell = 2m$ (which one has to do if $k = 2m - 1$) then one can choose σ arbitrarily close to 2m which implies that the continuity assertion (2) is rather strong.

(g) The regularity assumption (1) can be weakened. In particular it suffices to assume that f satisfies only an appropriate Hölder condition with respect to $x \in \Omega$. □

The proofs of Theorem (2.1), its corollaries and the assertions contained in Remarks (2.2) are given in [3]. The main ideas are the following: Problem $(P)_{(t_0,u_0)}$ is considered as an abstract evolution equation of the form

$$\text{(3)} \qquad \begin{aligned} \dot{u} + A(t)u &= F(t,u), \qquad t_0 < t \leqq T, \\ u(t_0) &= u_0 \end{aligned}$$

in L_p, where $A(t)$ is the L_p-realization of $(\mathcal{A}(t), \mathcal{B}(t))$. Then it is shown that $(P)_{(t_0,u_0)}$ is equivalent to (3) and (3) is equivalent to an integral-evolution equation of the form

$$\text{(4)} \qquad u(t) = U(t,t_0)u_0 + \int_{t_0}^{t} U(t,\tau)F(\tau,u(\tau))d\tau, \quad t_0 \leqq t \leqq T.$$

Here U is a parabolic evolution operator for $\{A(t) \mid 0 \leqq t \leqq T\}$ in

L_p whose existence is guaranteed by general results of Yagi [17] and Kato and Tanabe [8]. The main difficulties stem from the facts that the domain of $A(t)$ is not constant (in general) and that F is an "unbounded nonlinear operator", that is, it is only densely defined in L_p.

If the domain of $A(t)$ were independent of t, the equation (4) could be treated by the method of fractional powers (e.g. [17, 14]). However, in our situation this method turns out to be not appropriate. In fact there seems to be no general results in the literature for $(P)_{(t_0,u_0)}$ guaranteeing existence <u>and</u> regularity for time-dependent boundary conditions (not even for a single equation, i.e. for $N = 1$). In our approach we study (4) directly in the Sobolev-Slobodeckii space W_p^{σ} using the fact that it can be characterized as an appropriate interpolation space. (More generally, we consider abstract equations of the form (4) in general interpolation spaces.) In order to obtain the stated regularity results we show that U restricts to an evolution operator on W_p^s which is, however, not strongly continuous for $t = t_0$. But we can establish the following fundamental regularity properties:

$$U(\cdot,t_0) \in C((t_0,T],\ \mathcal{L}_s(W_p^s,W_p^{2m+s}))$$

and

$$(t \longmapsto \int_{t_0}^{t} U(t,\tau)g(\tau)d\tau) \in C((t_0,T],W_p^{2m+s})$$

for every $g \in C^{\nu}([t_0,T],W_p^s)$ with $s/2m < \nu < 1$, where $\mathcal{L}_s(X,Y)$ denotes the space of all continuous linear operators from X into Y endowed with the strong topology (that is, the topology of pointwise convergence).

As already mentioned Theorem (2.1) seems to be the only existence and regularity result for semilinear parabolic equations which applies to general time-dependent boundary conditions and does not presuppose any structural condition for the nonlinearity f whatsoever, in particular no compatibility conditions. In the case of Dirichlet boundary conditions for a single equation ($N = 1$) von Wahl [15] has proved the existence of a classical solution without compatibility conditions for f. However, his result applies only to a restricted class of parabolic operators. Recently Da Prato and his students developed an abstract method to prove existence and regularity results for parabolic evolution operators without compatibility conditions for the nonlinearity (for the case that $D(A(t))$ is constant). Their main idea is to drop the assumption that $D(A(t))$ be dense in the underlying Banach space X. However they always assume that the resolvent satisfies an estimate of the form

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} = O((1 + |\lambda|^{-1}) \text{ for } \operatorname{Re} \lambda \geqq \lambda_0.$$

This restricts to applicability of their method considerably (for example to the case that X equals $C(\bar{\Omega})$ or an appropriate subspace of $C^{\mu}(\bar{\Omega})$). If we let $X = W_p^s$, as in our situation, it can only be shown that

$$\|(\lambda + A(t))^{-1}\|_{\mathcal{L}(X)} = O((1 + |\lambda|)^{-1+s/2m}) \text{ for } \operatorname{Re} \lambda \geqq \lambda_0.$$

Thus their method does not apply to the spaces W_p^s. Moreover for many questions concerning problem $(P)_{(t_0,u_0)}$ the spaces W_p^s are a natural setting as will be seen in the next section. For more detailed references to the literature we refer to [3].

3. <u>Global Existence.</u> The basis for our global existence results is the following lemma, where $\|\cdot\|_{s,p}$ denotes the norm

in W_p^s.

(3.1) **Lemma** : **Suppose that** $1 \leq p_0 < \infty$, $1 < p < \infty$ **and** $0 \leq s_0, \sigma_0 < 2m + \ell$, **and that** $s_0 = 0$ **if** $p_0 = 1$. **Let** $s_0 < s \leq 2m + \ell$ **and suppose that**

$$p_0(s_0 - \sigma_0) \leq n.$$

Finally let $0 < \alpha < \gamma$ **and**

$$1 \leq \gamma < \alpha + p_0 \quad \frac{\alpha(s - \sigma_0) + (1 - \alpha)n/p}{n + (\sigma_0 - s_0)p_0}$$

Then there exists a constant c **such that**

$$\|u\|^{\gamma}_{\sigma_0, p\gamma} \leq c \|u\|^{\alpha}_{s,p} \|u\|^{\gamma-\alpha}_{s_0,p_0} \qquad \forall u \in W^{s_0}_{p_0} \cap W^s_p$$

provided

$$\frac{1}{p} \leq \frac{\gamma - \alpha}{p_0} + \frac{\alpha}{p} .$$

The proof of this Lemma, which can be considered as an extension of the Gagliardo-Nirenberg inequality, follows from the interpolation space characterizations of the Sobolev-Slobodeckii spaces and from Sobolev-type imbedding theorems.

It is now easy to prove the following general global existence theorem, where we let $t^+(t_0,u_0) := \sup J(t_0,u_0)$.

(3.2) **Theorem**: **Suppose that** $0 \leq s_0 < 2m$ and $1 \leq p_0 < \infty$, **and that** $s_0 = 0$ if $p_0 = 1$. **Let** $\varkappa \in \mathbb{Z}$ **satisfy** $\varkappa < s_0 - n/p_0 \leq \varkappa + 1$ **and suppose that there exist a continuous function** g (**depending only on** (t,x) **if** $\varkappa < 0$) **and constants** c **and** γ_j, $j = \varkappa + 1, \ldots$ $\ldots, k$ **such that**

$$|f(t,x,u,Du,\ldots,D^k u)| \leq g(t,x,u,\ldots,D^{\varkappa}u) + c \sum_{j=\varkappa+1}^{k} |D^j u|^{\gamma_j}$$

and

(1) $$1 \leq \gamma_j < 1 + p_o \frac{2m - j}{n + (j-s_o)p_o}, \quad j = \varkappa + 1, \ldots, k.$$

Finally suppose that, for some $(t_o,u_o) \in [0,T) \times W^{\sigma}_{p,\mathcal{B}(t_o)}$ *and* $t_1 \in (t_o, t^+(t_o,u_o))$,

(2) $$\sup_{t_1 \leq t < t^+(t_o,u_o)} \|u(t,t_o,u_o)\|_{s_o,p_o} < \infty .$$

Then $t^+(t_o,u_o) = T$.

Proof: (a) Let $s \in [\sigma, 2m)$ be arbitrary and suppose we can show that

(3) $$\sup_{t_1 \leq t < t^+} \|u(t)\|_{s,q} < \infty ,$$

where $t^+ := t^+(t_o,u_o)$ and $u(t) := u(t,t_o,u_o)$. Then $W^s_p \hookrightarrow W^{\sigma}_p$ implies

$$\sup_{t_1 \leq t < t^+} \|u(t)\|_{\sigma,p} < \infty .$$

Thus it is easily seen that

$$\sup_{t_1 \leq t < t^+} \|F(t,u(t))\|_{o,p} < \infty .$$

Since, by a continuity and compactness argument, F is bounded in L_p on $\{(t,u(t)) \mid t_o \leq t \leq t_1\}$, it follows that F is bounded in L_p on graph(u). Hence the assertion follows from Remark (2.2.b) provided we can show that (3) is true.

(b) It follows from (1) that we can find numbers $\alpha \in (0,1)$, and $s \in (k + n/p, 2m) \setminus (\mathbb{N} + 1/p)$ such that

$$\gamma_j < \alpha + p_o \frac{\alpha(s-j) + (1-\alpha)n/p}{n + (j-s_o)p_o}, \quad j = \varkappa + 1, \ldots, k.$$

By the results of Section 2 we can assume that $p \geq p_o$. Hence $1/p \leq ((\gamma_j - \alpha)/p_o) + (\alpha/p)$ and we obtain from Lemma (3.1)

$$\| |D^j u|^{\gamma_j} \|_{o,p} = \|D^j u\|^{\gamma_j}_{o,p\gamma_j} \leq c \|u\|^{\gamma_j}_{j,p\gamma_j} \leq$$

$$\leq c\,\|u\|^{\alpha}_{s,p}\,\|u\|^{\gamma_j-\alpha}_{s_o,p_o}$$

for $j = \varkappa+1,\dots,k$. Thus

(4) $$\|F(t,u(t))\|_{o,p} \leq \|g(t,\cdot,u(t),\dots,D^{\varkappa}u(t))\|_{o,p} + \\ + c\Big(\sum_{j=\varkappa+1}^{k}\|u(t)\|^{\gamma_j-\alpha}_{s_o,p_o}\Big)\|u(t)\|^{\alpha}_{s,p}$$

for $t_1 \leq t < t^+$ (since $u(t) \in W^s_p$ for $t_1 \leq t < t^+$ by Theorem (2.1)). Since $\varkappa < s_o - n/p_o \leq s_o - n/p$ the a priori estimate (2) and the imbedding $W^{s_o}_p \hookrightarrow C^{\varkappa}$ (for $\varkappa \in \mathbb{N}$) imply

(5) $$\sup_{t_1 \leq t < t^+} \|g(t,\cdot,u(t),\dots,D^{\varkappa}u(t))\|_{o,p} < \infty .$$

Now it follows from the results of Section 2 that the integral-evolution equation (2.4) is well defined in W^s_p for $t_1 \leq t < t^+$. Thus

(6) $$\|u(t)\|_{s,p} \leq \|U(t,t_1)u(t_1)\|_{s,p} + \\ + \int_{t_1}^{t}\|U(t,\tau)\|_{\mathcal{L}(L_p,W^s_p)}\,\|F(\tau,u(\tau))\|_{o,p}\,d\tau$$

for $t_1 \leq t < t^+$ (where we have used the unique solvability). Inserting (4),(2) and (5) in (6) we see that

$$\|u(t)\|_{s,p} \leq c\big(1 + \max_{t_1 \leq \tau \leq t_2}\|u(\tau)\|^{\alpha}_{s,p}\big)$$

for $t_1 \leq t \leq t_2 < t^+$, where c is independent of t_2 (due to the estimate $\|U(t,\tau)\|_{\mathcal{L}(L_p,W^s_p)} = O((t-\tau)^{-s/2m})$ for $0 \leq \tau < t \leq T$).
This implies (3), whence the assertion. □

(3.3) Remark: Suppose that $\mathcal{A}$, $\mathcal{B}$ and f are independent of t and that the spectrum of A (in L_2, for example) is contained in the open right half-plane. Then, given the assumptions of Theorem (3.2), it follows that $t^+(t_o,u_o) = \infty$ and that

$$\sup_{t_1 \le t < \infty} \|u(t,t_0,u_0)\|_{2m,p} < \infty .$$

Thus, if it is known that the positive semiorbit $\gamma^+(u_0) := \{u(t,0,u_0) \mid 0 \le t < t^+(0,u_0)\}$ is bounded in $W_{p_0}^{s_0}$ it is bounded in W_p^{2m} (for $t > 0$). Furthermore it is relatively compact in W_p^{2m}, which implies in particular that $\gamma^+(u_0)$ has a nonempty limit set in W_p^{2m}. If, moreover, $F(u) \in W_{p,\mathcal{B}}^{s}$ for some $s > n/p$ (which is no restriction if $\min\{m_{\rho,\Gamma} \mid 1 \le \rho \le mN, \Gamma \in \Gamma\} > 0$) then W_p^{2m} can be replaced by $C^{2m+\mu}(\bar{\Omega}, \mathbb{C}^N)$ for some $\mu \in (0,1)$. □

The above theorem generalizes (and simplifies) considerably numerous earlier results (e.g. [1, 9, 12, 13], cf. [4] for more detailed references. It should also be noted that, due to Remark 2.2.e, Theorem (3.2) is also true (modulo some regularity assumptions near infinity) if Ω is unbounded).

The main content of Theorem (3.2) is the assertion that we obtain global existence if we can obtain an a priori bound in some weak norm (in the $W_{p_0}^{s_0}$-norm) and if the nonlinearity satisfies an appropriate growth restriction. In the particularly important case that $k = 0$ it follows that $u(\cdot,t_0,u_0)$ exists globally if

(7) $$|f(t,x,\xi)| \le c(1 + |\xi|^{\gamma}) \qquad \forall (t,x,\xi) \in [0,T] \times \bar{\Omega} \times \mathbb{C}^N,$$

where

$$1 \le \gamma < 1 + \frac{2m\ p_0}{n - s_0\ p_0} = \frac{n + (2m-s_0)p_0}{n - s_0 p_0}$$

provided we know that

$$\sup_{t_1 \le t < t^+} \|u(t,t_0,u_0)\|_{s_0,p_0} < \infty$$

for some $t_1 \in (t_0, t^+)$.

There is a quite general class of problems for which every solution can be bounded a priori in the W_2^m-norm. To describe this class we restrict ourselves to the real case and introduce the following splitting assumption:

(SP) There are continuous functions g and h such that

$$f(t,x,u,\dots,D^k u) = g(t,x,u,\dots,D^k u) + h(x,u),$$

a constant c with

$$|g(t,x,u,\dots,D^k u)| \leq c(1 + \sum_{j=0}^{m} |D^j u|),$$

and a function $H \in C^{0,1}(\bar{\Omega} \times \mathbb{R}^N, \mathbb{R})$ such that $h = \nabla_\xi H$, where ∇_ξ denotes the gradient with respect to $\xi \in \mathbb{R}^n$.

In addition we consider the following definitness assumption:

$(D)_{(t_0,u_0)}$ For each $t_1 \in (t_0, t^+(t_0,u_0))$ there are constants $\lambda_0 > 0$ and c, $c_0 \geqq 0$ such that

$$\lambda_0 \|u(t)\|^2_{m,2} - c_0 \|u(t)\|^2_{0,2} \leq 2 \int_{t_1}^{t} (A(\tau)u(\tau)|\dot{u}(\tau))d\tau + c(1 + \int_{t_1}^{t} \|u(\tau)\|^2_{m,2} d\tau)$$

for $t_1 \leq t < t^+$.

By taking the L_2-inner product of $\dot{u}(t)$ and the equation $\dot{u} + A(t)u = F(t,u)$ it is not difficult to deduce an a priori bound for $\|u(t)\|_{m,2}$ on the basis of (SP) and $(D)_{(t_0,u_0)}$ by means of Gronwall's inequality. Then Theorem (3.2) implies the following

(3.4) Theorem: Let $(t_0,u_0) \in [0,T) \times W^{\sigma}_{p,\mathcal{B}(t_0)}$ be given and suppose that (SP) and $(D)_{(t_0,u_0)}$ are satisfied. Moreover, suppose that

$$\overline{\lim_{|\xi| \to \infty}} \frac{H(x,\xi)}{|\xi|^2} < \infty$$

uniformly with respect to $x \in \bar{\Omega}$, and that

$$|h(\cdot,\xi)| \leq c(1 + |\xi|^{\gamma}) \qquad \forall \xi \in \mathbb{R}^N,$$

where

$$(8) \qquad 1 \leq \gamma < 1 + \frac{4m}{n-2m} = \frac{n+2m}{n-2m} .$$

Then $t^+(t_o,u_o) = T$.

It can be shown that $(D)_{(t_o,u_o)}$ is satisfied for every $(t_o,u_o) \in [0,T) \times W^{\sigma}_{p,\mathcal{B}(t_o)}$ if $(\mathcal{A}(t),\Omega)$, $0 \leq t \leq T$, is a strongly parabolic system and $(\mathcal{B}(t),\Gamma)$, $0 \leq t \leq T$, is the Dirichlet boundary operator (cf. Example (1.1.b)). Thus Theorem (3.4) generalizes a result of Pecher and von Wahl [11], where it has been assumed that $N = 1$, $f(t,x,u,\dots,D^k u) = f(u)$ and that $\int_0^{\xi} f(s)\,ds \leq c\,\xi^2$, that is, $H(\xi) \leq c\,\xi^2$ for $\xi \in \mathbb{R}$.

It can also be shown that $(D)_{(t_o,u_o)}$ is satisfied for every $(t_o,u_o) \in W^{\sigma}_p$ in the situation of Example (1.1.c) provided the matrices a_{jk} are symmetric. Thus Theorem (3.4) generalizes considerably recent results of Cosner [6] and Alikakos [2]. These authors assumed the stronger ellipticity condition

$$\sum_{r,s=1}^{N} \sum_{j,k=1}^{m} a^{rs}_{jk}(x)\, \xi^j_r \xi^k_s \geq c_o \sum_{r=1}^{N} \sum_{j=1}^{n} |\xi^j_r|^2$$

for all $x \in \bar{\Omega}$ and $\xi^j_r \in \mathbb{R}$, $1 \leq j \leq n$, $1 \leq r \leq N$, considered Dirichlet boundary conditions and the autonomous case, assumed that g is a linear differential operator and thet $(h(\cdot,\xi)\,|\,\xi) < 0$ for $\xi \in \mathbb{R}^N \setminus \{0\}$ and

$$\overline{\lim_{|\xi| \to \infty}} \frac{(h(x,\xi) \quad)}{|\xi|^2} \leq -\beta$$

uniformly in $x \in \Omega$, where β is a sufficiently large positive constant. Then Cosner proved global existence under the growth restriction $1 \leq \gamma < n/(n-2)$. Alikakos obtained global existence if γ satisfies (8) (with $m = 1$) but he had to assume that the matrices $a_{jk}(x)$, $1 \leq j, k \leq n$, commute for every $x \in \Omega$.

It is natural to ask whether the equality sign in (1) can be permitted. Von Wahl [15, 16] has shown that this is the case if $N = 1$ and $\mathcal{B}$ is the Dirichlet boundary operator, provided $p_0 = 2$ and f satisfies an appropriate monotonicity condition. By means of the continuity argument employed in [15, 16] similar results can be obtained in our general setting.

Detailed proofs of the assertions of this section are given in 4 .

References

[1] ALIKAKOS, N.D.: L^p bounds of solutions of reaction-diffusion equations, Comm. in Partial Diff. Equat. 4(1979), 827-868.

[2] ALIKAKOS, N.D.: Quantitative maximum principles and strongly coupled gradient-like reaction-diffusion systems, Proc. Royal Soc. Edinburgh 94(1983), 265-286.

[3] AMANN, H.: Existence and regularity for semilinear parabolic evolution equations. To appear.

[4] AMANN, H.: Global existence for semilinear parabolic systems. In preparation.

[5] BROWDER, F.E.: On the spectral theory of elliptic operators, I. Math. Annalen 141(1961), 22-130.

[6] COSNER, CH.: Pointwise a priori bounds for strongly coupled semi-linear systems of parabolic partial differential equations, Indiana Univ. Math. J. 30(1981), 607-620.

[7] FRIEDMAN, A.: Partial Differential Equations, Holt, Rinehart and Winston, New York, 1969.

[8] KATO, T. and TANABE, H.: On the abstract evolution equation, Osaka Math. J. 14(1962), 107-133.

[9] KIELHOEFER, H.: Global solutions of semilinear evolution equations satisfying an energy inequality, J.Diff. Equat. 36(1980), 188-222.

[10] LIONS, J.L., MAGENES, E.: Non-Homogeneous Boundary Value Problems and Applications I, Springer Verlag, Berlin, 1972.

[11] PECHER,H., von WAHL, W.: Klassische Lösungen im Grossen semilinearer parabolischer Differentialgleichungen, Math. Z. 145(1975), 255-265.

[12] ROTHE, F.: Uniform bounds from bounded L_p-functionals in reaction-diffusion equations, J. Diff. Equat. 45 (1982), 207-233.

[13] ROTHE, F.: A priori estimates, global existence and asymptotic behaviour for weakly coupled systems of reaction-diffusion equations, Habilitationsschrift, Univ. Tübingen, 1983.

[14] SOBOLEVSKII, P.E.: Equations of parabolic type in a Banach space, Amer. Math. Soc. Transl., Ser. 2, 49(1966), 1-62.

[15] von WAHL, W.: Lineare und semilineare parabolische Differentialgleichungen in Räumen hölderstetiger Funktionen, Abh. Math. Sem. Univ. Hamburg 43(1975), 234-262.

[16] von WAHL, W.: Semilinear elliptic and parabolic equations of arbitrary order, Proc. Royal Soc. Edinburgh, Sect. A, 78(1978), 193-207.

[17] YAGI, A.: On the abstract linear evolution equations in Banach spaces, J. Math. Soc. Japan 28(1976),290-303.

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(Oblatum 25.5. 1984)