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THE SOLUTION OF A FUČÍK'S CONJECTURE

Rudolf ŠVARC

Dedicated to the memory of Svatopluk FUČÍK

Abstract: In his book [3] Fučík had formulated an open problem on the equations with jumping nonlinearity. Roughly speaking, having in mind the special kind of the nonlinearity, there could be some nontrivial relations between the Leray-Schauder degree and the number of solutions to such equations. By a method of geometrical visualization of $\mathbb{R}^4$, this article shows that it is not the case.

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Introduction

Let H be a Hilbert space with a cone C of "non-negative" elements, i.e., for each $u \in H$ there exist

$$u^+ = \max\{u, 0\} \in C,$$
$$u^- = \max\{-u, 0\} \in C,$$
$$u = u^+ - u^-.$$

Let the mappings $u \longmapsto u^+$,

$u \longmapsto u^-$

be continuous. Let $S : H \to H$ be a linear completely continuous selfadjoint operator. Let λ and μ be two real parameters. We define the operators $S_{\lambda,\mu} : H \to H$ as follows:

$$S_{\lambda,\mu} u = u + \lambda S u^{+} - \mu S u^{-} .$$

An operator of this type is said to be an operator with jumping nonlinearity. First results concerning this type of operators were probably obtained by Ambresetti and Prodi in [1,2]. Some other papers concerning this subject are quoted in the references to this article. A list of references can be found in [3] also.

The operator $S_{\lambda,\mu}$ being positively homogeneous ($S_{\lambda,\mu}(tu) = tS_{\lambda,\mu} u$), one can easily prove the

Assertion: Let B be a ball centred in $0 \in H$. Let the Leray-Schauder degree of $S_{\lambda,\mu}$ w.r.t. the point 0 and the ball B

$$\deg(S_{\lambda,\mu}, 0, B)$$

be defined.

Then the equation

(*) $$S_{\lambda,\mu} u = f$$

has at least $|\deg(S_{\lambda,\mu}, 0, B)|$ solutions for each $f \in H$.

In [3] Fučík had formulated the following

Conjecture: Let B be a ball centered in $0 \in H$. Let the Leray-Schauder degree of $S_{\lambda,\mu}$ w.r.t. the point 0 and the ball B

$$\deg(S_{\lambda,\mu}, 0, B) = 0 .$$

Then there exists some $f \in H$ such that the equation (*) has no solution.

This conjecture doesn't hold and we shall construct a counterexample.

1. The Representation of $\mathbb{R}^4$ by Moves in $\mathbb{R}^3$

We shall construct a counterexample to the Fučík's conjecture in the four-dimensional euclidean space $\mathbb{R}^4$. In order to achieve this goal, we need some geometrical intuition concerning the four-dimensional euclidean space. Fortunately we have good experiences with the four-dimensional space-time, because we all live in it. These experiences only need to be translated into the geometrical terms and assertions concerning the $\mathbb{R}^4$.

For the sake of better understanding the corresponding construction, at first we shall investigate the relations between $\mathbb{R}^3$ and the three-dimensional "plane-time". Then we shall proceed by analogy in the more interesting case of $\mathbb{R}^4$ and the space-time.

Let us have the cartesian coordinates (O, x_1, x_2, t) in the three-dimensional euclidean space. Let ϱ^t be the plane perpendicular to the t-axis, which intersects it in the point $O^t = (0, 0, t)$. Let the axes x_1^t, x_2^t be the perpendicular projections of the axes x_1 and x_2 into the plane ϱ^t. Let U be any geometrical object in the space. Let U^t be its section by the plane ϱ^t.

Let $\tilde{\varrho}$ be a plane with the cartesian coordinates $(\tilde{O}, \tilde{x}_1, \tilde{x}_2)$. In the time t we can map ϱ^t (with the coordinates (O^t, x_1^t, x_2^t) and U^t) isometrically onto $\tilde{\varrho}$ so that the axis x_1^t is mapped onto the axis $\tilde{x}_1$, x_2^t onto $\tilde{x}_2$.

Mapping this way in each moment t the corresponding plane ϱ^t onto $\tilde{\varrho}$, we get in $\tilde{\varrho}$ some moving object, which will be in the time t conformable to the section U^t.

So we can visualize any geometrical object in the space as a moving object in the plane. It's worth of mentioning explicitly, how can be interpreted the point, the straight-line, the half-line and the plane in $\mathbb{R}^3$ by means of two-dimensional moving pictures.

Let us choose some point in $\mathbb{R}^3$. Then all but one plane perpendicular to the t-axis are disjoint with the point. That's why the plane $\tilde{\varrho}$ will be "void" in any time t, only in one moment we shall see there one distinguished point. A straight-line may be either perpendicular to the t-axis or not. If not, then every t-section is a plane with a distinguished point. So in the plane $\tilde{\varrho}$ we shall see one moving point. Because this point corresponds to a straight-line, it will move with a constant velocity (in special cases this velocity can be 0). If the straight-line is perpendicular to the t-axis, all but one t-sections are void, the remaining section contains all the line. So the plane $\tilde{\varrho}$ will be void in all but one time moments, in one moment we shall see some straight-line in it. As for the half-line, the situation in $\tilde{\varrho}$ will be similar. If it is not perpendicular to the t-axis, then we shall see in $\tilde{\varrho}$ either a point moving with a constant velocity until it disappears in some time moment, then $\tilde{\varrho}$ remains void. Or $\tilde{\varrho}$ will be void for an infinite time, but in some moment there appears a point moving with a constant velocity, which can't disappear any more.

To a plane in $\mathbb{R}^3$ corresponds in general in $\tilde{\varrho}$ a straight-line moving with constant velocity through $\tilde{\varrho}$. Probably the reader visualizes the moving pictures in $\tilde{\varrho}$ as black objects in a white plane. The visualization of such a type is necessary in the case of a plane perpendicular to the t-axis. Such a plane can be represented in $\tilde{\varrho}$ as follows: $\tilde{\varrho}$ is white in all but one moments, in one moment it is black.

By analogy one can visualize geometrical objects in $\mathbb{R}^4$ as moving geometrical objects in $\mathbb{R}^3$. E.g., to a three-dimensional hyperplane in $\mathbb{R}^4$ there corresponds in general a plane in $\mathbb{R}^3$, moving itself with a constant velocity. Another example: Two two-dimensional planes in $\mathbb{R}^4$ have in general one common point. This fact can be visualized as follows: To a plane in $\mathbb{R}^4$ corresponds a moving straight-line in $\mathbb{R}^3$. To two planes in $\mathbb{R}^4$ correspond in general two moving straight-lines in general setting, i.e., they are not parallel and they have not the same velocity. Thus they intersect in just one point in just one moment.

Of course, we can represent geometrical objects in $\mathbb{R}^n$ as moving objects in $\mathbb{R}^{n-1}$ by this way, but we shall need this representation only in the cases $n = 3$ and $n = 4$.

2. Brouwer Degree of a Map

Because we shall work in euclidean spaces we don't need the concept of the Leray-Schauder degree of a map. As concerns the Brouwer degree, we have to make some comments about its application to the special type of problems, we are dealing with.

The Brouwer degree of a continuous mapping $F: \bar{D} \to \mathbb{R}^n$ with respect to the (non-void open bounded) set $D \subset \mathbb{R}^n$ with the boundary ∂D and a point $f \in \mathbb{R}^n - F(\partial D)$ will be denoted as

$$\deg(F, f, D).$$

$J(F(x))$ means the Jacobi determinant of F in x .

Definition: $f \in \mathbb{R}^n - F(\partial D)$ is said to be a regular value of F iff there exists $J(F(x)) \neq 0$ whenever $F(x) = f$.
$f \in \mathbb{R}^n - F(\partial D)$ is said to be a singular value of F if it is not a regular value of F .

Remark: In the definition of regular values of F one usually supposes that F is C^1 , so $J(F(x))$ always exists. Unfortunately, the operators with jumping nonlinearity are continuous, but not C^1 in general. That is why the above definition suits better for our aims.

It is a well-known fact that

$$(1) \qquad \deg(F,f,D) = \sum_{x \in F^{-1}(f)} \operatorname{sign} J(F(x)) ,$$

whenever $F \in C^1(D) \cap C(\bar{D})$ and $f \in \mathbb{R}^n - F(\partial D)$ is a regular value of F . For $F^{-1}(f) = \emptyset$ we have $\deg(F,f,D) = 0$.

Let $B_r(0)$ be a ball with radius r centred in 0 .

Let

$$(2) \qquad S_{\lambda,\mu} : \mathbb{R}^n \to \mathbb{R}^n$$

be defined by the equation

$$(3) \qquad S_{\lambda,\mu} u = u + \lambda S u^+ - \mu S u^- .$$

$J(S_{\lambda,\mu}(x))$ does not exist in general, if

$$x \in \bigcup_{i=1}^{n} \varrho_i ,$$

where

$$\varrho_i = \{ x \in \mathbb{R}^n \mid x_i = 0 \} , \quad i = 1,2,\dots,n .$$

Let P_n be the system of all subsets of the set

$$\mathcal{N} = \{1,2,3,\dots,n\} .$$

The hyperplanes ϱ_i divide $\mathbb{R}^n$ into 2^n components K_N, $N \in P_n$ defined as follows:

$$K_N = \{x \in \mathbb{R}^n \mid x_i > 0 \quad \text{for } i \in N, x_i < 0 \quad \text{for } i \in \mathcal{N} - N\} .$$

ϱ_i can be divided into the subsets

$$\varrho_{i,N} = \{x \in \mathbb{R}^n \mid x_i = 0 ; \; x_j \geq 0 \qquad \text{for } j \in N , j \neq i ,$$
$$x_j \leq 0 \quad \text{for } j \in \mathcal{N} - N , j \neq i\} .$$

$S_{\lambda,\mu}$ is evidently linear on the closure of any component K_N and

$$\dim \varrho_{i,N} = n-1 \qquad \text{for every } i,N .$$

Thus

$$\dim S_{\lambda,\mu}(\varrho_{i,N}) \leq n-1$$

and the Lebesque n-dimensional measure

$$\operatorname{meas} S_{\lambda,\mu}(\varrho_{i,N}) = 0$$

for every i, N . But then

$$(4) \qquad \operatorname{meas} S_{\lambda,\mu}\Big(\bigcup_{i,N} \varrho_{i,N}\Big) = \operatorname{meas} S_{\lambda,\mu}\Big(\bigcup_{i=1}^{n} \varrho_i\Big) = 0 ,$$

too.

<u>Lemma 1.</u> Let $J(S_{\lambda,\mu}(x)) = 0$. Then there exists
$$y \in S^{-1}_{\lambda,\mu}(S_{\lambda,\mu}(x)) \cap \bigcup_{i=1}^{n} \varrho_i .$$
<u>Proof</u> is trivial, one only needs to recall that $S_{\lambda,\mu}$ is linear on the closure of any component K_N . As a consequence of Lemma 1. and (4) we have

Lemma 2. The set of singular values of $S_{\lambda,\mu}$ has zero Lebesque measure.

So for almost all $f \in \mathbb{R}^n$ we can define

$$\deg(S_{\lambda,\mu}, f, B_r(0))$$

by (1). In the singular values we can define $\deg(S_{\lambda,\mu}, f, B_r(0))$ using the continuity of the Brouwer degree on $\mathbb{R}^n - S_{\lambda,\mu}(\partial B_r(0))$.

In particular $\deg(S_{\lambda,\mu}, 0, B_r(0))$ is defined this way, whenever $0 \notin S_{\lambda,\mu}(\partial B_r(0))$. Having in mind the positive homogeneity of $S_{\lambda,\mu}$, we see that $\deg(S_{\lambda,\mu}, 0, B_r(0))$ is independent of r.

Further, f being a regular value of $S_{\lambda,\mu}$ and $0 \notin S_{\lambda,\mu}(\partial B_r(0))$, the values tf are also regular for all $t > 0$ small enough. Thus we get for any regular value f

$$\deg(S_{\lambda,\mu}, 0, B_r(0)) = \lim_{t \to 0^+} d(S_{\lambda,\mu}, tf, B_r(0)) =$$

$$= \lim_{t \to 0^+} \sum_{v \in S^{-1}_{\lambda,\mu}(tf) \cap B_r(0)} \operatorname{sign} J(S_{\lambda,\mu}(v)) =$$

$$= \lim_{t \to 0^+} \sum_{v/t \in S^{-1}_{\lambda,\mu}(f) \cap B_{r/t}(0)} \operatorname{sign} J(tS_{\lambda,\mu}(v/t))$$

$$= \lim_{t \to 0^+} \sum_{u \in S^{-1}_{\lambda,\mu}(f) \cap B_{r/t}(0)} \operatorname{sign} J(S_{\lambda,\mu}(u))$$

$$= \sum_{u \in S^{-1}_{\lambda,\mu}(f)} \operatorname{sign} J(S_{\lambda,\mu}(u)).$$

From now on we are interested only in the degree of $S_{\lambda,\mu}$ w.r.t. 0 and a ball centred in 0 and we shall use a shorter notation, namely $d(S_{\lambda,\mu})$. In this notation we have:

(5) $$d(S_{\lambda,\mu}) = \sum_{u \in S^{-1}_{\lambda,\mu}(f)} \operatorname{sign} J(S_{\lambda,\mu}(u))$$

for any regular value f of $S_{\lambda,\mu}$.

Convention:

The vectors considered in the sequel will be tacitly supposed to be regular. If it would not be this case, we could always take a regular value of $S_{\lambda,\mu}$ near enough to the singular one in the question.

Remark: All we have done till now, we could do with any neighbourhood of 0 instead of $B_r(0)$. In particular, instead of $B_r(0)$ we could consider the unit ball in the non-euclidean norm

(6) $$\|u\|_1 = \sum_{i=1}^{n} |u_i| ,$$

i.e., the set

$$\tilde{B}_1 = \{ u \in \mathbb{R}^n \mid \Sigma |u_i| \le 1 \} .$$

$S_{\lambda,\mu}$ being linear on $\bar{K}_N$, $N \in P_n$ and $\bar{K}_N \cap \partial\tilde{B}_1$ lying in a hyperplane in $\mathbb{R}^n$, the set

$$S_{\lambda,\mu}(\bar{K}_N \cap \partial\tilde{B}_1)$$

lies also in a hyperplane in $\mathbb{R}^n$. From the point of view of geometrical visualization we can take advantage of this fact. In the sequel we shall work with $\tilde{B}_1$ rather than $B_r(0)$, because $S_{\lambda,\mu}(\partial\tilde{B}_1)$ can be better visualized then $S_{\lambda,\mu}(B_r(0))$. In particular, all pictures are to be understood in the norm (6), i.e., as the images of $\partial\tilde{B}_1$. Also, instead of $\partial\tilde{B}_1$ (or $\partial B_r(0)$) we shall use only the symbol ∂ .

3. The Geometrical Rules for Computing the Brouwer Degree

∂ in the norm (6) is not a smooth surface. Nevertheless it can be oriented as shown for $n=2$ on the picture 1A. The vector of the outer normal n_x in a point $X \in \partial$ (with all coordinates different from zero) can be chosen so that it belongs to the same $K_N \subset \mathbb{R}^n (N \in P_n)$ as the point X. This way the outer normal is defined in almost all points of ∂.

Remark: ∂ can be smoothened on a small neighbourhood of the points some coordinates of which are zero. Choosing an appropriate orientation on this regularization of ∂, we get on ∂ the just defined orientation by taking the limit.

On a neighbourhood of X in ∂ we can choose a local system of coordinates $(X, \xi_1, \xi_2, \dots, \xi_{n-1})$ in such a way, that the system of coordinates $(X, \xi_1, \dots, \xi_{n-1}, n_x)$ in $\mathbb{R}^n$ is oriented positively. $S_{\lambda,\mu}$ maps the system $(X, \xi_1, \dots, \xi_{n-1})$ on a system $(X', \xi'_1, \dots, \xi'_{n-1})$. This system is regular whenever X' is a regular value of $S_{\lambda,\mu}$. Now we can choose the normal $n_{x'}$ to $S_{\lambda,\mu}(\partial)$ in X' so that the system $(X', \xi'_1, \dots, \xi'_{n-1}, n_{x'})$ is oriented positively (see picture 1B). Let $n'_x = S_{\lambda,\mu}(n_x)$. Then $J(S_{\lambda,\mu}(X)) > 0$ iff $(X', \xi'_1, \dots, \xi'_{n-1}, n'_x)$ has a positive orientation. ($J(S_{\lambda,\mu}(X)) \neq 0$ because X' is supposed to be a regular value of $S_{\lambda,\mu}$.)

Thus

$$\operatorname{sign} J(S_{\lambda,\mu}(x)) = \operatorname{sign}(n'_x, n_{x'}),$$

$(.,.)$ denoting the inner product. But $X = \alpha n_x + \sigma$, $\alpha > 0$ and σ is parallel with that part $K_N \cap \partial$ of ∂ which contains X. Thus

$$X' = S_{\lambda,\mu}(X) = \alpha n'_X + \gamma ,$$

γ being parallel with $S_{\lambda,\mu}(K_N \cap \partial)$ and $X' \in S_{\lambda,\mu}(K_N \cap \partial)$, i.e., $(n_{X'}, \gamma) = 0$ and

(7) $$\operatorname{sign} J(S_{\lambda,\mu}(X)) = \operatorname{sign}(X', n_{X'}) .$$

Let f be a regular value of $S_{\lambda,\mu}$. Let us take the half-line

$$p = \{ tf \mid t \geq 0 , t \in \mathbb{R} \} .$$

On the picture 2B this half-line intersects $S_{\lambda,\mu}(\partial)$ in the points M'_1, M'_2, M'_3 with preimages M_1, M_2, M_3 on the picture 2A. From the picture and (7)

$$\operatorname{sign} J(S_{\lambda,\mu}(M_1)) = 1 ,$$
$$\operatorname{sign} J(S_{\lambda,\mu}(M_2)) = 1 ,$$
$$\operatorname{sign} J(S_{\lambda,\mu}(M_3)) = -1 .$$

Now for each $u \in \mathbb{R}^n$ and $t > 0$

$$\operatorname{sign} J(S_{\lambda,\mu}(u)) = \operatorname{sign} J(S_{\lambda,\mu}(tu)) ,$$

in particular,

$$\operatorname{sign} J(S_{\lambda,\mu}(u)) = \operatorname{sign} J\left(S_{\lambda,\mu}\left(\frac{u}{\|u\|}\right)\right)$$

for $u \neq 0$. Now, from (5) we can deduce the equation

$$d(S_{\lambda,\mu}) = \sum_{u \in S^{-1}_{\lambda,\mu}(f)} \operatorname{sign} J\left(S_{\lambda,\mu}\left(\frac{u}{\|u\|}\right)\right)$$

and using also (7) we get

$$d(S_{\lambda,\mu}) = \sum_{\substack{X' = tf,\, t>0 \\ X' \in S_{\lambda,\mu}(\partial)}} \operatorname{sign}(X', n_X) ,$$

but $X' = tf$, $t > 0$ means that $X' \in p$. So we have finally

(8) $d(S_{\lambda,\mu}) = \sum_{X' \in S_{\lambda,\mu}(\partial) \cap p} \operatorname{sign}(X', n_{X'})$.

In the case of picture 2 we have

$$d(S_{\lambda,\mu}) = 1 + 1 - 1 = 1 .$$

In general we have the following rule for computing $d(S_{\lambda,\mu})$: First we choose a suitable half-line p starting in O'. Then we construct the set of all intersection points $\{M'_i ;\ i = 1, 2, \ldots, k\} = p \cap S_{\lambda,\mu}(\partial)$. Then we have to sum the signs of $(M'_i , n_{M'_i})$ for $i = 1, 2, \ldots, k$.

Further we shall denote $\partial' = S_{\lambda,\mu}(\partial)$.

Remark: Regularizing ∂ and $S_{\lambda,\mu}$, we get certain regularization $\tilde{\partial}'$ of ∂' . Then the orientation of the whole $\tilde{\partial}'$ is defined by choosing $n_{X'}$ in a single point X' of $\tilde{\partial}'$. Taking a limit, we get an orientation of ∂' , which is of the type defined above. Thus if we are interested only in $|d(S_{\lambda,\mu})|$ and not in $d(S_{\lambda,\mu})$, we do not need to care about ∂ . We only have to choose $n_{X'}$ in a point $X' \in \partial'$ and ∂' will be oriented. If we take the orientation opposite to that one induced by $S_{\lambda,\mu}$ from ∂ , the formula (8) gives $-d(S_{\lambda,\mu})$ instead of $d(S_{\lambda,\mu})$. From now on we shall suppose that

(9) $S_{\lambda,\mu}(\partial \cap \{x \in \mathbb{R}^n \mid x_n = 0\}) \subset \varrho$,

where ϱ is a hyperplane in $\mathbb{R}^n$. Let ϱ' be a hyperplane parallel with ϱ and passing through O' . For computing $d(S_{\lambda,\mu})$ according to the above-written rule, we can take p lying in ϱ' .

Then $p \subset \varrho^1$ and the points M_i' lie in $\partial' \cap \varrho^1$. Let $n_{x'}^1$ be the perpendicular projection of $n_{x'}$ in ϱ^1. Then

$$\operatorname{sign}(X', n_{x'}^1) = \operatorname{sign}(X', n_{x'})$$

and (see (8))

(10) $$d(S_{\lambda,\mu}) = \sum_{X' \in \partial' \cap p} \operatorname{sign}(X', n_{x'}^1) .$$

Clearly $n_{x'}^1$ is a normal vector to $\partial' \cap \varrho^1$ in ϱ^1.

Let $A^+ = S_{\lambda,\mu}(a_n)$, $A^- = S_{\lambda,\mu}(-a_n)$

(a_n is the unit vector in the direction of the axis x_n).
We shall distinguish two cases:

$\boxed{1}$ A^+ and A^- lie in opposite half-spaces, defined by ϱ^1.
Let, e.g., ϱ lie in the half-space $A^-_{\varrho^1}$.
With the help of a rotation of the coordinates in $\mathbb{R}^n$ the axis y_n can be made perpendicular to ϱ^1. Then $\partial' \cap \varrho^1$ is $F(\partial^1)$, where $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $F(0) = 0$, F is positively homogeneous and "linear on quadrants" (∂^1 is the boundary of the unit ball in $\mathbb{R}^{n-1}$). So F is of the same type as $S_{\lambda,\mu}$, in particular, a formula analogous to (8) and (10) holds for F.
In the preimage M_i of the point M_i' we can define a positively oriented system of coordinates ($M_i, \xi_{i,1}, \ldots, \xi_{i,n}$) such that the axes $\xi_{i,1}', \ldots, \xi_{i,n-2}'$ lie in ϱ^1, that $\xi_{i,n-1}'$ has the direction of the vector $M_i' A^+$ and $\xi_{i,n}$ the direction of OM_i. (This is possible for $n \geq 3$; if $n \leq 2$ everything is easier.) Now all the axes $\xi_{i,n-1}'$ are oriented into the half-space $A^+_{\varrho^1}$ and all the other axes lie in ϱ^1. That is why either all the

systems $(M_i', \xi_{i,1}', \dots, \xi_{i,n}')$ have the same orientation as the corresponding systems $(M_i', \xi_{i,1}', \dots, \xi_{i,n-2}', \xi_{i,n}')$ in ϱ^1 or all have the opposite orientation to the corresponding systems. So up to the sign $d(S_{\lambda,\mu})$ is equal to $d(F)$. In the situation drawn in picture 3 $(n=2)$ we can compute $d(S_{\lambda,\mu})$ up to the sign from the picture 4, whenever we know in which component of ϱ^1, defined by the points B_1', D_1', lies the point O'. In the situation on the pictures 5A, resp. 5B $(n=3)$ we can compute $d(S_{\lambda,\mu})$ up to the sign from the pictures 6A, resp. 6B, of course we must know, in which one of the components of ϱ^1 lies O'.

$\boxed{2}$ A^+ and A^- lie in the same half-space defined by ϱ^1 (see picture 8). With the exception of not interesting cases, $\partial' \cap \varrho^1$ is a pair of homothetic objects similar to (9). Let us denote these objects as U^+ and U^-, U^+ being that one which is contracted with the growing x_n into the point A^+ (see section 1), U^- being the other one. The points M_i' can be divided into two subsets, one subset containing those ones lying on U^+, the other subset that ones lying on U^-.

Making a rotation of coordinates, U^+ can be considered as $F^+(\partial^1)$, U^- as $F^-(\partial^1)$. (F^+ and F^- are certain maps $\mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1}$ with the properties of F defined in the part 1 Further we can choose these maps so that

$$d(F^+) = \sum_{x' \in p \cap U^+} \operatorname{sign}(x', n_{x'}^1),$$

$$d(F^-) = \sum_{x' \in p \cap U^-} \operatorname{sign}(x', n_{x'}^1),$$

then we get of course

(11) $d(S_{\lambda,\mu}) = d(F^+) + d(F^-)$.

Lemma 3. Let $u,v \in S^{-1}_{\lambda,\mu}(f)$ (f is a regular value of $S_{\lambda,\mu}$) . Let

$$\operatorname{sign} u_i = \operatorname{sign} v_i \qquad \text{for } i = 1,2,\dots,n-1,$$

(12) $$\operatorname{sign} u_n = -\operatorname{sign} v_n .$$

Then

(13) $$\operatorname{sign} J(S_{\lambda,\mu}(u)) = -\operatorname{sign} J(S_{\lambda,\mu}(v)) .$$

Proof. Let $u \in K_{N_1}$, $v \in K_{N_2}$, $N_1, N_2 \in P_n, n \in N_2, N_1 = N_2 \cup \{n\}$. On K_{N_1} resp. K_{N_2} the operator $S_{\lambda,\mu}$ coincides with some regular linear operator S_{λ,μ,N_1} , resp. S_{λ,μ,N_2} . The matrices of these two linear operators differ ony in the n -th column and

(14)
$$J(S_{\lambda,\mu}(u)) = \det S_{\lambda,\mu,N_1},$$
$$J(S_{\lambda,\mu}(v)) = \det S_{\lambda,\mu,N_2} .$$

For u and v we have the linear equations
$$S_{\lambda,\mu,N_1} u = f , \quad S_{\lambda,\mu,N_2} v = f .$$
According to the Frobenius' theorem

(15) $$u_n = \frac{\det_{n,f} S_{\lambda,\mu,N_1}}{\det S_{\lambda,\mu,N_1}} , \qquad v_n = \frac{\det_{n,f} S_{\lambda,\mu,N_2}}{\det S_{\lambda,\mu,N_2}} ,$$

$\det_{n,f} S_{\lambda,\mu,N_i}$ are the determinants of the matrices which we get from the matrices S_{λ,μ,N_i} substituting the n-th column by f. But S_{λ,μ,N_i} are different only in the n-th column, so after this substitution the two matrices will coincide. Thus, we have

$$\det_{n,f} S_{\lambda,\mu,N_1} = \det_{n,f} S_{\lambda,\mu,N_2}$$

and

$$\operatorname{sign} \det S_{\lambda,\mu,N_1} = - \operatorname{sign} \det S_{\lambda,\mu,N_2}$$

according to (12) and (15). Now according to (14) we get (13). On U^+ and U^- the orientation is defined by means of the projections $n^1_{X'}$ of the vectors $n_{X'}$ into ϱ^1. U^+ and U^- are homothetic, so to each point of U^+ corresponds just one point of U^-, further the vectors $n^1_{X'}$ and $n^1_{Y'}$ in corresponding points X', Y' are parallel.

From lemma 3 follows that these two vectors have always opposite orientations:

The points X', Y' define a straight-line q. Let us choose a point $Q' \in q$ not in the interval $X'Y'$. Let us make a shift of the system of coordinates in order to get the origin into the point Q'. That means: we have to subtract the vector $O'Q' \in \mathbb{R}^{n-1}$ from all the columns of the matrices F^+, F^- and also from all the columns of the matrix S (in the case of the matrix S we consider $O'Q'$ as a vector in $\mathbb{R}^n$). This way we get certain maps $F^+_Q, F^-_Q, S_{Q,\lambda,\mu}$. In the new coordinate system neither the geometrical form nor the orientation of ∂' is subjected to a change. (The orientation of ∂' is now induced from ∂ by means of $S_{Q,\lambda,\mu}$ instead of $S_{\lambda,\mu}$.) Thus, the orientation of U^+ resp. U^- remains the same, too.

Choosing now f in the direction of the half-line $Q'X' = Q'Y'$, lemma 3 applied to $S_{Q,\lambda,\mu}$ gives the assertion.

Remark: Excluding the cases, in which the centre of the homothety of U^+ and U^- lies on $U^+ \cap U^-$ (we have made this assumption tacitly in the foregoing considerations), we can choose just this centre for Q', because this centre never lies in $X'Y'$.

U^+ divides ϱ' into certain components $\varkappa^+_m$, U^- into components $\varkappa^-_m$. (Between $\varkappa^+_m$ and $\varkappa^-_m$ there is a one-to-one correspondence because of the homothety.)

To each of the components $\varkappa^+_m$ there corresponds an integer d_m such that $d(F^+)$ equals d_m, if O' lies in $\varkappa^+_m$. Defining similar integers d^-_m for $\varkappa^-_m$, we get according to the previous reasoning the equation

$$d^-_m = -d_m .$$

Thus we can formulate the following rule for computing $|d(S_{\lambda,\mu})|$:

We choose some orientation of U^+ and we find the integers d_m corresponding to the components $\varkappa^+_m$. To the components $\varkappa^-_m$ correspond the integers $-d_m$. If $O' \in \varkappa^+_{m_1} \cap \varkappa^-_{m_2}$, we have according to (11) either

$$d(S_{\lambda,\mu}) = d_{m_1} - d_{m_2} \tag{16}$$

or

$$d(S_{\lambda,\mu}) = -d_{m_1} + d_{m_2} .$$

In the case $n = 3$, if U^+, U^- are as in the picture 7A (see picture 8), we take the picture 9 at first. In the hyperplane

ϱ' there is U^+ , orienting it properly, the integers d_m will be as in the picture. In the hyperplane ϱ'' there is U^- , to the components $\varkappa_m^-$ of ϱ'' correspond the integers $-d_m$. Covering ϱ'' with ϱ' and computing the sums of the corresponding integers for each component of $\varrho' - \partial'_1$ we get the picture 10, from which we can read $d(S_{\lambda,\mu})$ up to the sign, if we know, in which of its components lies O'.

If U^+, U^- have the form of the picture 7B, see pictures 11 and 12 which have been obtained just the same way from 7B as 9 and 10 have been obtained from 7A.

Remark: The most interesting feature of the picture 12 is the presence of the integer -2 in it, i.e., for $n = 3$ there are operators with $|d(S_{\lambda,\mu})| = 2$.

Remark: Let us mention once more the assumption (9)!

4. The Surjectivity of the Operator $S_{\lambda,\mu}$

Let us continue in the investigation of the last example in the preceding section. What is the two-dimensional visualization according to the section 1? At first we have a void plane. In a certain moment there turns up the picture 7B. This picture immediately splits into two copies of itself. Each of these copies then begins to contract and shift uniformly to a point. In some other moment one of these copies disappears in the corresponding point and then we have only one copy of the picture 7B in the plane. This remaining copy continues in contracting and shifting, at last it disappears in the corresponding point, too. Then the plane remains void.

Further, in the time 0 in the plane will be one point more (it corresponds to the origin of the coordinate system).

So in this moment we shall see something like the picture 12 (without the integers but with one more point). What does it mean, that the equation (*) has no solution? In the usual setting it means that there is a half-line going from the origin which does not intersect ∂'.

Thus, in the plane visualization (in general) either in the time 0 there turns up the image of the origin which does not disappear any more, but moves with a constant velocity into infinity. Or, vice-versa, there is a point coming with a constant velocity from infinity into the origin, in some moment it falls into the origin and disappears for ever.

In neither of the two cases the moving point passes across the moving lines drawn in the picture 12 .

The components of the picture 12 with the integer 0 represent the cases in which (*) has no solution for some right-hand side. Really, let us imagine the development of the picture with reversely oriented time axis. In the time 0 we shall have just the picture 12 with one more point in a 0-component. We can choose such a velocity for this point, that it won't touch any of the lines of the picture in any time between 0 and the moment, when u^+ and u^- coincide. But then it cannot touch these lines either, because after this moment the plane will be completely void except of that one point. (Thus we have constructed a half-line not intersecting ∂'.)

This cannot happen in any other component. If, e.g., the image of the origin falls into the component with -2 , it cannot get out of it without passing across some of the moving lines, because all the component contracts into one point and our moving point will be necessarily "caught".

The difference between these two examples is caused by the fact, that all the $\mathcal{O}$-components in the picture 12 are coverings of some component of 7B by itself, all the other components are coverings of some component of 7B by another component of 7B.

5. The Existence of the Counterexample in $\mathbb{R}^4$

Now we want to find an operator $S_{\lambda,\mu}$, such that $d(S_{\lambda,\mu}) = 0$ and (★) has some solution for any right-hand side f. We want to use the results of the preceding sections, thus we will seek it among the operators fulfilling the assumption (9). It has no sense to seek it among the operators considered in the part 1 of the section 3. If there were an operator corresponding to this part of section 3 in the dimension n, there would exist an operator with the same properties in the dimension $n-1$, too.

So let $S^1, \varkappa_m^+, \varkappa_m^-, \mathcal{O}^1, d_m$ be as in the part 2 of the section 3.

Let $\mathcal{O}' \in \varkappa_{m_1}^+ \cap \varkappa_{m_2}^-$. We require (see (16)) that $0 = d_{m_1} - d_{m_2}$, i.e.,

$$d_{m_1} = d_{m_2},$$

according to the section 4 it must be

$$\varkappa_{m_1}^+ \neq \varkappa_{m_2}^+ .$$

Thus we look in the dimension $n-1$ for such an operator $S^1_{\lambda_1,\mu_1}$ for which $\mathbb{R}^{n-1} - S^1_{\lambda_1,\mu_1}(\partial^1)$ has two different components with the same corresponding integers.

We can't find a counterexample in the dimension 3, because it is clear, that $\mathbb{R}^2 - S^1_{\lambda_1,\mu_1}(\partial^1)$ has one of the forms in the pictures 7A, 7B. In neither case it has two different components with the same integers d_m . But it is just as clear now that a counterexample exists for $n = 4$. Namely, the picture 15 shows that it is possible to construct on operator

$$S^1_{\lambda_1,\mu_1} : R^3 \to R^3$$

such that $R^3 - S^{1\,1}_{\lambda_1,\mu_1}(\partial^1)$ has two components with the same d_m . We only have to cover, e.g., the component with 1 on the left-hand side of the picture 12 with such a component on the right-hand side. One only needs to know that these two components are really disjoint in R^3 (see picture 13, they are drawn there). It does not follow at once from the fact that their two-dimensional sections in the picture 12 are disjoint! E.g., all the 0-components in the picture 12 are parts of the section of the single component of $R^3 - S_{\lambda,\mu}(\partial)$ with $d_m = 0$.

Remark: $S^1_{\lambda_1,\mu_1}$ corresponds to F from the section 3.

Remark: By the same reasoning as above we can show that for $n = 4$ there exists $S_{\lambda,\mu}$ with $|\, d\,(S_{\lambda,\mu})| = 3$.

An attentive reader has probably noticed that we haven't reached our goal yet. In fact, till now we have proved only the existence of an operator with required properties in a more general class of operators than defined in the beginning. Namely, it's the class of operators of the type

(17) $$Au^+ + Bu^- ,$$

A and B being two linear operators. (Let us notice that the i-th column of the matrix A is the image of a_i , the i-th column of B is the image of $-a_i$.)

But

$$Au^+ + Bu^- = \frac{1}{2}(A+B)u^+ + \frac{1}{2}(A-B)u^+ +$$
$$+ \frac{1}{2}(A-B)u^- - \frac{1}{2}(A-B)u^- =$$
$$= \frac{1}{2}(A-B)u + \frac{1}{2}(A+B)u^+ + \frac{1}{2}(A+B)u^- .$$

From the construction of the counterexamples one sees at once that they form an open set in the class of operators (17). So taking another example if necessary, we can suppose that $A-B$ is a regular matrix. Then, making a regular change of coordinates in the space $\mathbb{R}^4$, the operator (17) can be transformed into the form

$$u + (A-B)^{-1}(A+B)u^+ + (A-B)^{-1}(A+B)u^- , \tag{18}$$

which is $\left[(A-B)^{-1}(A+B)\right]_{1,-1}$ in our notation.
A regular transformation of coordinates can only change the sign of $d(S_{\lambda,\mu})$ (it does not matter) and does not concern the solvability of (*).

One can also show that the matrices A, B in (18) can be chosen so that $(A-B)^{-1}(A+B)$ is symmetric.

Remark: We could continue with the covering construction into higher dimensions. Of course, it would be technically more difficult. Nevertheless it's almost clear now that we can construct examples with $|d(S_{\lambda,\mu})|$ as big as we please. Also we could construct examples with $d(S_{\lambda,\mu}) = 0$ and as many solutions for every right-hand side, as we please. One only must take n big enough.

6. One Counterexample

We can take $\lambda = 1$, $\mu = -1$ and

$$S = \begin{pmatrix} 3{,}5 & -1 & -1 & -1 \\ -1 & 3{,}5 & -1 & -1 \\ -1 & -1 & 3{,}5 & -1 \\ -1 & -1 & -1 & 2{,}5 \end{pmatrix}. \tag{19}$$

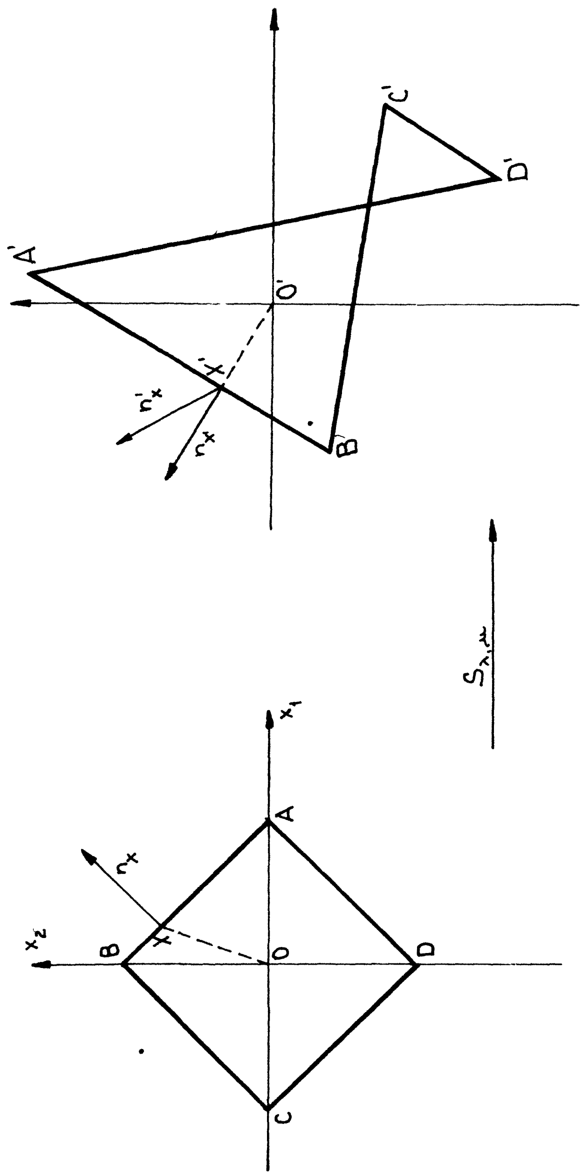
Remark: The matrix S has a double eigenvalue. But the counterexamples to the Fučík's conjecture form an open set, thus there exist matrices without multiple eigenvalues which also give counterexamples for some λ and μ.

Remark: If we knew the above-written matrix S and the numbers λ and μ, the proof, that it gives a counterexample to the Fučík's conjecture, could be done much shorter. But, according to my opinion, the method used in the construction of the example gives a better understanding of it (see the concluding remarks to section 5).

Problem: The inverse matrix to S in (19) is

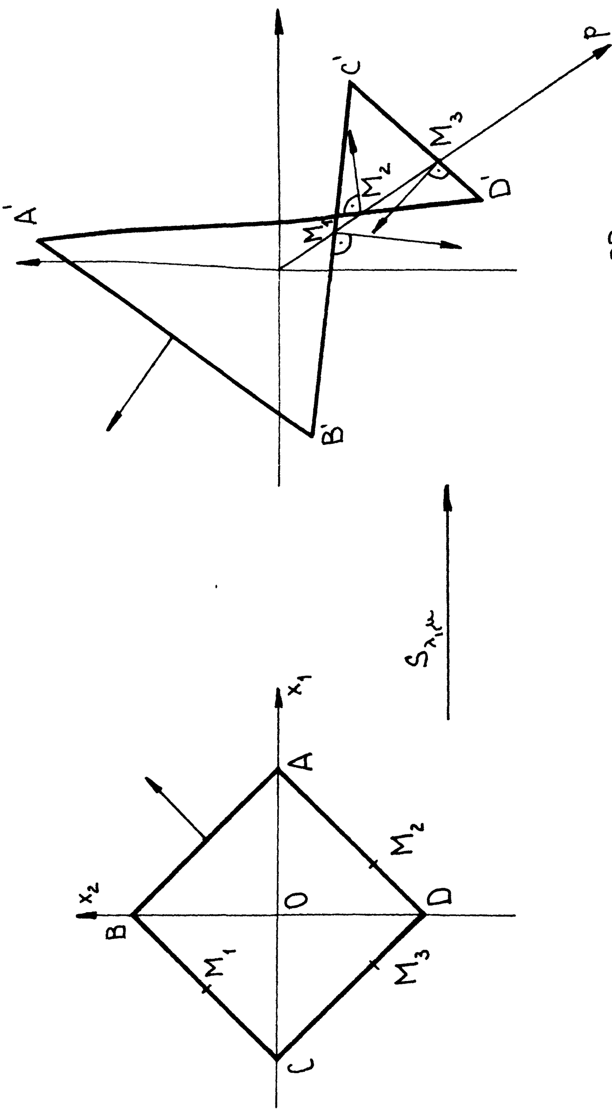
$$S^{-1} = \begin{pmatrix} \frac{34}{27} & \frac{28}{27} & \frac{28}{27} & \frac{4}{3} \\ \frac{28}{27} & \frac{34}{27} & \frac{28}{27} & \frac{4}{3} \\ \frac{28}{27} & \frac{28}{27} & \frac{34}{27} & \frac{4}{3} \\ \frac{4}{3} & \frac{4}{3} & \frac{4}{3} & 2 \end{pmatrix}$$

Thus all its entries are positive. It would be interesting to know whether this fact is important for the construction of the counterexample or not.



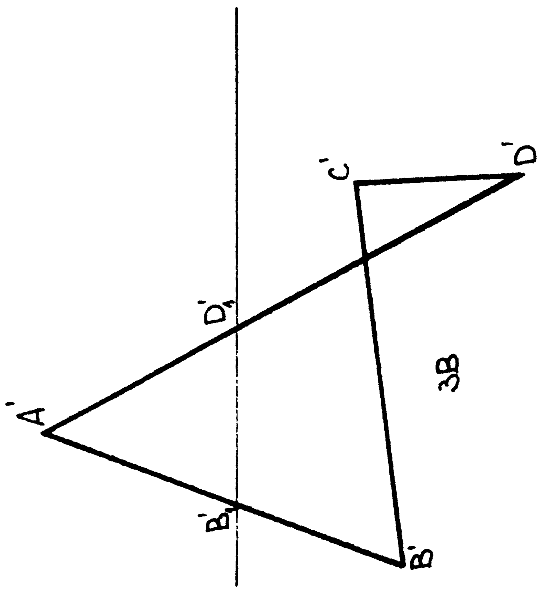
1B

1A

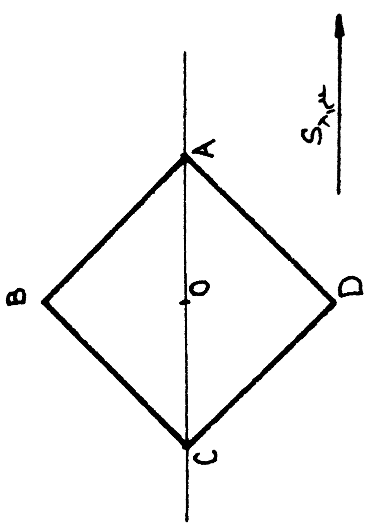


2B

2A

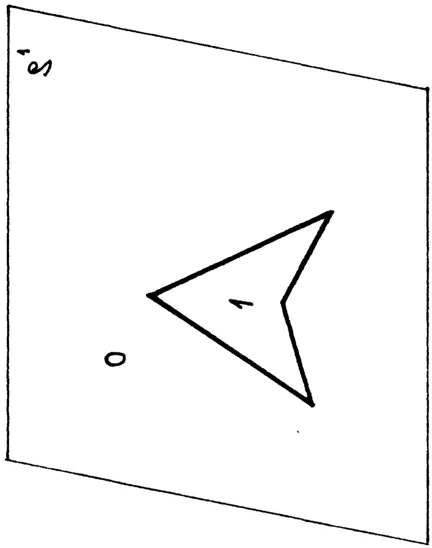


3B

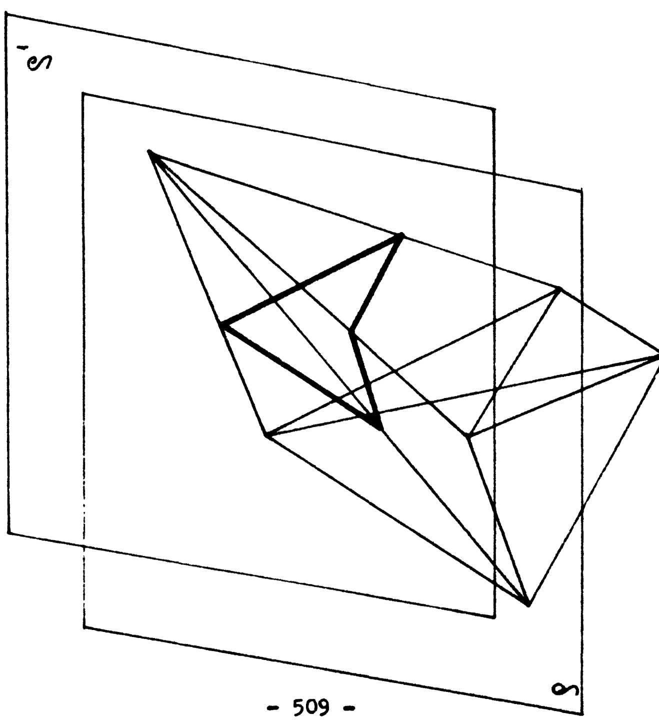


3A

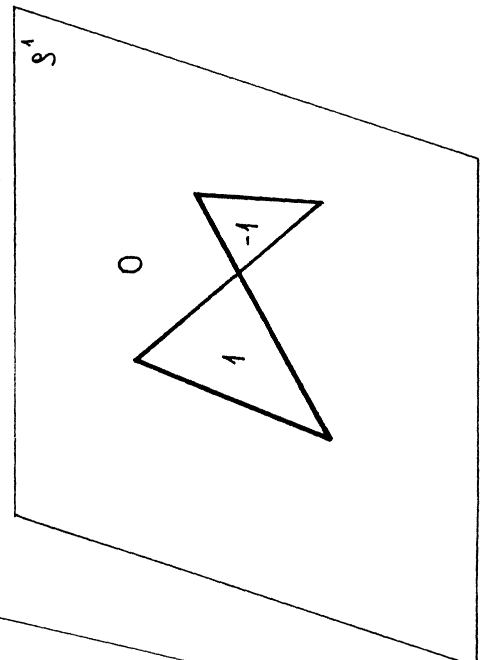
4



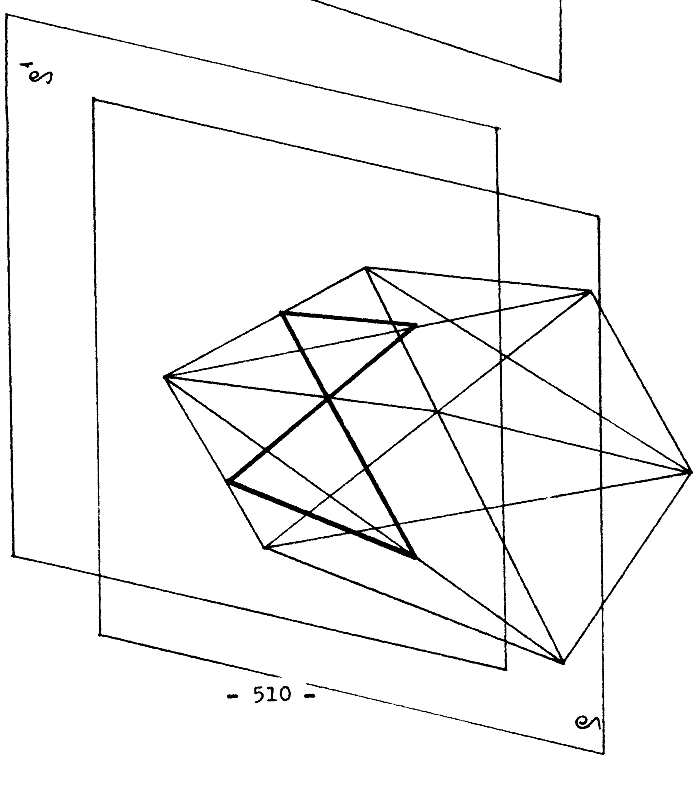
6A



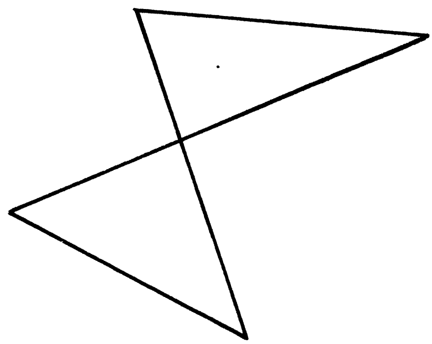
5A



6B



5B



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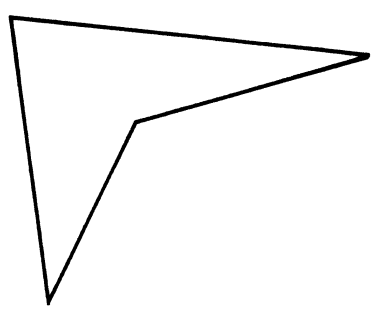

7A

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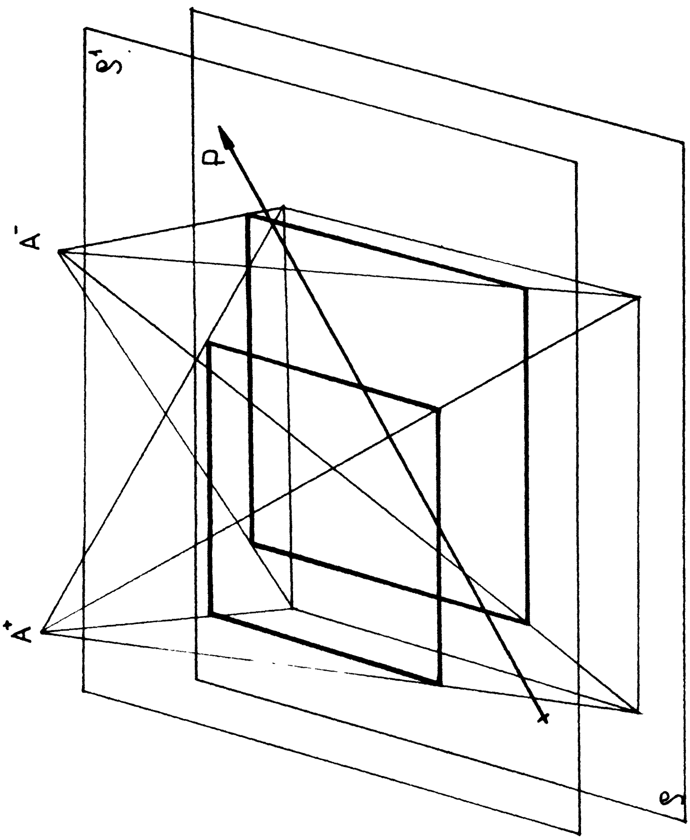

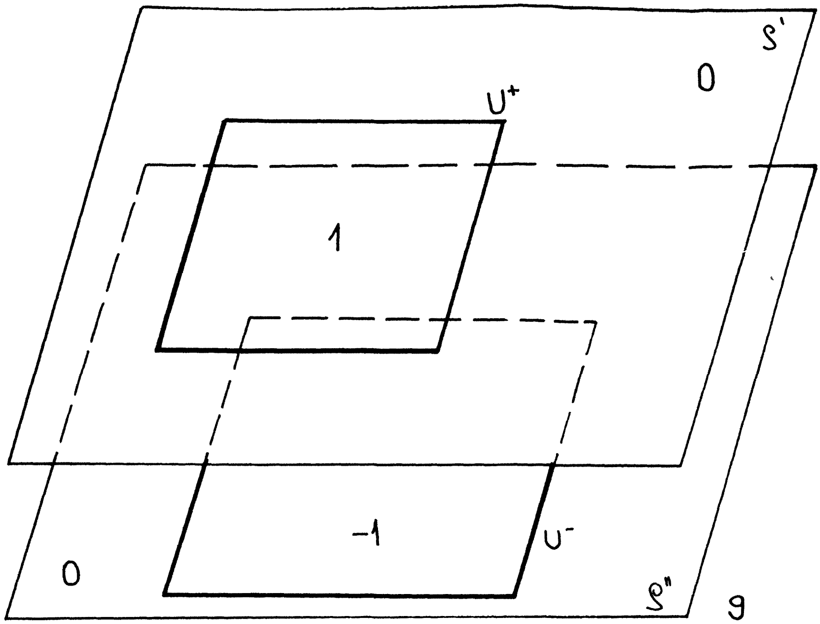

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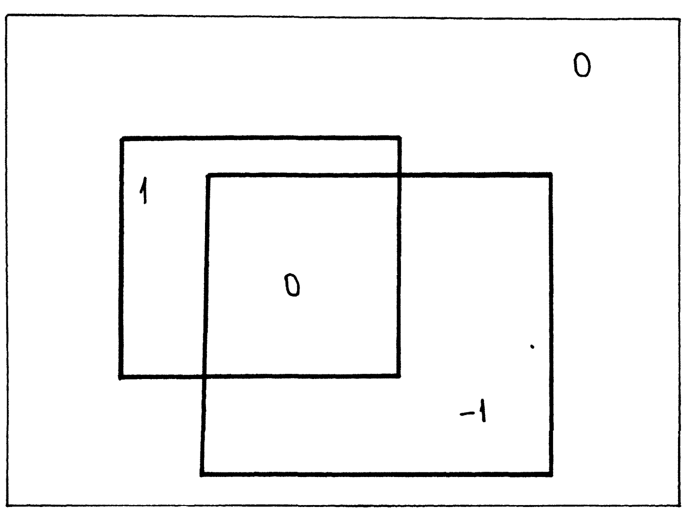

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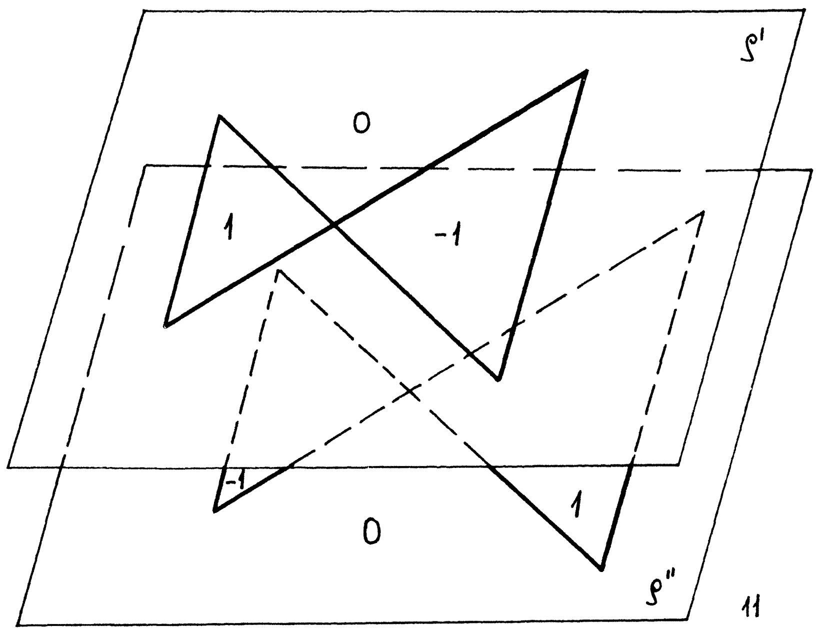

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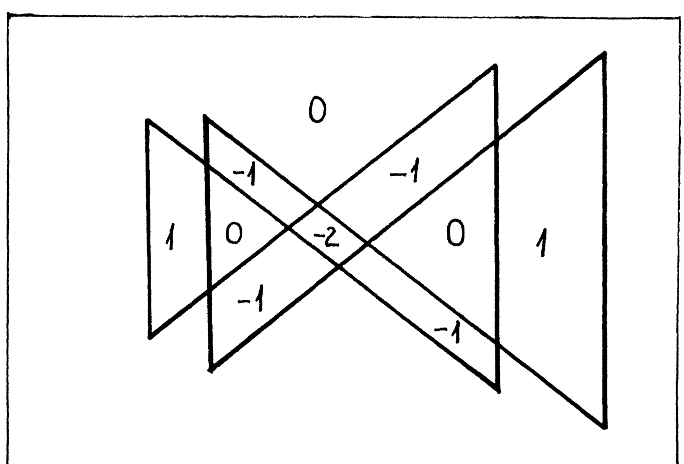

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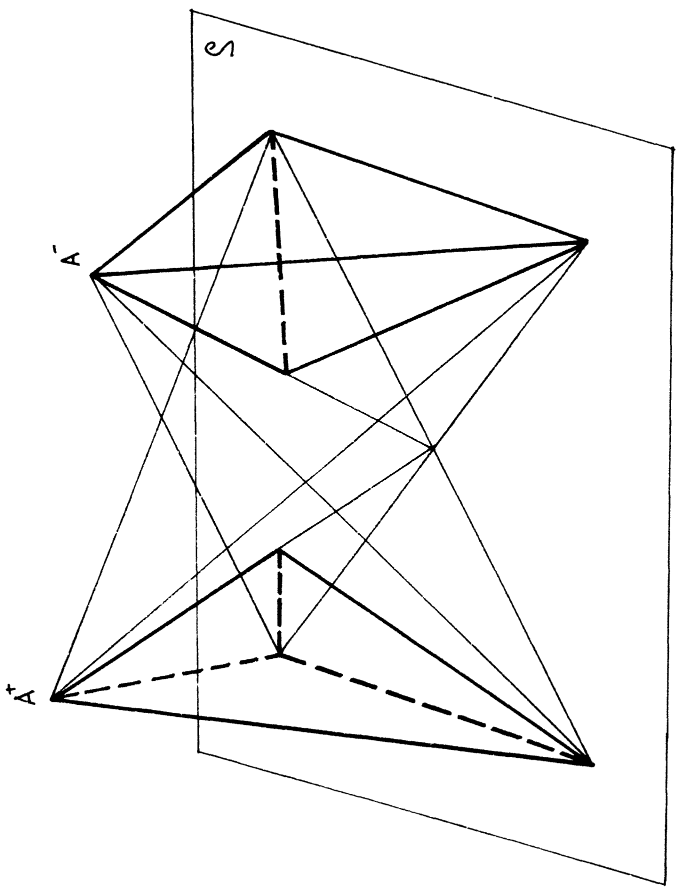

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