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THE DIRICHLET PROBLEM IN THE DISPERSION OF GAS EXHALATIONS OVER A WET HILLY SURFACE

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Dedicated to the memory of Svatopluk FUČÍK

Abstract: Sutton [8], Berliand [1] and Marchuk [6] have studied the dispersion of gas exhalations over a flat surface. In this paper, we consider the dispersion of gas exhalations over a wet hilly surface.

Key words: Elliptic equation, Dirichlet problem, weak solution, "very weak" solution.

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Introduction. The main goal of this paper is to study a question of the existence, unicity and regularity of the "very weak" solution of the boundary value problem describing the air pollution in the case of the wet hilly surface. The case of flat surface was considered by many authors (e.g. Berliand [1], Marchuk [6], and Sutton [8]). The case of a hilly surface was considered by Hino [4] and Berliand and collective [2] (under some simplifying assumptions). As far as the author knows, the problem of the dispersion of gas exhalations over general hilly surface has not been considered.

Under the assumptions formulated in part I the process of exhalation dispersion corresponds to the Dirichlet problem for an elliptic equation of the second order with the right-hand side given by Dirac distribution. So we must seek the solution of the boundary value problem in the "very weak" sense (see Definition 3). Its existence and unicity is proved in part II. Part III deals with the regularity of the "very weak" solution.

I. Formulation of the problem. The general continuity equation has the form

$$(1) \qquad \frac{\partial c}{\partial t} - \operatorname{div}(K.\operatorname{grad} c) + (\vec{V}, \operatorname{grad} c) + \sigma c = f(t, \xi)$$

where $c = c(t, \xi)$ is a concentration of the exhalations,

$$K = \begin{pmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_z \end{pmatrix}$$

is a matrix of the coefficients of turbulent diffusion, $\vec{V} = [v_x, v_y, v_z]$ is a vector of the wind velocity, $f = f(t, \xi)$ is a density of a given source of exhalations and $(.,.)$ is the inner product in $\mathbb{R}^3$.

Assume that during the process of exhalation dispersion the following conditions are satisfied:

1. The earth hilly surface over which the exhalations are extended is described by a twice continuously differentiable function $z = \alpha(x,y)$.

2. The exhalated gas reacts chemically with the atmosphere. The loss due to the chemical reaction is characterized by a non-negative constant σ.

3. The source of exhalations is situated in the point $\xi_0 = [0,0,h]$, where h is its effective height and Q is its emission for time unit.

4. The process is stationary, i.e. $\frac{\partial c}{\partial t} = 0$.

5. The earth surface is wet. It means that the most of exhalations are absorbed by the earth surface. Thus, the adequate boundary condition is

$$c(x,y,\alpha(x,y)) = 0$$

for every $[x,y] \in \mathbb{R}^2$.

6. The wind velocity satisfies the mass conservation law

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0.$$

7. The concentration of the gas exhalations vanishes in the infinity, i.e.

$$\lim_{|x|+|y|+z \to \infty} c(x,y,z) = 0.$$

Under those assumptions we can formulate the following boundary value problem

$$Lc = -\operatorname{div}(K,\operatorname{grad}c) + (\vec{v},\operatorname{grad}c) + \sigma c = Q\delta_{\xi_0}(\xi)$$

$$\lim_{|x|+|y|+z \to \infty} c(x,y,z) = 0,$$

$$c(x,y,\alpha(x,y)) = 0.$$

II. <u>Existence and unicity of the solution.</u> We consider our problem in the bounded domain $\Omega \subset \{[x,y,z] \in \mathbb{R}^3 : z > \alpha(x,y)\}$ for which:

(i) $\partial\Omega$ is a twice continuously differentiable boundary, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_2 is described by the function $z = \alpha(x,y)$ and $\Gamma_1 = \partial\Omega - \Gamma_2$.

(ii) $\xi_0 \in \Omega$ and Ω is sufficiently large. It enables us to put, approximately,

$$c|_{\Gamma_1} = 0.$$

In the domain Ω our boundary value problem is the Dirichlet problem in the form

(2) $Lc \equiv -\operatorname{div}(K.\operatorname{grad}c) + (\vec{\nabla}, \operatorname{grad}c) + \sigma c = Q\,\delta_{\xi_o}(\xi)$

(3) $$c\,|_{\partial\Omega} = 0.$$

By the symbols $W_2^k(\Omega)$ and $\overset{\circ}{W}{}_2^k(\Omega)$ we denote the Sobolev spaces (see [3]).

Let A be an elliptic differential operator defined by the relation

(4) $$Au = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(\xi)\,\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{3} b_i(\xi)\,\frac{\partial u}{\partial x_i} + a(\xi)\,u\,.$$

Define a bilinear form

(5) $$((u,v)) = \sum_{i,j=1}^{3} \int_\Omega a_{ij}\,\frac{\partial u}{\partial x_j}\,\frac{\partial v}{\partial x_i} + \sum_{i=1}^{3} \int_\Omega b_i\,\frac{\partial u}{\partial x_i}\,v + \int_\Omega a(\xi)u$$

for every $u,v \in W_2^1(\Omega)$.

<u>Definition 1</u>. We say that the function $u \in W_2^1(\Omega)$ (resp. $u \in \overset{\circ}{W}{}_2^1(\Omega)$) is a weak solution of the differential equation

(6) $$Au = f$$

(resp. of the Dirichlet problem

(7) $$Au = f,$$

(8) $$u\,|_{\partial\Omega} = 0)$$

if for all $v \in \overset{\circ}{W}{}_2^1(\Omega)$ the relation

(9) $$((u,v)) = \langle f,v \rangle$$

holds, where $\langle .,. \rangle$ is the duality between $W_2^{-1}(\Omega)$ and $\overset{\circ}{W}{}_2^1(\Omega)$.

<u>Proposition 1</u>. Let Ω be a domain with a twice continuously differentiable boundary and $\Omega_1 \subset \overline{\Omega}_1 \subset \Omega \cup \Gamma$, where Γ is any open part of $\partial\Omega$. Further, let the following condi-

tions be satisfied:

(i) There exists $\gamma > 0$ such that

$$\gamma^{-1} \|\eta\|^2 \leq \sum_{i,j=1}^{3} a_{ij}(\xi)\eta_i\eta_j \leq \gamma\|\eta\|^2$$

for a.e. $\xi \in \Omega$ and every $\eta = [\eta_1, \eta_2, \eta_3] \neq 0$.

(ii) The functions a_{ij} have Lipschitz derivatives of the first order and b_i, a are Lipschitz functions on $\overline{\Omega}$.

Then every weak solution u of the equation (7) with the right-hand side $f \in L_2(\Omega)$ which on Γ is given by some function u_o from $W_2^2(\Omega_1)$, i.e.

$$Au = f,$$

$$u|_{\Gamma} = u_o|_{\Gamma}, \quad u_o \in W_2^2(\Omega_1)$$

belongs to $W_2^2(\Omega_1)$, while for every subdomain $\Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1 \cup \Gamma$ there exists a constant $M = M(\Omega_1, \Omega_2)$ such that the following inequality holds:

$$\|u\|_{W_2^2(\Omega_2)} \leq M(\Omega_1, \Omega_2)(\|f\|_{L_2(\Omega_1)} + \|u_o\|_{W_2^2(\Omega_1)}).$$

Proof (see [5]).

Theorem 1 (Existence and unicity of the weak solution). Let the following conditions be satisfied:

(i) There exists a constant $\gamma > 0$ such that the inequality

$$\gamma^{-1}\|\eta\|^2 \leq k_x(\xi)\eta_x^2 + k_y(\xi)\eta_y^2 + k_z(\xi)_z^2 \leq \gamma\|\eta\|^2$$

holds for a.e. $\xi \in \Omega$ and every $\eta = [\eta_x, \eta_y, \eta_z] \neq 0$.

(ii) k_x, k_y, k_z have Lipschitz derivatives of the first order and v_x, v_y, v_z are Lipschitz functions on $\overline{\Omega}$.

Then the Dirichlet problem

(10) $$Lc = f,$$

(11) $$c|_{\partial\Omega} = 0$$

has one and only one weak solution c for an arbitrary right-hand side $f \in W_2^{-1}(\Omega)$. Moreover, if the functional f belongs to $L_2(\Omega)$, then the weak solution $c \in W_{2,o}^2(\Omega) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ and the inequality

(12) $$\| c \|_{W_2^2(\Omega)} \leq M(\| f \|_{L_2(\Omega)} + \| c \|_{L_2(\Omega)})$$

holds, where the constant M is independent of f.

<u>Proof</u>. The existence and the unicity is an immediate consequence of Lax-Milgram's Theorem, because under the assumptions on the coefficients k_x, k_y, k_x and v_x, v_y, v_z our problem is coercive.

The inequality (12) follows from Proposition 1.

Analogically we get the following results:

<u>Theorem 1*</u> . Let the conditions of Theorem 1 be satisfied. Then the adjoint problem

(10*) $$L^* c^* = - \operatorname{div} (K. \operatorname{grad} c^*) - (\vec{\vartheta}, \operatorname{grad} c^*) + \sigma c^* = f^*$$

(11*) $$c^*|_{\partial\Omega} = 0$$

has one and only one weak solution c^* which belongs to $W_{2,o}^2(\Omega)$ for an arbitrary right-hand side $f^* \in W_2^{-1}(\Omega)$, while the following inequality is satisfied:

(12*) $$\| c^* \|_{W_2^2(\Omega)} \leq (\| f^* \|_{L_2(\Omega)} + \| c^* \|_{L_2(\Omega)}).$$

The constant M is independent of f^*.

<u>Remark</u>. According to Theorem 1, the Green operator

$$G^* : L_2(\Omega) \longrightarrow W_{2,o}^2(\Omega)$$

is defined by the relation

$$G^* f^* = c^* ,$$

where c^* is a weak solution of the problem (10^*)-(11^*).

Let Ω_o be a subdomain of Ω such that $\xi_o \in \Omega_o \subset \overline{\Omega}_o \subset \Omega$. Denote

$$W(\Omega_o) = W_2^2(\Omega_o) \cap L_2(\Omega),$$

equipped with the norm

$$\| u \|_{W(\Omega_o)} = \| u \|_{W_2^2(\Omega_o)} + \| u \|_{L_2(\Omega)}.$$

<u>Definition 2</u>. The problem (8)-(9) is called W-correct, if for every $f \in L_2(\Omega)$ there exists one and only one weak solution u from $W(\Omega_o)$.

Now, let v^* be a weak solution of the adjoint problem

$$(7^*) \quad A^* u^* = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(a_{ij}(\xi)\,\frac{\partial v^*}{\partial x_i}\right) - \sum b_i(\xi)\,\frac{\partial v^*}{\partial x_i} + av^* = f^*,$$

$$(8^*) \qquad v^*\big|_{\partial\Omega} = 0$$

with the right-hand side $f^* \in L_2(\Omega)$ and u is a weak solution of the problem (7)-(8) with the right-hand side $f \in W_2^{-1}(\Omega)$. According to the definition of the weak solution we have

$$((v^*,u)) = (f^*,u)$$

and

$$((u,v^*)) = \langle f,v^* \rangle .$$

We have easily

$$(u,f^*) = \langle f,v^* \rangle .$$

Taking account of the notation introduced in Remark, we can rewrite the last relation as

$$(13) \qquad (u,f^*) = \langle f,G^* f^* \rangle .$$

From this we go to

Definition 3. Let the boundary value problem (7*)-(8*) be W-correct. The function $c \in L_2(\Omega)$ is called a "very weak" solution of the problem (7)-(8) with the right hand side $f \in W^*(\Omega_0)$, if for every $f^* \in L_2(\Omega)$ the equality (13) holds, where $\langle .,. \rangle$ means the duality $W^*(\Omega_0)$ and $W(\Omega_0)$.

Proposition 2. If Ω is a domain with Lipschitz boundary then the imbedding of the space $W_2^2(\Omega)$ to $C(\bar{\Omega})$ is continuous.

Proof (see [7]).

Lemma 1. Dirac distribution δ_{ξ_0} belongs to $W^*(\Omega_0)$.

Proof. It is obvious that $\delta_{\xi_0} \in C^*(\bar{\Omega}_0)$. Proposition 2 implies that the imbedding of the space $C^*(\bar{\Omega}_0)$ to $W^*(\Omega_0)$ is continuous. From this, our proposition follows.

Lemma 2. If we define a function

$$f_n(\xi) = \frac{n^3}{\pi^{3/2}} \exp\left\{- n^2 \| \xi - \xi_0 \|^2\right\}$$

for every $n \in N$ and $\xi \in \mathbb{R}^3$, then the sequence $\{f_n\}$ converges to the Dirac distribution δ_{ξ_0} in the space $W^*(\Omega_0)$.

Proof. The statement follows both from the continuity of the imbedding of the space $C^*(\bar{\Omega}_0)$ into $W^*(\Omega_0)$ and the convergence of $\{f_n\}$ to δ_{ξ_0} in $C^*(\bar{\Omega}_0)$.

Theorem 2 (Existence and unicity of a "very weak" solution). Let the conditions (i),(ii) from Theorem 1 be satisfied. Then the Dirichlet problem (2)-(3) has one and only one "very weak" solution $c \in L_2(\Omega)$.

Proof. Let $\{f_n\}$ be a sequence of functions introduced in Lemma 1. According to Theorem 1, for every $n \in N$ there exists a weak solution of the problem

$$Lc_n = f_n \tag{14}$$

$$c_n \big|_{\partial\Omega} = 0. \tag{15}$$

Since c_n is a weak solution, it is also a very weak solution of the problem (14)-(15). Then

$$(c_n, f^*) = \langle Qf_n, G^* f^* \rangle \tag{16}$$

for every n and $f^* \in L_2(\Omega)$. Moreover, we have

$$|\langle Qf_n, G^* f^* \rangle| \leq Q \, \|f_n\|_{W^*(\Omega_0)} \|G^* f^*\|_{W(\Omega_0)} \tag{17}$$

$$= Q \, \|f_n\|_{W^*(\Omega_0)} (\|G^* f^*\|_{W_2^2(\Omega_0)} + \|G^* f^*\|_{L_2(\Omega_0)}).$$

According to Theorem 1* and the coerciveness we have

$$\|G^* f^*\|_{W_2^2(\Omega_0)} \leq M(\|f^*\|_{L_2(\Omega)} + \|G^* f^*\|_{L_2(\Omega)}) \tag{18}$$

$$\leq M_1 \, \|f^*\|_{L_2(\Omega)}.$$

From (16)-(18) we get

$$\|c_n\|_{L_2(\Omega)} \leq M_1 \, \|f_n\|_{W^*(\Omega_0)} \, .$$

The last inequality implies convergence of the sequence $\{c_n\}$ to some c in $L_2(\Omega)$.

It is easy to show that c is a "very weak" solution of our problem (2)-(3).

Unicity follows from the definition of the "very weak" solution.

III. Regularity of the "very weak" solution

Proposition 3. Let $\{\varphi_n^*\}$ be a sequence of distributions such that $\varphi_n^* \longrightarrow \varphi^*$ in $\mathcal{D}'(\Omega)$ and

$$\| D^{\beta} \varphi^{*}_{n} \|_{L_2(\Omega)} \leq K$$

for every $n \in N$. Then $D^{\beta} \varphi^{*} \in L_2(\Omega)$ and

$$\| D^{\beta} \varphi^{*} \|_{L_2(\Omega)} \leq K.$$

Proof (see [7]).

Theorem 3 (Regularity of the "very weak" solution up to the boundary). Let the assumptions (i),(ii) of Theorem 1 be satisfied. Then the very weak solution c belongs to $W_2^2(\Omega_1)$ for every subdomain $\Omega_1 \subset \Omega - \overline{\Omega}_o$, $\overline{\Omega}_1 \cap \overline{\Omega}_o = \emptyset$.

Proof. Let c be a "very weak" solution of the problem (2)-(3). Choose the subdomain Ω'_1 in such a way that $\Omega'_1 \subset \subset \Omega - \overline{\Omega}_o$, $\overline{\Omega}_1 \subset \Omega'_1 \cup \partial\Omega$ and $\overline{\Omega}'_1 \cap \overline{\Omega}_o = \emptyset$.

Let c_n be a weak solution of the problem (14)-(15). According to Proposition 1, there exists a constant $M = M(\Omega_1, \Omega'_1)$ such that

$$\| c_n \|_{W_2^2(\Omega_1)} \leq M(\Omega_1, \Omega'_1)(\| f_n \|_{L_2(\Omega'_1)} + \| c_n \|_{L_2(\Omega)})$$

holds.

This estimate together with the choice of the sequence $\{f_n\}$ and the convergence of the $\{c_n\}$ in $L_2(\Omega)$ yields our proposition.

Proposition 4. Let the assumption (i) from Proposition 1 be satisfied. Let further, the functions a_{ij} have Lipschitz derivatives of the k-th order and functions b_i, a have Lipschitz derivatives of the order $(k-1)$ in $\overline{\Omega}$ (for $k \geq 1$).

Then every weak solution u of the equation (7) with the right-hand side $f \in W_2^{k-1}(\Omega)$ belongs to $W_2^{k+1}(\Omega_1)$ for any $\Omega_1 \subset \subset \overline{\Omega}_1 \subset \Omega$. There exists a constant $M = M(\Omega_1)$ such that

$$\|u\|_{W_2^{k+1}(\Omega_1)} \leq M(\Omega_1)(\|u\|_{W_2^1(\Omega)} + \|f\|_{W_2^{k-1}(\Omega)})$$

holds.

Proof (see [7]).

Theorem 4 (Interior regularity of the "very weak" solution). Let the assumption (i) from Theorem 1 be satisfied and let the coefficients k_x, k_y, k_z have Lipschitz derivatives of the k-th order and the functions v_x, v_y, v_z have Lipschitz derivatives of order (k-1) in $\overline{\Omega}$ (for $k \geq 1$).

Then the "very weak" solution c belongs to $W_2^{k+1}(\Omega_2)$ for every subdomain $\Omega_2 \subset \overline{\Omega}_2 \subset \Omega - \overline{\Omega}_o$ such that $\overline{\Omega}_2 \cap \overline{\Omega}_o = \emptyset$.

Proof. Let Ω'_2 be a subdomain of Ω such that $\overline{\Omega}_2 \subset \Omega'_2 \subset \overline{\Omega}'_2 \subset \Omega - \overline{\Omega}_o$ and let c_n be a weak solution of the problem (14)-(15). According to Proposition 4 there exists a constant $M = M(\Omega_2, \Omega'_2)$ such that the inequality

$$(17) \qquad \|c_n\|_{W_2^{k+1}(\Omega_2)} \leq M(\Omega_2, \Omega'_2)(\|f_n\|_{W_2^{k-1}(\Omega'_2)} + \|c_n\|_{W_2^1(\Omega'_2)})$$

holds for every n. Moreover, let $\zeta = \zeta(\xi)$ be an infinitely differentiable function such that

$$\zeta(\xi) = \begin{cases} 1 \text{ for } \xi \in \Omega'_2 \\ 0 \text{ for } \xi \in \Omega''_2 \end{cases}$$

where $\overline{\Omega}'_2 \subset \Omega''_2 \subset \overline{\Omega}''_2 \subset \Omega - \overline{\Omega}_o$.

From the coerciveness we obtain

$$(18) \qquad \|\zeta c_n\|_{W_2^1(\Omega)} \leq ((\zeta c_n, \zeta c_n))$$
$$\leq |((\zeta c_n, \zeta c_n)) - ((c_n, \zeta^2 c_n))| + |((c_n, \zeta^2 c_n))| .$$

The definition of the weak solution gives

$$(19)\qquad |((c_n, \varsigma^2 c_n))| = |(f_n, \varsigma^2 c_n)| = |(\varsigma f_n, \varsigma c_n)|$$

$$\leq \| f_n \|_{L_2(\Omega''_2)} \| c_n \|_{L_2(\Omega''_2)} .$$

Performing an easy computation, we obtain

$$(20)\qquad |((\varsigma c_n, \varsigma c_n)) - ((c_n, \varsigma^2 c_n))| \leq M_1 \| c_n \|^2_{L_2(\Omega)} .$$

From the choice of $\{f_n\}$ and the convergence of the $\{c_n\}$ in $L_2(\Omega)$ and by means of (17)-(19) we get

$$\| c_n \|_{W_2^{k+1}(\Omega_2)} \leq M_2$$

for some constant M_2 independent of n.

From this and from Proposition 4 we get the assertion.

<u>Corollary</u>. If the coefficients k_x, k_y, k_z and v_x, v_y, v_z are infinitely differentiable, then the "very weak" solution c is infinitely differentiable with the exception of the points where the sources are replaced.

<u>Proof</u> follows from Theorem 4.

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