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SOME NOWHERE DENSELY GENERATED TOPOLOGICAL PROPERTIES

Robert L. BLAIR

Abstract: $[\varkappa,\lambda]$-compactness is characterized in such a way that the following result of Mills and Wattel is an immediate consequence:
(*) $[\varkappa,\lambda]$-compactness is nowhere densely generated in the class of T_1-spaces without isolated points. (The special case of (*) for compactness is due to Katětov.) In addition, characterizations of α-closed-completeness, α-compactness, and pseudo-$(\alpha,\varkappa)$-compactness are obtained with consequences similar to (*). Among these consequences, for example: Closed-completeness is nowhere densely generated in the class of T_1-spaces without closed discrete subsets of Ulam-measurable cardinality.

Key words and phrases: Nowhere densely generated, $[\varkappa,\lambda]$-compact, α-closed-complete, α-compact, pseudo-$(\alpha,\varkappa)$-compact, relatively pseudo-$(\alpha,\varkappa)$-compact, closed-complete, realcompact, screenable, measurable cardinal.

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0. Introduction. As in [MW], a property Φ is nowhere densely generated in a class C of topological spaces if, for every $X \in C$, X has property Φ whenever every nowhere dense closed subset of X has property Φ. For example:

0.1. Theorem (Mills and Wattel [MW, Theorem 2]). For all infinite cardinals $\varkappa$ and λ, $[\varkappa,\lambda]$-compactness is nowhere densely generated in the class of T_1-spaces without isolated points.

0.2. Theorem (Blair [Bl, 4]). Realcompactness is nowhere densely generated in the class of normal T_1-spaces without closed screenable subsets of Ulam-measurable cardinality.

(For definitions of terms used here, see §1-3 below.)

Theorem 0.1 generalizes the following earlier result of Katětov: Compactness is nowhere densely generated in the class of T_1- spaces without isolated points [K]. (Katětov's theorem is reproved in [VW, 2.4].) For other results closely related to 0.1 and 0.2, see Mills and Wattel [MW], van Douwen [vD, 11.1], van Douwen, Tall, and Weiss [vDTW], and [Bl]. I am indebted to Eric van Douwen for calling [K] and [MW] to my attention.

The main results of this paper can be summarized as follows:

(i) 1.2 is a characterization of $[\varkappa, \lambda]$-compactness that (a) quickly yields 0.1, (b) requires no separation hypothesis, and (c) has a proof much simpler than that of 0.1 in [MW]. (In connection with (c), see the remarks of [vD_2] (in which the present author's initials are erroneously printed as "D.E.").)

(ii) 2.4 is a characterization of α-compactness (in the sense of Herrlich [He]) in the class of normal T_1-spaces that implies 0.2 and cardinal generalizations thereof, and it is also a characterization of α-closed-completeness that implies both the Katětov theorem cited above (see 2.8) and the following: Closed-completeness (= a-realcompactness [D]) is nowhere densely generated in the class of T_1-spaces without closed screenable subsets of Ulam-measurable cardinality (see 2.5).

(iii) 3.1 is a characterization of pseudo-$(\alpha, \varkappa)$-compactness that implies, for example, the following: If X is T_1 and without isolated points, and if every nowhere dense subset of X is relatively pseudocompact in X, then X is pseudocompact (see 3.4). (Thus pseudocompactness is, in a strong sense, nowhere densely generated in the class of T_1-spaces without isolated points.)

The same basic technique (namely, an appropriate choice of a maximal family of pairwise disjoint open sets) underlies the proof of each of the three main results 1.2, 2.4, and 3.1.

Cardinals are initial ordinals. The smallest infinite cardinal is denoted by ω, and if α is a cardinal, then α^+ denotes the smallest cardinal β such that $\alpha < \beta$. The power set of a set X is denoted by $\mathcal{P}(X)$. If $A \subset X$ and $\mathcal{U} \subset \mathcal{P}(X)$, we set $\mathcal{U}(A) = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

No separation properties are assumed unless explicitly mentioned.

I am indebted to M. Hušek for suggestions for improving the exposition of an earlier version of this paper.

1. $[\varkappa, \lambda]$ -compactness. For infinite cardinals $\varkappa$ and λ, a space X is $[\varkappa, \lambda]$-compact if for every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \lambda$, there exists $\mathcal{V} \subset \mathcal{U}$ with $|\mathcal{V}| < \varkappa$ and $X = \cup \mathcal{V}$. (Thus $[\omega, \omega^* |\mathcal{P}(X)|]$-compact = compact, $[\omega^+, \omega^* |\mathcal{P}(X)|]$-compact = Lindelöf, and $[\omega, \omega]$-compact = countably compact.) Obviously $[\varkappa, \lambda]$-compactness is closed-hereditary, and X is trivially $[\varkappa, \lambda]$-compact if $\varkappa > \lambda$.

We shall say that a subset D of a space X is screenable (resp. strongly screenable) in X if there exists a pairwise disjoint family $(G_x)_{x\in D}$ of open subsets of X such that $x\in G_x$ (resp. $\mathrm{cl}\{x\}\subset G_x$) for every $x\in D$ (cf. [Bl]); and that D is <u>almost closed</u> in X if $\cup\{\mathrm{cl}\{x\}:x\in D\}$ is closed in X. Clearly every strongly screenable subset of X is screenable, and every closed subset of X is almost closed; and if X is T_1, then screenable = strongly screenable and closed = almost closed. The following is also obvious:

1.1. Lemma. If D is a strongly screenable subset of X, then $\cup\{\mathrm{cl}\{x\}:x\in D\}$ is the sum of the family $(\mathrm{cl}\{x\})_{x\in D}$.

The main result of this section is as follows:

1.2. Theorem. Let $\varkappa$ and λ be infinite cardinals with $\varkappa\le\lambda$. If X is a topological space, then the following are equivalent:

(1) X is $[\varkappa,\lambda]$-compact.

(2) Every almost closed, strongly screenable subset of X has cardinality $<\varkappa$, and every nowhere dense closed subset of X is $[\varkappa,\lambda]$-compact.

Proof. (1) $\Rightarrow$ (2): If D is an almost closed, strongly screenable subset of X, then $\cup\{\mathrm{cl}\{x\}:x\in D\}$ is closed in X and hence $[\varkappa,\lambda]$-compact. Since $\varkappa\le\lambda$, it follows from 1.1 that $|D|<\varkappa$. The remaining assertion of (2) is clear.

(2) $\Rightarrow$ (1): Let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}|\le\lambda$, and let $\mathcal{G}$ be a maximal family of pairwise disjoint open subsets of X such that, for every $G\in\mathcal{G}$, $G\subset U_G$ for some $U_G\in\mathcal{U}$. The maximality of $\mathcal{G}$ implies that $X-\cup\mathcal{G}$ is nowhere dense in X and hence $[\varkappa,\lambda]$-compact. Thus there exists $\mathcal{V}\subset\mathcal{U}$ such that $|\mathcal{V}|<\varkappa$ and $X-\cup\mathcal{G}\subset\cup\mathcal{V}$. Let $F=X-\cup\mathcal{V}$ and

$\mathcal{G}^* = \mathcal{G}(F)$. For each $G \in \mathcal{G}^*$, choose $x_G \in F \cap G$ and let $D = \{x_G : G \in \mathcal{G}^*\}$ and $A = \cup\{cl\{x_G\} : G \in \mathcal{G}^*\}$. Since $cl\, A \subset F \subset \cup \mathcal{G}$ and $\mathcal{G}$ is pairwise disjoint, it follows easily that A is closed in X. Moreover, for each $G \in \mathcal{G}$ we have $F \cap G = F \cap (X - \cup(\mathcal{G} - \{G\}))$, so $F \cap G$ is closed in X. Thus for each $G \in \mathcal{G}^*$, $cl\{x_G\} \subset F \cap G \subset G$, and we conclude that D is almost closed and strongly screenable. Hence $|\mathcal{G}^*| = |D| < \varkappa$. Let $\mathcal{W} = \mathcal{V} \cup \{U_G : G \in \mathcal{G}^*\}$ and note that $\mathcal{W} \subset \mathcal{U}$ and $|\mathcal{W}| < \varkappa$. Since $X - \cup \mathcal{V} = F \subset \cup \mathcal{G}^* \subset \cup\{U_G : G \in \mathcal{G}^*\}$, $\mathcal{W}$ covers X and the proof is complete. □

A space X is <u>essentially</u> T_1 [WW] if for each $x, y \in X$, either $cl\{x\} \cap cl\{y\} = \emptyset$ or $cl\{x\} = cl\{y\}$.

1.3. Corollary. Let $\varkappa$ and λ be infinite cardinals with $\varkappa \leq \lambda$. If every nowhere dense closed subset of X is $[\varkappa, \lambda]$-compact, then $X - \cup\{int\, cl\{x\} : x \in X\}$ is $[\varkappa, \lambda]$-compact, and the converse holds if X is essentially T_1.

Proof. Let $Y = X - \cup\{int\, cl\{x\} : x \in X\}$ and assume first that every nowhere dense closed subset of X is $[\varkappa, \lambda]$-compact. Let $D \subset Y$ be almost closed and strongly screenable in Y. Then there exists a pairwise disjoint family $(G_x)_{x \in D}$ of open subsets of Y such that $cl\{x\} \subset G_x$ for every $x \in D$, and the set $A = \cup\{cl\{x\} : x \in D\}$ is closed in Y and hence also in X. Suppose there is a nonempty open set U in X with $U \subset A$. Then $U \cap cl\{x\} \neq \emptyset$ for some $x \in D$, and then $x \in U \cap G_x \subset A \cap G_x = cl\{x\}$. Note also that $U \cap G_x$ is open in X, and hence $x \in int\, cl\{x\}$. But then $x \notin Y$, a contradiction. Thus A is nowhere dense in X and hence $[\varkappa, \lambda]$-compact. Since $\varkappa \leq \lambda$, 1.1 implies that $|D| < \varkappa$. Moreover, every nowhere dense closed subset of Y is also nowhere dense and closed in X and therefore $[\varkappa,\lambda]$-compact. Hence Y is $[\varkappa,\lambda]$-compact by 1.2.

Assume next that X is essentially T_1, that Y is $[\varkappa,\lambda]$-compact, and that E is a nowhere dense closed subset of X. If there exists $y \in E - Y$, then $y \in \operatorname{int} \operatorname{cl}\{x\}$ for some $x \in X$. Since X is essentially T_1, we then have $\emptyset \neq \operatorname{int} \operatorname{cl}\{x\} \subset \operatorname{cl}\{x\} = \operatorname{cl}\{y\} \subset E$, a contradiction. Thus $E \subset Y$ and hence E is $[\varkappa,\lambda]$-compact. □

1.4. Remarks. (a) Let $\varkappa$ and λ be as in 1.3 and let X be T_1. Then, by 1.3, the set of nonisolated points of X is $[\varkappa,\lambda]$-compact if and only if every nowhere dense closed subset of X is $[\varkappa,\lambda]$-compact. The theorem of Mills and Wattel (0.1) is an immediate consequence.

(b) It is worth remarking that the proof of 1.2 can be adapted to give a very brief direct proof of 0.1. For this purpose we first note the following well-known (and easily proved) fact:

1.5. Proposition. **If D is a discrete subspace of a T_1-space X, and if no point of D is isolated in X, then D is nowhere dense in X.**

Proof of 0.1. Assume that X is T_1 and without isolated points, that $\varkappa \leq \lambda$, and that every nowhere dense closed subset of X is $[\varkappa,\lambda]$-compact. Let $\mathcal{U}$ be an open cover of X with $|\mathcal{U}| \leq \lambda$, and choose $\mathcal{G}$, $\mathcal{V}$, F, $\mathcal{G}^*$, D, and $\mathcal{W}$ precisely as in the proof of (2) $\Longrightarrow$ (1) of 1.2. Since X is T_1 and $\operatorname{cl} D \subset F \subset \cup \mathcal{G}$, D is closed in X; and D is nowhere dense in X by 1.5. Thus D is a $[\varkappa,\lambda]$-compact discrete space, so $|D| < \varkappa$. As before, $|\mathcal{W}| < \varkappa$, $\mathcal{W} \subset \mathcal{U}$, and $\mathcal{W}$ covers X, and thus X is

$[\varkappa,\lambda]$-compact. □

We define the <u>cellular extent</u> ce(X) of a space X as follows: $ce(X) = \omega \cdot \sup\{|D|: D$ is an almost closed, strongly screenable subset of $X\}$. (The special case of ce(X) for T_1-spaces is defined in [Bl, 8(d)]. Clearly $ce(X) \leq \min\{c(X), \Delta(X)\}$, where c(X) is the cellularity of X [J] and $\Delta(X)$ is the discreteness character of X [Ho, §3]. As noted in [Bl, 8(d)], this inequality can be strict.)

As an immediate consequence of 1.2 we have:

1.6. Corollary. If λ is an infinite cardinal and if every nowhere dense closed subset of X is $[ce(X)^+,\lambda]$-compact, then X is $[ce(X)^+,\lambda]$-compact.

1.7. Corollary. If λ is an infinite cardinal and if every nowhere dense subset of X is $[c(X)^+,\lambda]$-compact, then X is hereditarily $[c(X)^+,\lambda]$-compact.

Proof. It follows from 1.2 that every open subset of X is $[c(X)^+,\lambda]$-compact. As a consequence, X is hereditarily $[c(X)^+,\lambda]$-compact. □

The special case of 1.7 for which $c(X) = \omega$ is noted by van Douwen, Tall and Weiss in [vDTW, p. 142] (cf. [MW, Corollary 3(b)]).

For infinite cardinals $\varkappa$ and λ with $\varkappa \leq \lambda$, we shall call X <u>iso-$[\varkappa,\lambda]$-compact</u> if every $[\varkappa,\varkappa]$-compact closed subset of X is $[\varkappa,\lambda]$-compact. (Thus iso-$[\omega,\omega\cdot|\mathcal{P}(X)|]$-compact = isocompact [Ba], i.e. every countably compact closed subset of X is compact.)

The following is an easy consequence of 1.2:

1.8. Corollary. For all infinite cardinals $\varkappa$ and λ with $\varkappa \leq \lambda$, iso-$[\varkappa,\lambda]$-compactness is nowhere densely generated in the class of all topological spaces.

2. α-closed-completeness and α-compactness. Let X be a topological space and α an infinite cardinal. If $\mathcal{S} \subset \mathcal{P}(X)$, $\mathcal{S}$ has the α-intersection property if $\cap \mathcal{A} \neq \emptyset$ for every $\mathcal{A} \subset \mathcal{S}$ with $|\mathcal{A}| < \alpha$, and $\mathcal{S}$ is fixed (resp. free) if $\cap \mathcal{S} \neq \emptyset$ (resp. $\cap \mathcal{S} = \emptyset$). By a closed ultrafilter (resp. z-ultrafilter) on X we mean a maximal filter in the lattice of closed subsets (resp. zero-sets) of X; and a space (resp. a Tychonoff space) X is α-closed-complete (resp. α-compact (in the sense of Herrlich [He])) if every closed ultrafilter (resp. z-ultrafilter) on X with the α-intersection property is fixed. (Thus ω-closed-complete = compact, ω-compact = compact Hausdorff, ω^+-closed-complete = closed-complete (= a-realcompact [D]), and ω^+-compact = realcompact.)

A cardinal $\varkappa$ is measurable if there exists a free ultrafilter on (the discrete space) $\varkappa$ with the $\varkappa$-intersection property [CN_2, p. 186]. For α an infinite cardinal, $m(\alpha)$ will denote the smallest measurable cardinal such that $\alpha \leq m(\alpha)$ (if such a cardinal exists; see the discussion in [CN_2, p. 203] and [J, A6.1]). Clearly $m(\omega) = \omega$.

The main result of this section is 2.4. For its proof we need the following three lemmas. (The easy proof of 2.1 is omitted; for 2.2 see e.g. [Hu] or [R, 2.4].)

2.1. Lemma. Every closed subspace of an α-closed-complete (resp. α-compact) space is α-closed-complete (resp. α-compact).

2.2. Lemma. A discrete space D is α-compact if and only if $|D| < m(\alpha)$.

2.3. Lemma. Let A be closed in X and let $\mathcal{F}$ be a closed ultrafilter (resp. z-ultrafilter) on X with the α-intersection property. If A meets every member of $\mathcal{F}$ (resp. and X is normal), then $\mathcal{F}' = \{F \cap A : F \in \mathcal{F}\}$ is a closed ultrafilter (resp. z-ultrafilter) on A with the α-intersection property.

Proof. Consider the case in which X is normal, $\mathcal{F}$ is a z-ultrafilter on X with the α-intersection property, and A meets every member of $\mathcal{F}$. Since A is C^*-embedded in X, it follows readily that $\mathcal{F}'$ is a z-ultrafilter on A. Suppose that $\beta < \alpha$ and that $(F_\xi)_{\xi<\beta}$ is a family of members of $\mathcal{F}$. If $\bigcap_{\xi<\beta} (F_\xi \cap A) = \emptyset$, then, by normality, there is a zero-set Z in X with $\bigcap_{\xi<\beta} F_\xi \subset Z$ and $Z \cap A = \emptyset$. Then for every $F \in \mathcal{F}$ we have $\emptyset \neq F \cap (\bigcap_{\xi<\beta} F_\xi) \subset F \cap Z$, and hence $Z \in \mathcal{F}$. But then $Z \cap A \neq \emptyset$, a contradiction, and we conclude that $\mathcal{F}'$ has the α-intersection property. The other case of the lemma can be verified in a straightforward way. □

2.4. Theorem. If α is an infinite cardinal and if X is T_1 (resp. normal T_1), then the following are equivalent:

(1) X is α-closed-complete (resp. α-compact).

(2) Every closed screenable subset of X has cardinality $< m(\alpha)$, and every nowhere dense closed subset of X is α-closed-complete (resp. α-compact).

Proof. (1) $\Longrightarrow$ (2): This is an immediate consequence of 2.1 and 2.2.

(2) $\Longrightarrow$ (1): Assume that X is T_1 (resp. normal T_1), let $\mathcal{F}$ be a closed ultrafilter (resp. z-ultrafilter) on X with

the α-intersection property, and suppose that $\cap\mathcal{F} = \emptyset$. Let $\mathcal{G}$ be a maximal family of pairwise disjoint open subsets of X such that, for every $G \in \mathcal{G}$, $G \cap F = \emptyset$ for some $F \in \mathcal{F}$. Since $\cap\mathcal{F} = \emptyset$, the maximality of $\mathcal{G}$ implies that $X - \cup\mathcal{G}$ is nowhere dense in X and therefore α-closed-complete (resp. α-compact). If $X - \cup\mathcal{G}$ meets every member of $\mathcal{F}$, then, by 2.3, $\mathcal{F}' = \{F \cap (X - \cup\mathcal{G}) : F \in \mathcal{F}\}$ is a closed ultrafilter (resp. z-ultrafilter) on $X - \cup\mathcal{G}$ with the α-intersection property. But then $\emptyset \neq \cap\mathcal{F}' \subset \cap\mathcal{F}$, a contradiction, and we conclude that $F^* \subset \cup\mathcal{G}$ for some $F^* \in \mathcal{F}$. Let $\mathcal{G}^* = \mathcal{G}(F^*)$, choose $x_G \in F^* \cap G$ for each $G \in \mathcal{G}^*$, and let $D = \{x_G : G \in \mathcal{G}^*\}$. Since $\mathrm{cl}\, D \subset F^* \subset \cup\mathcal{G}$, $\mathcal{G}$ is pairwise disjoint, and X is T_1, it follows that D is closed in X. Since D is clearly screenable, $|D| < m(\alpha)$.

Next, for each $\mathcal{H} \subset \mathcal{G}$, $\cup\mathcal{H}$ is a zero-set in $\cup\mathcal{G}$, so $F^* \cap (\cup\mathcal{H})$ is a zero-set in F^*. Thus $F^* \cap (\cup\mathcal{H})$ is closed in X, and if X is normal (so that F^* is C^*-embedded in X), then $F^* \cap (\cup\mathcal{H})$ is a zero-set in X. Thus we have:

(*) For every $\mathcal{H} \subset \mathcal{G}$, $F^* \cap (\cup\mathcal{H})$ is closed (resp. a zero-set) in X.

We show next that $\Phi = \{\mathcal{H} \subset \mathcal{G}^* : F^* \cap (\cup\mathcal{H}) \in \mathcal{F}\}$ is an ultrafilter on the discrete space $\mathcal{G}^*$ with the α-intersection property: Clearly $\emptyset \notin \Phi$. Suppose that $\beta < \alpha$ and that $(\mathcal{H}_\xi)_{\xi<\beta}$ is a family of members of Φ. Since $\mathcal{G}$ is pairwise disjoint, $\cap_{\xi<\beta}(\cup\mathcal{H}_\xi) = \cup(\cap_{\xi<\beta}\mathcal{H}_\xi)$. Then $F^* \cap (\cup(\cap_{\xi<\beta}\mathcal{H}_\xi)) = \cap_{\xi<\beta}(F^* \cap (\cup\mathcal{H}_\xi)) \neq \emptyset$, and hence $\cap_{\xi<\beta}\mathcal{H}_\xi \neq \emptyset$. Next, it follows from (*) that if $\mathcal{H} \in \Phi$ and $\mathcal{K} \subset \mathcal{G}^*$ with $\mathcal{H} \subset \mathcal{K}$, then $\mathcal{K} \in \Phi$. Finally, suppose that $\mathcal{H} \subset \mathcal{G}^*$ with $\mathcal{H} \notin \Phi$. By (*), $F^* \cap (\cup\mathcal{H})$ is closed (resp. a zero-set) in X and not in $\mathcal{F}$, so $F \cap F^* \cap (\cup\mathcal{H}) = \emptyset$ for some $F \in \mathcal{F}$. Since $F^* \subset \cup\mathcal{G}$, it then follows that $F \cap F^* \subset F^* \cap (\cup(\mathcal{G}^* - \mathcal{H}))$. By $(*)$, $\mathcal{G}^* - \mathcal{H} \in \Phi$, and thus Φ is, in fact, an ultrafilter on $\mathcal{G}^*$ with the α-intersection property.

Now $|\mathcal{G}^*| = |D| < m(\alpha)$, so by 2.2 there exists $G \in \cap \Phi$. Since $G \in \mathcal{G}$, we have $G \cap F = \emptyset$ for some $F \in \mathcal{F}$, and hence $F \cap F^* \subset F^* \cap (\cup(\mathcal{G}^* - \{G\}))$. Then, by $(*)$, $\mathcal{G}^* - \{G\} \in \Phi$ and hence $G \in \mathcal{G}^* - \{G\}$, a contradiction. The proof is now complete. □

Since a cardinal $\varkappa$ is Ulam-nonmeasurable if and only if $\varkappa < m(\omega^+)$ [CN, 8.31], we have:

2.5. Corollary. If X is T_1 (resp. normal T_1), then the following are equivalent:

(1) X is closed-complete (resp. realcompact).

(2) Every closed screenable subset of X has Ulam-nonmeasurable cardinality, and every nowhere dense closed subset of X is closed-complete (resp. realcompact).

The realcompact case of 2.5 (see 0.2 above) is proved by a different technique in [Bl, 4]. As noted in [Bl, 8(c)], the hypothesis of normality cannot be omitted in 2.5.

2.6. Corollary. If X is a Tychonoff cb-space [M], then the following are equivalent:

(1) X is realcompact.

(2) Every closed screenable subset of X has Ulam-nonmeasurable cardinality, and every nowhere dense closed subset of X is real-compact.

Proof. In view of 2.5, we need only note that every realcompact space is closed-complete [D. 1.16], every closed subspace of a cb-space is cb [M], and every Tychonoff closed-complete cb-space is realcompact [D, 1.10]. □

2.7. Corollary. If α is an infinite cardinal and if X is T_1 (resp. normal T_1), then the following are equivalent:

(1) The set of nonisolated points of X is α-closed-complete (resp. α-compact).

(2) Every nowhere dense closed subset of X is α-closed-complete (resp. α-compact).

Moreover, if (2) holds and the set of isolated points of X has cardinality $< m(\alpha)$, then X is α-closed-complete (resp. α-compact).

Proof. This follows readily from 2.1, 2.2, 2.4 and 1.5. □

2.8. Remark. We note that the ω-closed-complete (= compact) case of 2.7 implies (once again) the Katětov theorem cited in the Introduction.

3. <u>Pseudo-(α, $\varkappa$)-compactness</u>. Let α and $\varkappa$ be infinite cardinals with $\varkappa \le \alpha$ and let $Y \subset X$. We shall say that Y is <u>relatively pseudo-(α, $\varkappa$)-compact</u> in X if for every locally $< \varkappa$ family $\mathcal{U}$ of open subsets of X, $|\{U \cap Y : U \in \mathcal{U}\}| < \alpha$; and that Y is <u>relatively pseudo-α-compact</u> in X if Y is relatively pseudo-(α, ω)-compact in X. The space X is <u>pseudo-(α, $\varkappa$)-compact</u> (resp. <u>pseudo-α-compact</u>) if X is relatively pseudo-(α, $\varkappa$)-compact (resp. relatively pseudo-α-compact) in itself (see [CN_1]). (Thus Y is relatively pseudocompact in X (resp. X is pseudocompact) if and only if Y is relatively pseudo-ω-compact in X (resp. X is pseudo-ω-compact).)

3.1. Theorem. Let α and $\varkappa$ be infinite cardinals with $\varkappa \leq \alpha$. If either α is regular or $\varkappa < \alpha$, then the following are equivalent:

(1) X is pseudo-$(\alpha, \varkappa)$-compact.

(2) Every screenable subset of X is relatively pseudo-$(\alpha, \varkappa)$-compact in X.

Proof. (1) $\Longrightarrow$ (2): This implication is trivial.

(2) $\Longrightarrow$ (1): If there is a locally $< \varkappa$ family $\mathcal{U}$ of nonempty open subsets of X with $|\mathcal{U}| = \alpha$, choose Φ maximal (relative to inclusion) such that Φ is an injective function, $\mathcal{G} = \mathrm{dom}\, \Phi$ is a pairwise disjoint family of nonempty open subsets of X, and, for every $G \in \mathcal{G}$, $G \subset \Phi(G) \in \mathcal{U}$ and $|\mathcal{U}(G)| < \varkappa$. Then $\mathcal{G}$ is locally $< \varkappa$ and (by maximality of Φ) $\mathcal{U} = \mathcal{U}(\cup \mathcal{G})$. It follows that $|\mathcal{U}| = \Sigma\{|\mathcal{U}(G)| : G \in \mathcal{G}\}$, so the hypotheses on α imply that $|\mathcal{G}| = |\mathcal{U}|$. For each $G \in \mathcal{G}$, pick $x_G \in G$. Clearly $\{x_G : G \in \mathcal{G}\}$ is screenable, but not relatively pseudo-$(\alpha, \varkappa)$-compact, in X. □

3.2. Remark. The word "relatively" cannot be omitted in the implication (1) $\Longrightarrow$ (2) of 3.1: The ordinal space ω^+ is pseudocompact, but ω is screenable in ω^+ and nonpseudocompact.

3.3. Corollary. If α is an infinite cardinal and if every screenable subset of X is relatively pseudo-α-compact in X, then X is pseudo-α-compact.

Various other corollaries can of course be deduced. For example, from 3.1 and 1.5 we have:

3.4. Corollary. If X is T_1 and without isolated points, and if every nowhere dense subset of X is relatively pseudocompact in X, then X is pseudocompact.

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