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EQUIVALENCE OF K-IRREDUCIBILITY CONCEPTS

Ivo MAREK and Karel ŽITNÝ

Dedicated to Prof. Dr. F.L. BAUER Dr.h.c. on the occasion of the 60th anniversary of his birth.

Abstract: The equivalence of various concepts of irreducibility of positive operators in partially ordered Banach spaces introduced by G. Frobenius (Fr), H. Geiringer (Ge), König (Ko), H.H. Schaefer (Sc), I. Sawashima (Sa), J.S. Vandergraft (VS), V.Ya. Stecenko (St) and I. Marek and K. Žitný (MZ) is analyzed. All the concepts considered are equivalent if the dimension of the spaces under consideration is at least two. In one-dimensional spaces these concepts split into two classes - the criterion being a classification of the zero map as reducible ((MZ),(Sa),(Sc)) or irreducible ((Fr),(Ge),(Ko),(St),(VS)), respectively.

Key words: Normal generating cone, positive operator, irreducibility.

Classification: Primary 47A99
Secondary 15A48, 46A40

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1. Introduction. As well known, the concept of irreducibility of a matrix has been originated by G. Frobenius in the fundamental paper [2]. The role of irreducibility and its relationship to the concept of full indecomposability of a matrix are elucidated in the paper of H. Schneider [10], where the approaches of G. Frobenius, D. König and A.A. Markov to the theory of matrices with nonnegative real entries are compared. Schneider also gives a deep analysis of the concepts mentioned above and presents new proofs of some irreducibility and full indecomposability results. His main tool is the (elementary) graph theory leading to final definitive results in a very natural way.

The situation is rather different if one considers irreducibility concepts of cone preserving maps that in general have no direct relations to the "standard" order in the appropriate spaces, in particular, such maps cannot be represented by matrices with nonnegative real entries.

The concept of irreducibility of a matrix with nonnegative reals has been generalized in many directions by many authors. This is not the case of the concept of full indecomposability, however. The reason for this may be connected with the fact that the concept of full indecomposability of a matrix is equivalent to a property which has essentially a finite dimentsional character, whilst the generalized irreducibility concepts are dimension independent.

In this paper we are going to study several concepts of irreducibility. Our goal is that we show that all these concepts are equivalent if the dimension of the space under consideration is at least two. In the one-dimensional case these concepts split into two groups. The first group contains those concepts which admit the zero map to be irreducible, the second group conversely does treat the zero map as reducible.

2. Definitions and notation. Let Y be a real Banach space generated by a closed normal cone K [5], i.e. let (i)-(vi) hold, where (i) $K + K \subset K$, (ii) $aK \subset K$ for $a \in R^1_+ = \{b \in R^1 : b \geqq 0\}$, (iii) $K \cap (-K) = \{0\}$, (iv) $\overline{K} = K$ (here $\overline{K}$ denotes the norm closure of K), (v) $Y = K - K$, (vi) there is a $b \in R^1_+$, $b \neq 0$, such that $\|x + y\| \geqq b\|x\|$ whenever $x, y \in K$.

Let Y' be the dual space of Y. We denote by K' the dual cone of K defined as $K' = \{y' \in Y' : \langle x, y'\rangle \geqq 0$ for all $x \in K\}$. (We write $\langle x, y'\rangle$ in place of $y'(x)$.) We assume that K has a nonempty dual interior K^d; $K^d = \{x \in K: \langle x,x'\rangle \neq 0$ for all $x' \in K'$, $x' \neq 0\}$. A linear form $x' \in K'$ is called strictly positive, if $\langle x,x'\rangle \neq 0$ whenever $x \in K$, $x \neq 0$.

Let $B(Y)$ denote the space of bounded linear operators on Y. We call $T \in B(Y)$ K-positive, or shortly positive, if $TK \subset K$.

A subcone $F \subset K$ is called face of K [12], if $x \in F$ implies that $y \in F$ whenever $x - y \in K$. We denote by F_x the set defined as $F_x = \{y \in K: ax - y \in K$ for some $a \in R^1_+\}$. Obviously F_x is a face. An element $e \in K$ is called order unit of K, if for every $x \in K$, $x \neq 0$, there is a positive number $a = a(x)$, such that $a(x)e - x \in K$, i.e. $F_e = K$.

Let $T \in B(Y)$, then there exists the limit $\lim\limits_{k \to \infty} \|T^k\|^{\frac{1}{k}} = r(T)$ and it is called spectral radius of T.

To a given operator (matrix) $T \in B(Y)$, $T = (t_{jk})$, $j,k = 1,2,\dots$, we associate an oriented graph $G = (V,H)$ (graph of the matrix T) as follows: Every index $j \in \mathcal{N} = (1,2,\dots)$ is a vertex, i.e. an element of V and any couple (j,k) forms an edge, i.e. an element of H if and only if $t_{jk} \neq 0$.

As usual, a sequence of edges $\{(j,k_1),(k_1,j_2),\dots,(k_p,j_p)\}$, $p = 1,2,\dots$ is called a path from (j,k_1) to (k_p,j_p). A graph G is called strongly connected if for every two vertices $a,b \in V$ there is a path $h \in H$ connecting a and b.

3. K-irreducibility. A K-positive operator $T \in B(Y)$ is called K-irreducible, or more precisely (xx)-K-irreducible, where the bracket contains the symbol of the corresponding concept, if T has the following property:

(Sa) (I. Sawashima [8]). For every couple $x \in K$, $x \neq 0$, $x' \in K'$, $x' \neq 0$, there is a positive integer $p = p(x,x')$ such that $\langle T^p x,x'\rangle \neq 0$.

(Sc) (H.H. Schaefer [9]). For every $x \in K$, $x \neq 0$ and each $\lambda \in R^1$, $\lambda > r(T)$, the vector $y = T(\lambda I-T)^{-1}x$ belongs to K^d.

Let $\varphi(\lambda) = \sum_{k=1}^{\infty} a_k \lambda^k$ be a power series such that $a_k \in R^1$, $a_k > 0$ for $k \geq 1$, and whose radius of convergence $R(\varphi) > r(T)$.

(MZ) For every $x \in K$, $x \neq 0$, the vector $y = \varphi(T)x \in K^d$.

(St) (V.Ya. Stecenko [11]). Let $a \in R^1$, $a > r(T)$, $u \in K$, $u \neq 0$. The relation $au - Tu \in K$ implies that $u \in K^d$.

(VS) For every $x \in K$, $x \neq 0$, the relation $Tx \in F_x$ implies that $F_x \cap K^d \neq \emptyset$.

It should be noticed that the definition (VS) is a modified version of original definition given by J.S. Vandergraft [12]. The reason for this modification is a dimensionality aspect. If the cone K contains an order unit, then (VS) is equivalent to the original Vandergraft's definition [12]:

(JV) For every $x \in K$, $x \neq 0$, the relation $Tx \in F_x$ implies that $F_x = K$.

In particular, (VS) is equivalent to (JV) if $\dim X < +\infty$.

4. Equivalence of the concepts (Sa),(Sc),(MZ).

(Sa) $\Longleftrightarrow$ (Sc)

For $x \in K$, $x \neq 0$, $x' \in K'$, $x' \neq 0$, $a > r(T)$ we have that
$$\langle T(aI - T)^{-1}x, x' \rangle = a^{-1} \sum_{k=1}^{\infty} a^{-k} \langle T^k x, x' \rangle$$
and the equivalence of (Sa) and (Sc) easily follows.

More generally,

(Sa) $\Longleftrightarrow$ (MZ),

because $\langle \varphi(T)x, x' \rangle = \sum_{k=1}^{\infty} a_k \langle T^k x, x' \rangle$.

In particular, if $\varphi(a) = a(1 - a)^{-1}$, $|a| < 1$, we get (Sc) as a special case of (MZ).

We also see that the zero operator T = 0 cannot be K-irreducible for any of the concepts (Sa),(Sc) and (MZ).

5. <u>The equivalence of (St) and (VS).</u> First, let T be (St)-K-irreducible. Let $0 \neq x \in K$ be such that $Tx \in F_x$. We deduce that for some $a \in R^1$, $a > 0$, $ax - Tx \in K$. By (St) we conclude that $x \in K^d$, and thus (VS) holds.

Conversely, let T be (VS)-K-irreducible. If for some $a \in R^1$, $a > 0$, $ax - Tx \in K$, $x \neq 0$, then by (VS) there is a $y \in F_x \cap K^d$. It follows that $x \in K^d$ and hence (St) holds.

It is easy to see that the zero operator in Y, $\dim Y = 1$, is (St)-K-irreducible and also (VS)-K-irreducible as well. We return to this question again in connection with the irreducibility concepts in the sense of Frobenius and Geiringer.

6. <u>Equivalence of the concepts of irreducibility for</u> Y <u>with</u> $\dim Y \geqq$. In this section we show that all the five K-irreducibility concepts shown in Section 3 are equivalent if $\dim Y \geqq 2$. It is enough to show that

(Sa)$\Longleftrightarrow$ (St).

First, let T be (Sa)-K-irreducible and let $au - Tu \in K$, $u \in K$, $u \neq 0$, where $a \in R'$, $a > r(T)$. Let $x' \in K'$, $x' \neq 0$. Then

$$\langle u,x' \rangle \geqq a^{-1} \langle Tu,x' \rangle \geqq \dots \geqq a^{-k} \langle T^k x,x' \rangle .$$

By (Sa), there is a $p = p(u,x')$ such that $\langle T^p u,x' \rangle \neq 0$. Since $x' \in K'$ is arbitrary, we conclude that $u \in K^d$. Thus, T is (St)-K-irreducible.

Conversely, let T be (St)-K-irreducible. Evidently, $T \neq 0$ (here the hypothesis $\dim Y \geqq 2$ is needed). If $Tx = 0$ for all $x \in K$, then we can take $y \in K$, $y \notin K^d$, such that $y - Ty = y \in K$, a contradiction.

Let us assume that T is not (Sa)-K-irreducible and, under this assumption, let us distinguish two cases:

a) there is an $x_0' \in K'$, $x_0' \neq 0$, such that $\langle Tx, x_0' \rangle = 0$ for all $x \in K$.

b) For every $x' \in K'$, $x' \neq 0$, there is an $x \in K$ such that $\langle Tx, x' \rangle \neq 0$.

In case a) we choose $x_0 \in K$ such that $Tx_0 \neq 0$, in case b) let $x_0 \in K$, $x_0 \neq 0$ and such that $\langle T^p x_0, x_0' \rangle = 0$ for all $p = 1,2,\ldots$

Let

$$u = \sum_{k=1}^{\infty} \frac{1}{(1 + \|T\|)^k} T^k x_0,$$

then $u \in K$ and

$$Tu = \sum_{k=1}^{\infty} \frac{1}{(1 + \|T\|)^k} T^{k+1} x .$$

It follows that

$$(1 + \|T\|)u - Tu = Tx_0$$

and $u \neq 0$ whilst

$$\langle u, x_0' \rangle = \sum_{k=1}^{\infty} \frac{1}{(1 + \|T\|)^k} \langle T^k x_0, x_0' \rangle = 0,$$

a contradiction to the fact that $u \in K^d$. This completes the proof.

Summarizing, we state

<u>Theorem 1</u>. The concept of K-irreducibility (Sa),(Sc),(MZ), (St) and (VS) are all equivalent in spaces Y with dim $Y \geqq 2$.

Moreover, the concepts of groups (I) and (II) are equivalent respectively also, if dim Y = 1 but each concept of (I) is not equivalent to any of the concepts of (II) for the case dim Y = 1, where (I) denotes the collection of (Sa),(Sc) and (MZ), whilst (II) contains the concepts (St) and (VS), respectively.

7. Irreducibility in the spaces of sequences. In the previous sections we considered arbitrary Banach spaces generated by quite general cones. In such situation there is no hope of being able to relate the concepts of irreducibility given by G. Frobenius, H. Geiringer and D. König to the generalized irreducibility concepts. To be able to do so and without restricting ourselves to the finite dimensional situation, we consider the following type of Banach spaces generated by a natural generalization of the cone $R_+^n = \{x \in R^n : x = (\xi_1, \dots, \xi_n),\ \xi_j \geqq 0, j = 1, \dots, n\}$.

Let Y be any Banach space of sequences of real numbers having the following properties:

(a) The finitely generated vectors are dense in Y, i.e. for every $x \in Y$, $x = (\xi_1, \xi_2, \dots)$ we have that $\lim_{k \to \infty} \|x - x_k\| = 0$, where $x_k = (\xi_1, \dots, \xi_k, 0, \dots)$, $\xi_k \in R^1$; $x_k \in Y$;

(b) $Y = K - K$, where $K = \{x \in Y : x = (\xi_1, \xi_2, \dots),\ \xi_k \leqq 0, k = 1, 2, \dots\}$;

(c) for every $x \in Y$, $x = (\xi_1, \xi_2, \dots)$ the vector $|x| = (|\xi|_1, |\xi|_2, \dots)$ belongs to Y.

For the sake of simplicity we are going to consider operators T represented in a fixed (say standard) basis by infinite matrices $T = (t_{jk})$, $j,k = 1,2,\dots$.

A linear operator $P \in B(Y)$ is called permutation operator, if $P = (p_{jk})$, where $p_{jk} \in \{0,1\}$, $\sum_{k=1}^{\infty} p_{jk} = \sum_{k=1}^{\infty} p_{kj} = 1$ and $P^{-1} \in B(Y)$.

We now present an infinite-dimensional analogue of the irreducibility concepts of G. Frobenius [2] and H. Geiringer [3].

An operator $T \in B(Y)$ is called irreducible, or more precisely (xx)-irreducible, where (xx)- denotes the symbol of the corresponding concept, if the following holds, respectively:

(Fr) There is no permutation operator P such that the operator $T = (t_{jk})$, $j,k = 1,2,\dots$ has the form

$$T = PT_F P,$$

where

$$T_F = \begin{pmatrix} T_1 & T_3 \\ \Theta & T_2 \end{pmatrix}$$

with $T_1 \in B(Y_1)$, $T_2 \in B(Y_2)$, $Y_1 \subset Y$, $Y_2 \subset Y$, $\min(\dim Y_1, \dim Y_2) \geqq 1$ and $P^* = (p_{jk})$, $p_{jk} = p_{kj}$, $j,k = 1,2,\dots$.

(Ge) There is no decomposition of the set $\mathcal{N} = \{1,2,\dots\}$ into two parts $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ and $t_{jk} = 0$ for $j \in \mathcal{N}_2$ and $k \in \mathcal{N}_1$ and where $T = (t$ $j,k = 1,2,\dots$.

(Ko) The graph of the operator $T \in B(Y)$ is strongly connected.

<u>Remark.</u> In general T_F does not belong to $B(Y)$.

8. <u>Equivalence of the irreducibility concepts (Fr) and (Ge).</u> Let us assume first that there are nonempty sets $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$, $\mathcal{N}_1 \cap \mathcal{N}_2 = \emptyset$ and $t_{jk} = 0$ for $j \in \mathcal{N}_2$ and $k \in \mathcal{N}_1$.

We let $p_{j\ell_j} = 1$ for $j = 1,\dots$ and $\ell_j \in \mathcal{N}_1 = \{\ell_1,\dots$ $\dots,\ell_N,\dots\}$; further $p_{jk} = 0$ for $k \in \mathcal{N}_1$, $k \neq \ell_j$ and for $k \in \mathcal{N}_2$. Similarly, $p_{j\ell_j} = 1$ for $j \in \mathcal{N}$ and $\ell_j \in \mathcal{N}_2$, $p_{jk} = 0$ for $k \in \mathcal{N}_2$, $k \neq \ell_j$ and $k \in \mathcal{N}_1$. Then for j N_2 we have

$$t^{(F)}_{jk} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} p_{jr} t_{rs} p_{ks} = \sum_{s=1}^{\infty} t_{\ell_j s} p_{ks},$$

where $\ell_j \in \mathcal{N}_1$ and for $k \in \mathcal{N}_2$

$$t^{(F)}_{jk} = t_{\ell_j \ell_k}, \quad \ell_j \in \mathcal{N}_2, \quad \ell_k \in \mathcal{N}_1.$$

According to our hypothesis, $t_{\ell_{j,k}} = 0$. In other words,

$$T_F = \left(t_{jk}^{(F)}\right) = \begin{pmatrix} T_1 & T_3 \\ \Theta & T_2 \end{pmatrix}$$

and we see that T does not fulfil the condition (Fr). Thus (Fr)-irreducibility implies the (Ge)-irreducibility.

Conversely, let T not satisfy condition (Fr), i.e. let

$$T_F = PTP^* = \begin{pmatrix} T_1 & T_3 \\ \Theta & T_2 \end{pmatrix} = \left(t_{jk}^{(F)}\right)$$

where $T_1 \in B(Y_1)$ and $T_2 \in B(Y_2)$, $Y_1 \subset Y$, $Y_2 \subset Y$ and $P = (p_{jk})$ is a permutation operator. We let $\mathcal{N}_1 = \{\ell_1, \dots, \ell_N \dots\}$, where ℓ_j is such that $p_{j\ell_j} = 1$, $j \in \mathcal{N}_1$ and $\mathcal{N}_2 = \mathcal{N} \setminus \mathcal{N}_1$. Then for $\ell_q \in \mathcal{N}_2$ and $\ell_p \in \mathcal{N}_1$ we have that

$$t_{\ell_q \ell_p} = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} p_{r\ell_q} t_{rs}^{(F)} p_{\ell_s p} = \sum_{r=1}^{\infty} p_{r\ell_q} t_{r\ell_p}^{(F)} =$$
$$= t_{qp}^{(F)} = 0 .$$

Thus, (Ge)-irreducibility implies the (Fr)-irreducibility. The proof is complete.

9. <u>The equivalence of (Fr) and (St)</u>. In this section, when discussing the irreducibility concepts (St),(Sa) etc., we always assume that K is specified as in Section 7. In this case $K^d =$ $= \{x \in K : x = (\xi_1, \xi_2, \dots)^T : \xi_k > 0,\ k = 1,2,\dots\}$.

Let us assume that there is a permutation operator P such that

$$P\,T\,P^* = T_F = \begin{pmatrix} T_1 & T_3 \\ \Theta & T_2 \end{pmatrix}$$

with $T_1 \in B(Y)$, $T_2 \in B(Y_2)$, $Y_1 \subset Y$, $Y_2 \subset Y$. We let $x = (\xi_1, \dots, \xi_N, 0 \dots)$, $x' = (0, \dots, 0, \xi'_{N+1}, \dots)$ with $\xi_j > 0$,

$j = 1,\dots,N$, $\xi'_{N+1} > 0$, and $\xi'_k \geqq 0$, $k > N$. Then $\langle T_F x, x' \rangle = 0$. Let $u = F^* x$. Then for some $a \in R'$, $a > r(T)$, $au - Tu \in K$ and $u \notin K^d$. We conclude that non (Fr) implies non (St), that is (St)-irreducibility implies the (Fr)-irreducibility.

Let T be (Fr)-irreducible. We let $x = (\xi_1, \dots, \xi_n, 0, \dots)$ with $\xi_j > 0$ for $j \leqq n$ and define

$$x_{k+1} = (I + T)x_k, \; x_0 = (\xi_1, \dots, \xi_n, 0, \dots)^T.$$

Furthermore, let

$$T = \begin{pmatrix} T_1 & T_3 \\ T_4 & T_2 \end{pmatrix}$$

where $T_1 \in B(Y_1)$, $T_2 \in B(Y_2)$, $Y_1 \subset Y$, $Y_2 \subset Y$.

We see that the (n+1)-st component of x_1 is positive, otherwise from

$$x_1 = \begin{pmatrix} T_1 \bar{x} \\ T_4 \bar{x} \end{pmatrix} + x_0$$

it would follow that $T_4 = \Theta$ and that would contradict the hypothesis. Hence, generally,

$$x_{k+1} = (\xi_1^{(k+1)}, \dots, \xi_n^{(k+1)}, \xi_{n+1}^{(k+1)}, \dots, \xi_{n+k+1}^{(k+1)} \; \eta_{n+k+2}, \dots)^T$$

with $\xi_j^{(k+1)} > 0, j \leqq n+k+1$ and $\eta_\ell \geqq 0 \; \ell > n+k+1$. It follows that for every vector $x \in K$, $x \neq 0$, there is a power $p_j = p_j(x)$ such that $(T^{p_j} x)_j > 0$. Thus,

$$\sum_{k=0}^{\infty} \frac{1}{(r(T) + 1)^k} T^k x \in K^d.$$

Therefore, every $0 \neq y \in K$, for which $ay - Ty = x \in K$, $a > r(T)$, $x \neq 0$, must be in K^d and followingly, T is (St)-irreducible. The proof is complete.

10. <u>The equivalence of (Ko) and (Ge)</u>. Let $T \in B(Y)$ be not (Ko)-irreducible. We let the vertices into two disjoint classes

as follows: j and k belong to the same class $\mathcal{N}_1$ if and only if there is a path in the operator graph G connecting j and k and $\mathcal{N}_2 = \mathcal{N} \setminus \mathcal{N}_1$, where $\mathcal{N} = \{1,2,\dots\}$. This means that any $k \in \mathcal{N}_2$ cannot be connected with any $j \in \mathcal{N}_1$.

We see that T cannot be (Ge)-irreducible. Thus non (Ko) implies non (Ge), that is (Ge) implies (Ko).

If T is not (Ge)-irreducible and G is the graph of T, then G cannot be strongly connected.

The strong connectivity of G would imply the existence of a chain $(j,k_1),(k_1,k_2),\dots,(k_p,k)$ such that $t_{jk_1}\, t_{k_1k_2} \dots t_{k_pk} \neq 0$, that is j and k belong both either to $\mathcal{N}_1$ or $\mathcal{N}_2$. This contradiction shows the implication non (Ge) $\Longrightarrow$ non (Ko), that is (Ko) implies (Ge) and this completes the proof.

We conclude by stating

<u>Theorem 2</u>. The irreducibility concepts (Fr), (Ge) and (Ko) are equivalent. Moreover, each of these concepts is equivalent to each of the concepts of group (II) and consequently, to each of the concepts of group (I) if dim $Y \geqq 2$.

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