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HARD IMPLICIT FUNCTION THEOREM AND SMALL PERIODIC SOLUTIONS TO PARTIAL DIFFERENTIAL EQUATIONS

Pavel KREJČÍ

Dedicated to the memory of Svatopluk FUCIK

Abstract. In the first part we investigate the existence of small solutions of the equation $F(u) = h$ in a system of Banach spaces via "modified Newton's method". The abstract result is used in the second part for proving the existence of periodic solutions of partial differential equations of the second order.

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Introduction

Recently many new existence results have been obtained in the theory of partial differential equations by means of the Nash's iteration procedure which is a modification of the classical Newton's method. From a more general point of view it was developed e.g. by Schwartz [15], Moser [8],[9], Pták [13], Craven and Nashed [3], Shatah [16], Altman [1], issuing from the original Nash's paper [10]. These results are applied in the theory of PDE's for proving the existence of small solutions in the cases when one has some apriori estimates, but with a "lost of derivatives". This situation occurs frequently in the theory of nonlinear hyperbolic equations. The existence proofs are based either on the use of Moser's theorem (Rabinowitz [14], Craig [2], Petzeltová [12]), or on a direct application of the Nash's scheme (Hörmander [5], Klainerman [6], Shibata [17],[18]). All these results are obtained under the assumption that the data (i.e.

right-hand side, initial conditions, if any, etc.) are sufficiently small and sufficiently smooth.

In the present paper we want to emphasize that in fact, this theory can be put in a rather elementary framework. In the first part we derive sufficient conditions for the solvability of the abstract equation $F(u) = \dot{h}$ in a system of Banach spaces $\{X^{p,L}\}$. Our aim is to minimize the requirements on the smallness and "smoothness" of the data. In the second part we apply the abstract theorem for proving the existence of classical periodic solutions to the equation

$$\phi(u,\partial_t u,\partial_x u,\partial_t^2 u,\partial_x^2 u,\partial_x\partial_t u) = h(t,x)$$

with zero Dirichlet boundary conditions on $[0,1]$, where ϕ and h are given functions.

Throughout the paper, we denote all constants whose values depend essentially only on quantities $a,b,\dots$ by $c_{a,b,\dots}$. Especially, c_L denotes any constant depending essentially only on L.

I. Operator equation

1. Statement of the main theorem

(1.1) Assumptions.

(N) Let $2 \leqq q \leqq \dot{q}$ be given real numbers and let $N_\infty, N_q, N_{\dot{q}}, N_+, N_-, N_0, N$ be nonnegative integers such that

$$N_+ \geqq \max\{N_0, N_-\}$$

$$N_\infty \geqq N_{\dot{q}} \geqq N_q$$

$$N \geqq \max\{N_+ + N_0 + N_q + N_{\dot{q}} - 2N_-;\ N_0 + N_\infty;\ N_+ + N_\infty - N_-\}$$

and put

$$M = 2N+1+N_\infty+N_- - N_0 - N_q - N_{\dot{q}} .$$

(X) Let $\{X^{p,L},\ p=2,q,\dot{q},\infty,\ L=0,1,2,\dots\}$ be a system of Banach spaces endowed with norms $|.|_{p,L}$ and let the following relations hold (the symbol $\hookrightarrow$ denotes the continuous embedding):

$$X^{p,L+1} \hookrightarrow X^{p,L} \quad \text{for each } p \text{ and each } L \geqq 0$$

$$X^{\dot p,L} \hookrightarrow X^{p,L} \quad \text{for } \dot p \geqq p,\ L\geqq 0$$
$$X^{2,L+N_p} \hookrightarrow X^{p,L} \quad \text{for } p=q,\dot q,\infty\ ,\ L\geqq 0\ .$$
Let X_o^{2,N_-} be a closed subspace of X^{2,N_-} and for $L\geqq N_-$ put $X_o^{2,L} = X_o^{2,N_-} \cap X^{2,L}$.

(S) Let $r>1$ be a given real number and let $\{S_n\}_{n=0}$ be a sequence of "smoothing operators" such that for each $L\geqq 0$, $K\geqq 0$, $u\in X^{p,K}$ there is $S_n u\in X^{p,L}$ and there exist constants c_L such that

(S1) $|S_n u|_{p,L} \leqq c_L\, r^{(L-K)n}\, |u|_{p,K}\ ,\quad L\geqq K,\quad n\geqq 0$

(S2) $|(I-S_n)u|_{p,L} \leqq c_L\, r^{(L-K)n}\, |u|_{p,K}\ ,\quad L\leqq K\leqq M+N_-,\ n\geqq 0.$

(F) Let $\delta_o>0$ be a given number and let $F\colon D_L(F)\to X^{\infty,L}$, where $D_L(F) = \{u\in X^{\infty,L+N_+}\ ,\ |u|_{\infty,N_o}<\delta_o\}$, be a continuous mapping for $0\leqq L\leqq M$, $F(0)=0$, which is twice Fréchet differentiable for $0\leqq L\leqq M-N_+-N_\infty+N_-$ and such that

(F1) for each $v\in D_L(F)$, $u_1,u_2\in X^{\infty,L+N_+}$, $0\leqq L\leqq M-N_+-N_\infty+N_-$ there is

$$|F''(v)(u_1,u_2)|_{2,L} \leqq c_L \sum_{\substack{\lambda=(\lambda_1,\lambda_2,\lambda_3)\\ |\lambda|=L+N_+-N_o}} (1+|v|_{\infty,\lambda_1+N_o})\cdot|u_1|_{q,\lambda_2+N_o}\cdot|u_2|_{\dot q,\lambda_3+N_o}\ ,$$

(F2) there exists some $\delta_->0$ such that for every $v\in X^{\infty,M+N_o+N_-}$, $|v|_{\infty,N_o+N_-}<\delta_-$ and for every $h\in X^{2,M}$ there exists a unique solution $u\in X_o^{2,M+N_-}$ to the equation $F'(v)\,u = h$, such that

$$|u|_{2,L+N_-} \leqq c_L\left(|h|_{2,L} + |v|_{\infty,L+N_o+N_-}\,|h|_{2,0}\right)$$

holds for each L, $0\leqq L\leqq M$.

(1.2) **THEOREM**. Let (1.1) hold. Then there exists some $\delta_N > 0$ such that for each $h \in X^{2,M}$, $|h|_{2,M} < \delta_N$ there exists at least one solution $u \in X_0^{2,N+N_-}$ to the equation $F(u) = h$.

(1.3) Remarks.

(i) Since there is $X^{2,N+N_-} \hookrightarrow X^{\infty,N_+}$, the value of $F(u)$ in the theorem is well defined.

(ii) In the applications, the number N_+ characterizes the "order" of the equation, N_+-N_- is the number of "lost derivatives", N_0 is the highest order occurring in the "argument of the nonlinearity".

2. Iteration scheme

The iteration process is almost the same as in [5], [6] or [17]. We are to solve the following sequence of linear equations

$$(2.1)\qquad F'(0)\, u_0 = h$$

$$(2.2)_0\qquad F'(S_0u_0)\, w_0 = h_0$$

$$\vdots$$

$$(2.2)_n\qquad F'(S_nu_n)\, w_n = h_n$$

$$\vdots$$

where
$$u_n = u_0 + \sum_{k=0}^{n-1} w_k$$
$$h_0 = S_0e_0, \quad e_0 = -F(u_0) + F'(0)\, u_0$$
$$h_n = S_ne_n + (S_n - S_{n-1}) \sum_{k=0}^{n-1} e_k$$
$$e_n = f_n + g_n$$
$$f_n = -F(u_n) + F(u_{n-1}) + F'(u_{n-1})\, w_{n-1}$$
$$g_n = (F'(S_{n-1}u_{n-1}) - F'(u_{n-1}))\, w_{n-1}\,.$$

We check easily that one has

$$(2.3)\qquad F(u_{n+1}) = h - e_{n+1} - (I-S_n) \sum_{k=0}^{n} e_k$$

First, let $h \in X^{2,M}$ be arbitrary. Following (1.1)(F2) we find the unique solution $u_o \in X_o^{2,M+N_-}$ of (2.1). There is

$$(2.4) \quad |u_o|_{2,L+N_-} \leqq c_L |h|_{2,L} , \quad 0 \leqq L \leqq M .$$

For the solvability of $(2.2)_o$ we require

$$(2.5) \quad \begin{aligned} |S_o u_o|_{\infty,N_o} &< \delta_o \\ |u_o|_{\infty,N_o} &< \delta_o \\ |S_o u_o|_{\infty,N_o+N_-} &< \delta_- . \end{aligned}$$

These conditions are satisfied if h is taken sufficiently small, say $|h|_{2,M} < \delta_1$. On the other hand there is

$$(2.6) \quad |h_o|_{2,L} = |S_o e_o|_{2,L} \leqq c_L \int_0^1 |F''(\sigma u_o)(u_o,u_o)|_{2,0}\, d\sigma \leqq$$

$$\leqq c_L \sum_{|\lambda|=N_+-N_o} (1 + |u_o|_{\infty,\lambda_1+N_o}) |u_o|_{q,\lambda_2+N_o} |u_o|_{\dot{q},\lambda_3+N_o} ,$$

so that the unique solution $w_o \in X_o^{2,M+N_-}$ of $(2.2)_o$ satisfies

$$|w_o|_{2,L+N_-} \leqq c_L \left(|h_o|_{2,L} + |u_o|_{2,0} |h_o|_{2,0}\right) , \quad 0 \leqq L \leqq M.$$

Let $\varepsilon > 0$ be for the present arbitrarily chosen. From (2.4) and (2.6) it follows that one can find some δ_ε, $0 < \delta_\varepsilon \leqq \varepsilon$ such that if

$$(2.7) \quad |h|_{2,M} < \delta_\varepsilon ,$$

then there is

$$(2.8)_o \quad |w_o|_{2,L+N_-} \leqq \varepsilon , \quad 0 \leqq L \leqq M.$$

Put

$$(2.9) \quad \gamma = N + \tfrac{1}{3} .$$

Our next goal is to choose ε in $(2.8)_o$ in such a way that the inequalities

$$(2.8)_k \quad |w_k|_{2,L+N_-} \leqq \varepsilon\, r^{(-\gamma+L)k}$$

hold for arbitrary integers $k \geqq 0$ and $0 \leqq L \leqq M$. The constant $r > 1$ is introduced in (1.1)(S).

For this purpose we proceed by induction over k. We make the following assumption:

(2.10) For some $\varepsilon > 0$ and $n \geq 0$ there exists the sequence $\{w_k\}_{k=0}^{n} \subset X_o^{2,M+N_-}$ of solutions of $\{(2.2)_k,\ k=0,1,\ \ldots\ ,n\}$, respectively, satisfying $(2.8)_k$, $k=0,1,\ \ldots\ ,n$.

3. Estimates

(3.1) **Proposition**. Let (2.10) hold. Then there is

(i) $|u_{n+1}|_{2,L+N_-} \leq c_L\, \varepsilon\,, \quad 0 \leq L \leq N$

(ii) $|u_{n+1}|_{2,L+N_-} \leq c_L\, \varepsilon\, r^{(-\gamma+L)(n+1)}, \quad N < L \leq M$

(iii) $|u_n|_{2,L+N_-} \leq c_L\, \varepsilon\,, \quad 0 \leq L \leq N$

(iv) $|u_n|_{2,L+N_-} \leq c_L\, \varepsilon\, r^{(-\gamma+L)(n+1)}, \quad N < L \leq M$

(v) $|(I-S_{n+1})u_{n+1}|_{2,L+N_-} \leq c_L\, \varepsilon\, r^{(-\gamma+L)(n+1)}, \quad 0 \leq L \leq M$

(vi) $|(I-S_n)u_n|_{2,L+N_-} \leq c_L\, \varepsilon\, r^{(-\gamma+L)(n+1)}, \quad 0 \leq L \leq M$

(vii) $|f_{n+1}|_{2,L} \leq c_L\, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)}, \quad 0 \leq L \leq M-N_+-N_\infty+N_-$

(viii) $|g_{n+1}|_{2,L} \leq c_L\, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)}, \quad 0 \leq L \leq M-N_+-N_\infty+N_-$

(ix) $|e_{n+1}|_{2,L} \leq c_L\, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)}, \quad 0 \leq L \leq M-N_+-N_\infty+N_-$

(x) $\left|\sum_{k=0}^{n} e_k\right|_{2,L} \leq c_L\, \varepsilon^2\,, \quad 0 \leq L < M-N_+-N_\infty+N_-$

(xi) $\left|\sum_{k=0}^{n} e_k\right|_{2,M-N_+-N_\infty+N_-} \leq c\, \varepsilon^2\, r^{(n+1)/3}$

(xii) $\left|(I-S_n)\sum_{k=0}^{n} e_k\right|_{2,L} \leq c_L\, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)},$
$0 \leq L \leq M-N_+-N_\infty+N_-$

(xiii) $\left|(I-S_{n+1})\sum_{k=0}^{n} e_k\right|_{2,L} \leq c_L\, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)},$
$0 \leq L \leq M-N_+-N_\infty+N_-$

(xiv) $|h_{n+1}|_{2,L} \leq c_L\, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)}, \quad 0 \leq L\,.$

P r o o f. Using (2.10), (2.4) and (2.6) we obtain

$$|u_0|_{2,L+N_-} + \sum_{k=0}^{n} |w_k|_{2,L+N_-} \leqq \varepsilon \left(c_L + \sum_{k=0}^{n} r^{(-\gamma+L)k}\right) .$$

In the case $L \leqq N$ there is $\sum_{k=0}^{n} r^{(-\gamma+L)k} \leqq \sum_{k=0}^{\infty} r^{(-\gamma+L)k} \leqq c_L$, for $L > N$ we have $\sum_{k=0}^{n} r^{(-\gamma+L)k} = r^{\gamma-L}\, r^{(-\gamma+L)(n+1)} \sum_{k=0}^{n} r^{(\gamma-L)k} \leqq$ $\leqq c_L\, r^{(-\gamma+L)(n+1)}$, and (i) - (iv) follow easily.

For proving the inequalities (v) and (vi) we use (1.1)(S) and (ii), (iv). There is

$$|(I-S_n)u_n|_{2,L+N_-} \leqq c_L\, r^{(L-M)n}\, |u_n|_{2,M+N_-} \leqq c_L\, \varepsilon\, r^{(-\gamma+L)(n+1)}$$

which yields (vi) ((v) is analogous).

Next, we express

$$f_{n+1} = -\int_0^1 (1-\sigma)\, F''(u_n+\sigma w_n)(w_n,w_n)\, d\sigma$$

$$g_{n+1} = -\int_0^1 F''(u_n+\sigma(S_n-I)u_n)((I-S_n)u_n,w_n)\, d\sigma$$

Using (1.1)(F1), (X) and (N) we obtain for $0 \leqq L \leqq M-N_+-N_\infty+N_-$

$$|f_{n+1}|_{2,L} \leqq c_L \sum_{|\lambda|=L+N_+-N_0} (1 + |u_n|_{2,\lambda_1+N_0+N_\infty} + |w_n|_{2,\lambda_1+N_0+N_\infty}) \cdot \cdot |w_n|_{2,\lambda_2+N_0+N_q}\, |w_n|_{2,\lambda_3+N_0+N_{\dot q}}$$

$$|g_{n+1}|_{2,L} \leqq c_L \sum_{|\lambda|=L+N_+-N_0} (1 + |u_n|_{2,\lambda_1+N_0+N_\infty})\, |w_n|_{2,\lambda_2+N_0+N_q} \cdot \cdot |(I-S_n)u_n|_{2,\lambda_3+N_0+N_{\dot q}} .$$

Therefore, we estimate both $|f_{n+1}|_{2,L}$ and $|g_{n+1}|_{2,L}$ (and hence $|e_{n+1}|_{2,L}$) from above by

$$c_L\, \varepsilon^2 \sum_{|\lambda|=L+N_+-N_0} (1 + r^{(-\gamma+\lambda_1+N_0+N_\infty-N_-)(n+1)}) \cdot r^{(-2\gamma+\lambda_2+\lambda_3+2N_0+N_q+N_{\dot q}-2N_-)(n+1)} \leqq$$

$$\leqq c_L\, \varepsilon^2\, r^{(-2\gamma+L+N_++N_0+N_q+N_{\dot q}-2N_-)(n+1)}$$

which is (vii). (viii) (ix)

The assertions (x) and (xi) are similar to (i) - (iv). We use the fact that the estimate

$$|e_k|_{2,L} \leqq c_L\, \varepsilon^2\, r^{(-2\gamma+L+N_++N_0+N_q+N_{\dot q}-2N_-)\,k}$$

holds for each $k=0,1,\ldots n+1$ and that one has

$M-N_+-N_\infty+N_- \;=\; 2N-N_+-N_o-N_q-N_{\dot q}+2N_-+1$.

The proof of (xii) and (xiii) is analogous to (v) and (vi) and we don't reproduce it here. For proving the last assertion (xiv) we observe that

$$|S_{n+1}e_{n+1}|_{2,L} \leqq c_L\, r^{L(n+1)}\, |e_{n+1}|_{2,0}$$

holds for arbitrary $L \geqq 0$. Further, for $L \leqq M-N_+-N_\infty+N_-$ we have

$$\Big|(S_{n+1}-S_n)\sum_{k=0}^{n} e_k\Big|_{2,L} \leqq \Big|(I-S_n)\sum_{k=0}^{n} e_k\Big|_{2,L} + \Big|(I-S_{n+1})\sum_{k=0}^{n} e_k\Big|_{2,L},$$

so that we can use (xii) and (xiii), for $L > M-N_+-N_\infty+N_-$ we obtain

$$\Big|(S_{n+1}-S_n)\sum_{k=0}^{n} e_k\Big|_{2,L} \leqq \Big|S_{n+1}\sum_{k=0}^{n} e_k\Big|_{2,L} + \Big|S_n\sum_{k=0}^{n} e_k\Big|_{2,L} \leqq$$

$$\leqq c_L\, r^{(L-M+N_++N_\infty-N_-)(n+1)} \Big|\sum_{k=0}^{n} e_k\Big|_{2,M-N_+-N_\infty+N_-}$$

and using (xi) the proof is complete.

4. Proof of the main theorem.

Up to now we have shown that for an arbitrarily chosen $\varepsilon > 0$ we can find some $\delta_\varepsilon > 0$ such that if $|h|_{2,M} < \delta_\varepsilon$, the equations (2.1) and $(2.2)_o$ have solutions $u_o, w_o \in X_o^{2,M+N_-}$ respectively, satisfying $|u_o|_{2,L+N_-} \leqq c_L\varepsilon$, $|w_o|_{2,L+N_-} \leqq \varepsilon$, $0 \leqq L \leqq M$. Further, assuming (2.10) we have derived the estimates (3.1).

By (1.1)(F2) the sufficient conditions for the solvability of $(2.2)_{n+1}$ are

$$(4.1)\qquad \begin{aligned} |S_{n+1}u_{n+1}|_{\infty,N_o+N_-} &< \delta_- \\ |u_n|_{\infty,N_o} &< \delta_o \\ |S_n u_n|_{\infty,N_o} &< \delta_o \\ |u_{n+1}|_{\infty,N_o} &< \delta_o \end{aligned}$$

Since there is $N + N_- \geqq N_o + N_\infty + N_-$, we see that by (3.1) (i), (iii) the conditions (4.1) are fulfilled provided ε is taken sufficiently small.

Following (1.1)(F2) the solution $w_{n+1} \in X_o^{2,M+N_-}$ of $(2.2)_{n+1}$ satisfies the inequality $(0 \leq L \leq M)$:

$$|w_{n+1}|_{2,L+N_-} \leq c_L \left(|h_{n+1}|_{2,L} + |S_{n+1}u_{n+1}|_{\infty,L+N_o+N_-} |h_{n+1}|_{2,0}\right) \leq$$

$$\leq c_L \left(|h_{n+1}|_{2,L} + r^{L(n+1)} |u_{n+1}|_{2,N+N_-} |h_{n+1}|_{2,0}\right) .$$

Using (3.1)(i),(xiv) we find the estimate

$$|w_{n+1}|_{2,L+N_-} \leq c_L \, \varepsilon^2 r^{(-2\gamma+L+N_++N_o+N_q+N_{\dot q}-2N_-)(n+1)}, \quad 0 \leq L \leq M \,;$$

On the other hand, by (1.1)(M) and (2.9) there is

$$-\gamma+N_++N_o+N_q+N_{\dot q}-2N_- \leq -\tfrac{1}{3} < 0 \text{ , hence}$$

(4.2) $|w_{n+1}|_{2,L+N_-} \leq c_L \, \varepsilon^2 r^{(-\gamma+L)(n+1)}, \quad 0 \leq L \leq M.$

The constant c_L in (4.2) is independent of n. Thus the choice

(4.3) $\varepsilon < (\max \{c_L, \ 0 \leq L \leq M\})^{-1}$

yields $(2.8)_{n+1}$.

By induction over n we conclude that we can construct the infinite sequence $\{w_n\}_{n=0}^{\infty} \subset X_o^{2,N+N_-}$ of solutions of $\{(2.2)_n\}_{n=0}^{\infty}$ provided $\delta_\varepsilon = \delta_N$ is taken sufficiently small (so that (2.5), $(2.8)_o$, (4.1), (4.3) are fulfilled) and each w_n satisfies the corresponding inequality $(2.8)_n$. Since the series $\sum_{n=0}^{\infty} |w_n|_{2,N+N_-} \leq \varepsilon \sum_{n=0}^{\infty} r^{-n/3}$ is convergent, we see that $\{u_n\}_{n=0}^{\infty}$ is a fundamental sequence in $X_o^{2,N+N_-}$ and hence it admits a limit $u \in X_o^{2,N+N_-}$. Especially, $u_n \longrightarrow u$ in X^{∞,N_+}. By continuity of F, $F(u_n) \longrightarrow F(u)$ in $X^{\infty,0}$. On the other hand, by (2.3) and (3.1) (ix), (xii) there is

$$|F(u_{n+1}) - h|_{\infty,0} \leq c \left(|e_{n+1}|_{2,N_\infty} + \left|(I-S_n) \sum_{k=0}^{n} e_k\right|_{2,N_\infty}\right) \leq$$

$$\leq c \, \varepsilon^2 r^{-2(n+1)/3},$$

hence F(u) = h, which was to be proved.

II. A nonlinear equation of the second order

5. Existence theorem.

As an application of the above theory we consider the problem of the existence of time-periodic solutions of the equation

$$(5.1) \qquad \phi(u,\ \partial_t u,\ \partial_x u,\ \partial_t^2 u,\ \partial_x^2 u,\ \partial_x\partial_t u) = h(t,x)$$

with period $\omega > 0$, $t \in R^1$, $x \in \Omega =]0,1[$, satisfying boundary conditions $u(t,0) = u(t,1) = 0$.

The spaces in which the equation (5.1) is to be solved are chosen in a natural way: we put for $L \geq 0$ and $1 \leq p < \infty$ $X^{p,L} = W_\omega^{p,L}(\Omega)$, $X^{\infty,L} = C_\omega^L(\overline{\Omega})$ and for $L \geq 1$ $X_0^{2,L} =$

$= \{u \in W_\omega^{2,L}(\Omega),\ u(t,0) = u(t,1) = 0\}$, where $W_\omega^{p,L}(\Omega)$ denotes the Sobolev space of all real functions $u(t,x)$, $t \in R^1$, $x \in \Omega$, ω-periodic with respect to t and having all derivatives up to the order L in $L_p(]0,\omega[\times\Omega)$, with norm

$$|u|_{p,L} = \sum_{K=0}^{L} \sum_{J=0}^{K} \left(\int_\omega\int_\Omega |\partial_t^J \partial_x^{K-J} u|^p \, dx\, dt \right)^{1/p}$$

(the symbol $\int_\omega$ denotes the integration with respect to t over any interval $]t_0, t_0+\omega[$).

Similarly, $C_\omega^L(\overline{\Omega})$ denotes the space of all continuously differentiable functions on $R^1 \times \overline{\Omega}$ up to the order L and ω-periodic with respect to t, endowed with the norm

$$|u|_{\infty,L} = \sum_{K=0}^{L} \sum_{J=0}^{K} \sup \{ |\partial_t^J \partial_x^{K-J} u(t,x)|,\ t \in R^1,\ x \in \Omega \} .$$

The existence theorem is stated as follows (the symbol $\partial_i\phi(0)$ denotes the derivative of ϕ with respect to the i-th variable at the point $(0,0,0,0,0,0)$).

(5.2) THEOREM. Let $\omega > 0$, $\delta_0 > 0$, $N \geq 4$ be given, N integer. Put $M = 2N$. Let ϕ be a mapping of class C^{M+2} in its domain of definition $D(\phi) = [-\delta_0, \delta_0]^6$ such that

(i) $\phi(0) = 0$

(ii) $\partial_2\phi(0) > 0$, $\partial_5\phi(0) < 0$, $\partial_3\phi(0) = 0$.

Then there exist positive constants $\delta_1 > 0$, $\delta_N > 0$ such that if $\partial_1\phi(0) > -\delta_1$, then for every $h \in X^{2,M}$, $|h|_{2,M} < \delta_N$ there exists at least one solution $u \in X_0^{2,N+1}$ to the equation (5.1).

The proof consists in verifying that for the operator

(5.3) $F(u) \equiv \phi(\Lambda u) = \phi(u, \partial_t u, \partial_x u, \partial_t^2 u, \partial_x^2 u, \partial_x \partial_t u)$

the conditions (1.1) are fulfilled.

Putting $N_+ = N_0 = 2$, $N_- = 1$, $q = \dot{q} = 4$, $N_q = N_{\dot{q}} = 1$, $N_\infty = 2$ we check immediately that (1.1)(N) and (X) hold.

Next, we state without proof two lemmas. The first one is an easy consequence of the Nirenberg inequality (cf. [11]), the proof of the second one can be found in [9] or [17].

(5.4) Lemma. Let $0 \leq J \leq K \leq I \leq L$, $1 \leq p,q \leq \infty$. Then there exist constants $c_{p,I}$, $c_{p,q,L}$ such that for every $u \in X^{p,I}$, $v \in X^{q,L-J}$ there is

(i) $|u|_{p,K} \leq c_{p,I} \; |u|_{p,I}^{(K-J)/(I-J)} \; |u|_{p,J}^{(I-K)(I-J)}$

(ii) $|u|_{p,K} \, |v|_{q,L-K} \leq c_{p,q,L} \left(|u|_{p,J} |v|_{q,L-J} + |u|_{p,I} |v|_{q,L-I} \right)$.

(5.5) Lemma. Let $\delta_0 > 0$, $0 \leq L \leq M+2$. There exists a constant $c_{\delta_0,L}$ independent of ϕ such that for each $v \in X^{\infty,L+2}$, $|v|_{\infty,2} < \delta_0$ there is $\phi(\Lambda v) \in X^{\infty,L}$ and

$|\phi(\Lambda v)|_{\infty,L} \leq c_{\delta_0,L} \, (1 + |v|_{\infty,L+2}) \; |||\phi|||_L$, where

$$|||\phi|||_L = \sup\left\{ |\partial_{i_1} \dots \partial_{i_K} \phi(s_1, \dots, s_6)|, \, |s_i| < \delta_0, \; 0 \leq K \leq L, \; 1 \leq i_J \leq 6 \right\}.$$

We introduce the smoothing operators $\{S_n\}$ following [5], [6], [17]. First we define the continuous linear prolongation operators $P_{p,L}: W_\omega^{p,L}(\Omega) \longrightarrow W_\omega^{p,L}(R^1)$, $1 \leq p \leq \infty$, $0 \leq L \leq M+1$ by the Hestenes formula (cf. [4]). Then we find a C^∞-function φ

with support in $]-1,1[$ such that (see [17])

(5.6) (i) $\int_{-\infty}^{\infty} \varphi(s)\, ds = 1$

(ii) $\int_{-\infty}^{\infty} s^k \varphi(s)\, ds = 0, \; k = 1,2, \dots ,M.$

Finally, for $u \in X^{p,L}$ and $n \geq 0$ we put

(5.7) $(S_n u)(t,x) = \int_{R^2} r^{2n} \varphi(r^n(t-s))\, \varphi(r^n(x-y))\, P_{p,L} u(s,y)\, dyds,$

where $r > 1$ is an arbitrary fixed real number. We can directly check using (5.6) and (5.4) that the sequence $\{S_n\}_{n=0}^{\infty}$ satisfies (1.1)(S).

Since the verification of (1.1)(F1) follows from a straightforward computation employing (5.4) and (5.5), for proving the Theorem (5.2) it remains to show that (1.1)(F2) holds.

6. Linear equations.

Let us consider the linear equation

(6.1) $a_1 u + a_2\, \partial_t u + a_3\, \partial_x u + a_4\, \partial_t^2 u + a_5\, \partial_x^2 u + a_6\, \partial_x \partial_t u = h$

and assume

(6.2) For $i=1, \dots ,6$ there is $a_i \in X^{\infty,M+1}$ and

$|a_i|_{\infty,1} \leq A_i$

$|\partial_t a_i|_{\infty,0} \leq \tau_i$

$|\partial_x a_i|_{\infty,0} \leq \xi_i$

$\inf \{a_2(t,x),\ t \in R^1,\ x \in \Omega\} \geq m_2 > 0$

$\inf \{-a_5(t,x),\ t \in R^1,\ x \in \Omega\} \geq m_5 > 0$

$a_1(t,x) \geq -1/(8\vartheta^2) \min\{\lambda m_2, m_5\}$, where ϑ is the constant such that for every $u \in X_0^{2,1}$ there is

$$\int_\omega \int_\Omega |u|^2\, dx\, dt \leq \vartheta^2 \int_\omega \int_\Omega (|\partial_t u|^2 + |\partial_x u|^2)\, dx\, dt \text{ , and}$$

$$\lambda = \frac{2A_4}{m_2} + \frac{A_6^2}{m_2 m_5} ,$$

$A_3 \leq 1/(4\lambda) \min\{m_5, \lambda m_2\}$

and the numbers τ_i, ξ_i satisfy the inequalities

$$\lambda(\tau_4 + \xi_6 + \xi_5 + \nu^2\tau_1) + \nu^2(\tau_2 + \xi_3) + \nu(3\tau_4 + \xi_5 + \tau_6) \leq \frac{\lambda}{4} m_2$$

$$\lambda(\tau_5 + \xi_5 + \nu^2\tau_1) + \nu^2(\tau_2 + \xi_3) + \nu(\tau_4 + 3\xi_5 + 3\tau_6) \leq \frac{1}{4} m_5$$

$$(2M + 1)\,\tau_5 + \xi_5 + M\,\tau_6 \leq \frac{1}{4\lambda} m_5$$

$$(2M - 1)\,\tau_4 + \xi_6 + \xi_5 + M\,\tau_6 \leq \frac{1}{4} m_2$$

(6.3) THEOREM. Let us assume (6.2). Then for each $h \in X^{2,M}$ there exists a unique solution $u \in X_0^{2,M+1}$ of (6.1) and the inequality

$$(6.4) \qquad |u|_{2,L+1} \leq c_{L,\{A_i, m_i, \tau_i, \xi_i\}} \Big(|h|_{2,L} + |h|_{2,0} \sum_{i=1}^{6} |a_i|_{\infty,L+1}\Big)$$

holds for every L, $0 \leq L \leq M$.

Proof. We use the classical Galerkin-type procedure. Let $h \in X^{2,M}$ be given. For $m \geq 1$ we put

$$\underset{m}{u}(t,x) = \sum_{k=-m}^{m} \sum_{j=1}^{m} u_{kj}\, w_{kj}(t,x), \quad t \in R^1,\ x \in \Omega, \text{ where}$$

$w_{kj}(t,x) = e^{i\frac{2\pi k}{\omega}t} \sin j\pi x$, i is the imaginary unit, $u_{kj} = \bar{u}_{-kj}$.
The constant vector $U_m = \{u_{kj},\ j=1, \dots, m,\ k=-m, \dots, m\}$ is required to satisfy the system ($\bar{w}_{kj}$ denotes the complex conjugate of w_{kj})

$$(6.5) \int_\omega\int_\Omega (a_1 \underset{m}{u} + a_2 \partial_t \underset{m}{u} + a_3 \partial_x \underset{m}{u} + a_4 \partial_t^2 \underset{m}{u} + a_5 \partial_x^2 \underset{m}{u} + a_6 \partial_x \partial_t \underset{m}{u}) \cdot \bar{w}_{kj}\, dx\, dt = \int_\omega\int_\Omega h\, \bar{w}_{kj}\, dx\, dt,$$

$k=-m, \dots, m$, $j=1, \dots, m$,

which is a linear algebraic equation of the form

$$(6.6) \qquad A_m U_m = H_m,$$

where A_m is a square matrix of the type $(3m+1)\times(3m+1)$.

Let us multiply the (k,j)-th equation in (6.5) successively by $\bar{u}_{kj}$ and by $-i\lambda\frac{2\pi k}{\omega}\bar{u}_{kj}$ and in both cases sum over $k=-m, \dots, m$ and $j=1, \dots, m$. Summing the two relations we obtain after integration by parts

$$\int_\omega\int_\Omega (\lambda a_2 |\partial_t \underset{m}{u}|^2 - a_5 |\partial_x \underset{m}{u}|^2 + a_1 |\underset{m}{u}|^2)\, dx\, dt =$$
$$= \int_\omega\int_\Omega \{ (\tfrac{\lambda}{2}\partial_t a_4 + \tfrac{\lambda}{2}\partial_x a_6 + a_4) |\partial_t \underset{m}{u}|^2 - \tfrac{\lambda}{2}\partial_t a_5 |\partial_x \underset{m}{u}|^2 +$$
$$+ \tfrac{1}{2}(\lambda\partial_t a_1 + \partial_t a_2 + \partial_x a_3) |\underset{m}{u}|^2 + (-\lambda a_3 + \lambda\partial_x a_5 + a_6)\partial_x \underset{m}{u}\, \partial_t \underset{m}{u} +$$
$$+ \partial_t a_4 \cdot \partial_t \underset{m}{u} \cdot \underset{m}{u} + (\partial_x a_5 + \partial_t a_6)\partial_x \underset{m}{u} \cdot \underset{m}{u} + h(\lambda\partial_t \underset{m}{u} + \underset{m}{u}) \}\, dx\, dt .$$

Estimating

$$(6.7) \quad \left| \int_\omega\int_\Omega a_6\, \partial_x \underset{m}{u}\, \partial_t \underset{m}{u}\, dx\, dt \right| \leq A_6 (\tfrac{\mu}{2} |\partial_x \underset{m}{u}|^2_{2,0} + \tfrac{1}{2\mu} |\partial_t \underset{m}{u}|^2_{2,0})$$

with $\mu = m_5/A_6$ and using (6.2) we obtain

$$(6.8) \quad |\partial_t \underset{m}{u}|_{2,0} + |\partial_x \underset{m}{u}|_{2,0} \leq c_{\{A_i, m_i, \tau_i, \xi_i\}} \| H_m \| \leq$$
$$\leq c_{\{A_i, m_i, \tau_i, \xi_i\}} |h|_{2,0} ,$$

where $\|\cdot\|$ is a norm in R^{3m+1}.

From (6.8) it follows that the matrix A_m in (6.6) is nonsingular, hence there exists a unique solution U_m of (6.6). Moreover, by (6.8) the sequence $\{\underset{m}{u}\}_{m=1}^{\infty}$ is bounded in $X_o^{2,1}$. Consequently, there exists a subsequence of $\{\underset{m}{u}\}$ which converges weakly in $X_o^{2,1}$ to some $u \in X_o^{2,1}$. Taking the limit in (6.5) we see that

$$(6.9) \quad \int_\omega\int_\Omega \{ a_1 uv + a_2 \partial_t u \cdot v + a_3 \partial_x u \cdot v - \partial_t u \cdot \partial_t (a_4 v) - \partial_x u \cdot \partial_x (a_5 v) -$$
$$- \partial_x u \cdot \partial_t (a_6 v) \}\, dx\, dt = \int_\omega\int_\Omega h \cdot v\, dx\, dt$$

holds for every $v = \bar{w}_{kj}$ and hence for every $v \in X_o^{2,1}$. Remark that (6.9) can be considered as the definition of the weak solution of (6.1).

The passage to the weak limit in (6.8) gives

$$(6.10) \quad |u|_{2,1} \leq c_{\{A_i, m_i, \tau_i, \xi_i\}} |h|_{2,0} .$$

In order to obtain further estimates we multiply the (k,j)-th equation in (6.5) consecutively by $(i\,\frac{2\pi k}{\omega})^{2L}\,\bar{u}_{kj}$ and $\lambda(i\,\frac{2\pi k}{\omega})^{2L+1}\,\bar{u}_{kj}$, L being an arbitrary integer $1 \leq L \leq M$. Summing again over k and j and integrating by parts we obtain respectively

$$(6.11)\quad \int_\omega\int_\Omega \{-a_4\,|\partial_t^{L+1}\underline{u}|^2 - a_5\,|\partial_t^L\partial_x\underline{u}|^2 - a_6\cdot\partial_t^L\partial_x\underline{u}\cdot\partial_t^{L+1}\underline{u}\}dxdt =$$

$$= \int_\omega\int_\Omega \Big\{ \sum_{K=1}^{L}\binom{L}{K}\partial_t^K a_5\cdot\partial_t^{L-K}\partial_x\underline{u}\cdot\partial_t^L\partial_x\underline{u} - \sum_{K=0}^{L}\binom{L}{K}\partial_t^K a_1\cdot\partial_t^{L-K}\underline{u}\cdot\partial_t^L\underline{u} +$$

$$+ \sum_{K=0}^{L-1}\binom{L-1}{K}(-\partial_t^K\partial_x a_5\cdot\partial_t^{L-K-1}\partial_x\underline{u} + \partial_t^K a_3\cdot\partial_t^{L-K-1}\partial_x\underline{u} + \partial_t^K a_2\cdot\partial_t^{L-K}\underline{u})\cdot$$

$$\cdot\partial_t^{L+1}\underline{u} + \sum_{K=1}^{L-1}\binom{L-1}{K}(\partial_t^K a_4\cdot\partial_t^{L-K+1}\underline{u} + \partial_t^K a_6\cdot\partial_t^{L-K}\partial_x\underline{u})\cdot\partial_t^{L+1}\underline{u} +$$

$$+ \partial_t^L h\cdot\partial_t^L\underline{u} \Big\}\, dx\,dt ,$$

and

$$(6.12)\quad \lambda\int_\omega\int_\Omega \{(a_2 + (L-\tfrac{1}{2})\partial_t a_4 - \tfrac{1}{2}\partial_x a_6)|\partial_t^{L+1}\underline{u}|^2 +$$

$$+ (L+\tfrac{1}{2})\partial_t a_5|\partial_t^L\partial_x\underline{u}|^2 + (a_3 - \partial_x a_5 + L\partial_t a_6)\partial_t^L\partial_x\underline{u}\cdot\partial_t^{L+1}\underline{u}\}\, dx\,dt =$$

$$= \lambda\int_\omega\int_\Omega \Big\{ \sum_{K=2}^{L}\binom{L}{K}(-\partial_t^K a_4\cdot\partial_t^{L-K+2}\underline{u} - \partial_t^K a_6\cdot\partial_t^{L-K+1}\partial_x\underline{u})\,\partial_t^{L+1}\underline{u} +$$

$$+ \sum_{K=1}^{L}\binom{L}{K}(\partial_t^K\partial_x a_5\cdot\partial_t^{L-K}\partial_x\underline{u} - \partial_t^K a_2\cdot\partial_t^{L-K+1}\partial_x\underline{u} - \partial_t^K a_3\cdot\partial_t^{L-K}\partial_x\underline{u})\partial_t^{L+1}\underline{u} -$$

$$- \sum_{K=1}^{L}\binom{L+1}{K+1}\partial_t^{K+1}a_5\cdot\partial_t^{L-K}\partial_x\underline{u}\cdot\partial_t^L\partial_x\underline{u} - \sum_{K=0}^{L}\binom{L}{K}\partial_t^K a_1\cdot\partial_t^{L-K}\underline{u}\,\partial_t^{L+1}\underline{u} +$$

$$+ \partial_t^L h\,\partial_t^{L+1}\underline{u} \Big\}\, dx\,dt \quad .$$

Now, adding (6.11) to (6.12) and using (6.2) and an estimate analogous to (6.7) with the same μ we derive the inequality

$$|\partial_t^{L+1}\underline{u}|_{2,0}^2 + |\partial_t^L\partial_x\underline{u}|_{2,0}^2 \leq c_{L,\{A_i,m_i,\tau_i,\xi_i\}}(|\partial_t^{L+1}\underline{u}|_{2,0} +$$

$$+ |\partial_t^L\partial_x\underline{u}|_{2,0}) \Big\{ |h|_{2,L} + \sum_{K=1}^{L}(\sum_{i=1}^{6}|a_i|_{\infty,K+1})(|\partial_t^{L-K+1}\underline{u}|_{2,0} +$$

$$+ |\partial_t^{L-K}\partial_x\underline{u}|_{2,0})\Big\}.$$

By induction over L using (6.10) and (5.4)(ii) we obtain

$$(6.13)\quad |\partial_t^{L+1} u_n|_{2,0} + |\partial_t^L \partial_x u_n|_{2,0} \leq$$
$$\leq c_{L,\{A_i,m_i,\tau_i,\xi_i\}} \left(|h|_{2,L} + |h|_{2,0} \sum_{i=1}^{6} |a_i|_{\infty,L+1}\right).$$

Hence, the weakly convergent subsequence $\{u_n\}$ of $\{u_m\}$ can be chosen in such a way that $\partial_t^L u_n$ converges weakly to $\partial_t^L u$ in $X_0^{2,1}$ and $|\partial_t^L u|_{2,1}$ is estimated from above by the right-hand side of (6.13) for all L, $0 \leq L \leq M$.

From (6.9) we see that the distributional derivative $\partial_x^2 u$ equals to some function from $X^{2,0}$, hence u is an element of $X_0^{2,2}$ and satisfies (6.1) a.e. in $R^1 \times \Omega$. Differentiating formally the equation (6.1) up to the order $M-1$ we show by induction (using (6.13) and (5.4)(ii)) that u is an element of $X_0^{2,M+1}$ and satisfies (6.4) for $0 \leq L \leq M$. Thus, the theorem (6.3) is proved.

7. Completing of the proof of (5.2) and final remarks.

The theorem (6.3) yields sufficient means for verifying that the operator (5.3) satisfies (1.1)(F2). In fact, we have

$$F'(v)\,u = \partial_1\phi(\Lambda v)\cdot u + \partial_2\phi(\Lambda v)\cdot\partial_t u + \partial_3\phi(\Lambda v)\cdot\partial_x u + \partial_4\phi(\Lambda v)\cdot\partial_t^2 u +$$
$$+ \partial_5\phi(\Lambda v)\cdot\partial_x^2 u + \partial_6\phi(\Lambda v)\cdot\partial_x\partial_t u\ .$$

If $\delta_- > 0$ is taken sufficiently small, then for $|v|_{\infty,3} < \delta_-$ the relations (6.2) hold e.g. for $m_2 = \frac{1}{2}\partial_2\phi(0)$, $m_5 = -\frac{1}{2}\partial_5\phi(0)$, $A_i = |\partial_i\phi(0)| + \delta_i$, $\delta_i = 1$ for $i \neq 1,3$, $\delta_3 = \frac{1}{4\lambda}\min\{\lambda m_2, m_5\}$, $\delta_1 = \frac{1}{16\nu^2}\min\{\lambda m_2, m_5\}$. Using (6.4) and (5.5) we obtain exactly (1.1)(F2). Thus, the proof of the theorem (5.2) is complete.

(7.1) Remarks.

(i) Another application of this method can be found in [7], where one investigates the existence of periodic solutions of the Maxwell equations in nonlinear media in the Sobolev spaces $H^{p,L}$ of divergence-free vector functions in three dimensions. In general, the proof in more space dimensions requires further considerations concerning the prolongation of domains of definition outside Ω and the regularity of solutions of linear elliptic equations.

(ii) If the operator (5.3) is quasilinear of the type

$$\partial_t(\phi_1(u, \partial_t u, \partial_x u)) + \partial_x(\phi_2(u, \partial_t u, \partial_x u)) + \phi_3(u, \partial_t u, \partial_x u) ,$$

we can "save" one derivative by putting $N_o = 1$.

(iii) The method remains valid for a nonhomogeneous equation $\phi(t,x,\Lambda u) = h$ provided we assume that the conditions (5.2) (i), (ii) hold uniformly with respect to $(t,x) \in R^1 \times \bar{\Omega}$, and $\partial_t \phi$ is sufficiently small.

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