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A GENERAL CONTINUITY PRINCIPLE

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Abstract: In this note we sharpen and generalize a theorem of I. Netuka concerning the continuity of signed potentials. The theorem is then applied to Riesz potentials.

Key words: axiomatic potential theory, convex cones, carrier.

Classification: Primary 31D05, 31B15
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Introduction. Answering a question posed by B. W. Schulze, I. Netuka [9] proved the following generalization of the classical Evans-Vasilesco theorem:

If μ is a signed measure on $\mathbb{R}^n$ with compact support K such that

$$p(x) = \int \frac{1}{|x-y|}^{n-2} \mu(dy)$$

is finite for any $x \in \mathbb{R}^n$, then p is already continuous on X provided $p_{|K}$ is continuous.

In fact, he even proved a generalization of this result to harmonic spaces which satisfy the strong domination axiom. Besides some results of the general theory of harmonic spaces his proof is based on Fuglede's fine minimum principle. In this note we give a generalization of Netuka's theorem to rather arbitrary convex cones of continuous functions on locally compact spaces. Our approach is based on an application of the Hahn-Banach theorem which occurred first in [1]. Surprisingly our theorem is not only more general but also sharper than Netuka's theorem in that p is no more assumed to be a signed potential.

In the sequel let P be a convex cone of non-negative continuous functions on a locally compact space X with a countable base. The only condition which we impose on P is that it contains a strictly positive element. We denote by $S := S_P$ the convex cone of all functions which are pointwise limits of increasing sequences in P.

K will always be compact subset of X and P_K a convex cone of functions $p \in P$ such that

$$s \in S,\ p \leq s \text{ on } K \Rightarrow p \leq s \text{ on } X.$$

We denote by P_K^* the linear space of all functions f on X for which there exist two sequences (p_n), (q_n) in P_K and a finite function $s \in S$ such that

(a) $|p_n - q_n| \leq s \qquad (n \in \mathbb{N})$.

(b) $\lim_{n\to\infty}(p_n - q_n)(x) = f(x) \qquad (n \in \mathbb{N})$.

Note that under our rather general assumptions a function $f \in P_K^*$ need even not be continuous on $X \setminus K$.

<u>1. Theorem.</u> A function $f \in P_K^*$ is already continuous throughout X if only its restriction to K is continuous.

For the proof we need the following

<u>2. Lemma.</u> For any $x \in X$ the set M_x of all measures μ on K such that

(i) $\int p\,d\mu \leq p(x) \qquad (p \in P)$,

(ii) $\int p\,d\mu = p(x) \qquad (p \in P_K)$,

is non-empty. Moreover, for any $\mu \in M_x$, we have

(iii) $\int s\,d\mu \leq s(x) \qquad (s \in S)$,

(iv) $\int f\,d\mu = f(x) \qquad (f \in P_K^*)$.

Proof. Let $\varphi : C(K) \to \mathbb{R}$ be the sublinear functional defined by

$$\varphi(f) := \inf\{(p-q)(x) : p \in P,\ q \in P_K,\ f \leq p - q \text{ on } K\}.$$

Note that the infimum is finite by the carrier property of P_K. By the Hahn-Banach theorem, there exists a linear functional l on $C(K)$ with $l \leq \varphi$. If $f \leq o$, then $l(f) \leq \varphi(f) \leq o$. Hence there exists a measure μ on K such that

$$l(f) = \int f d\mu \qquad\qquad (f \in C(K)).$$

To see (i), let $p \in P$ and observe that

$$\int p d\mu \leq \varphi(p_{|K}) \leq p(x).$$

For $p \in P_K$ we have also

$$\int -p d\mu \leq \varphi(-p_{|K}) \leq -p(x)$$

(iii) is an immediate consequence of (i).
To see (iv), let $f \in P_K^*$ and choose $(p_n),(q_n),s$ as in (a),(b). Using $s(x) < \infty$ and (a),(b) we may apply Lebesgue's dominated convergence theorem and (ii) to get

$$\int f d\mu = \lim_{n\to\infty} \int (p_n - q_n) d\mu = \lim_{n\to\infty} (p_n - q_n)(x) = f(x).$$

<u>Proof of the Theorem.</u> Let $f \in P_K^*$ such that $f_{|K}$ is continuous, $x \in X$ and (x_n) a sequence in X converging to x. We have to show

$$\lim_{n\to\infty} f(x_n) = f(x).$$

To accomplish this, we choose, for any $n \in \mathbb{N}$ a measure $\mu_n \in M_{x_n}$. If p_o is a strictly positive element in P with $p_o \geq 1$ on K, then we see from

$$\limsup_{n\to\infty} \mu_n(K) \leq \limsup_{n\to\infty} \int p_0 \, d\mu_n \leq \limsup_{n\to\infty} p_0(x_n) = p_0(x)$$

that (μ_n) is a bounded sequence of measures. Taking a subsequence, we may assume that (μ_n) converges vaguely to a measure μ on K. Since $p_{|K}, q_{|K} \in C(K)$ for any $p \in P$, $q \in P_K$ we have

$$\int p d\mu = \lim_{n\to\infty} \int p d\mu_n \leq \lim_{n\to\infty} p(x_n) = p(x)$$

$$\int q d\mu = \lim_{n\to\infty} \int q d\mu_n = \lim_{n\to\infty} q(x\ ) = q(x)$$

and therefore $\mu \in M_x$. Using 2 (iv) and $f_{|K} \in C(K)$ we finally get

$$f(x) = \int f d\mu = \lim_{n\to\infty} \int f d\mu_n = \lim_{n\to\infty} f(x_n)$$

From now on let (X,S) be a standard balayage space (cf. [3]). Denoting P the convex cone of continuous potentials we obviously have $S_p = S$. Thus the preceding notation coincides with the present notation. We denote further by P^m the convex cone of all potentials which are a countable sum of continuous potentials. In the notation of [6] this is exactly the band M. P_K will be the set of all continuous potentials with carrier in K. Obviously the difference of two finite potentials in P^m with carrier in K is contained in P_K^*. The converse, however, is only true in fairly trivial cases.

<u>3. Corollary.</u> Let $p,q \in P^m$ be finite such that $f := p - q$ is harmonic outside K. If $f_{|K}$ is continuous, then f is continuous on X.

Proof. In view of the preceding comments we have only to show that there exist $p', q' \in P^m$ with carrier in K and $f = p' - q' = p - q$. This problem was solved in [9] under rather restrictive assumptions In the general case we have to follow a device of B. Fuglede which is outlined at the end of [9]. We define inductively $p_0 = p, q_0 = q$

$$p_n := p_{n-1} - R(p_{n-1} - q_{n-1}),\ q_n = q_{n-1} - R(q_{n-1} - p_{n-1}).$$

Since P^m is a "cone of potentials" p_n, q_n belong again P^m and $p_n \leq q_{n-1}$, $q_n \leq p_{n-1}$. Moreover, $p_{n-1} - p_n$, $q_{n-1} - q_n$ are harmonic on $X \setminus K$. Hence (p_n) and (q_n) decrease to some limit u and $p' := p - u$, $q' := q - u$ are harmonic on $X \setminus K$. Since the sequences (p_n) and (q_n) are even specifically decreasing, the functions u, p',q' belong to P^m. Since $p' - q' = p - q = f$ and $S(p'), S(q') \subset K$ the assertion follows.

For more details of the above construction see [5], Proposition 1.2.

4. Corollary. Let $0 < \alpha \leq 2$, $n \geq 3$ and μ be a signed measure on $\mathbb{R}^n$ carried by a compact set K such that

$$f(x) = \int \frac{1}{|x-y|^{n-\alpha}} \mu(dy)$$

is finite for any $x \in \mathbb{R}^n$ and $f_{|K}$ is continuous. Then f is continuous throughout $\mathbb{R}^n$.

Proof. Since the kernel V^α defined by

$$V^\alpha f(x) = \int \frac{1}{|x-y|^{n-\alpha}} f(y)dy$$

is the potential kernel of a nice convolution semi-group (cf. [2], p. 136) the potentials

$$\int \frac{1}{|x-y|^{n-\alpha}} \nu(dy),$$

where ν runs through all positive measures, generate a standard balayage space (cf. [3]). Since the function kernel $\frac{1}{|x-y|^{n-\alpha}}$ satisfies the continuity principle (cf. [7], p. 189) every finite potential (of a measure) belongs to P^m. In particular, we have $f \in P^*_K$. The assertion follows now from the theorem.

Remarks. (a) It would be interesting to know whether there exists a direct proof of the last corollary which depends only on certain estimates of the kernel in question.

(b) For some illustrative examples the reader is refered to [9]. In particular, there is an example of a continuous Newtonian signed Potential with compact carrier which is not the difference of two continuous potentials.

(c) After having finished this paper A. Cornea made some important remarks: (1) The theory still works if X is an arbitrary topological space. (2) An element $f \in P_K^*$ is already lower semi-continuous if its restriction to K is lower semi-continuous. (3) If P is a convex cone of continuous functions on a compact space X, then a P-affine function (cf. [4], p. 500) is continuous on X if its restriction to a Shilov set (cf. [10]) is continuous.

All these assertions may be proved with the above method, one has only to replace sequences by filters.

(d) After having submitted a first version of this paper the author received the Preprint [8] of J. Král. In this paper Král extends his old (unpublished) method from 1975 to a new proof of Theorem 1. It should be remarked that his method is much simpler in special situations, Corollary 3 for example. On the other hand it seems not possible to get the second generalization indicated in (c) in this way.

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