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MONADS IN BASIC EQUIVALENCES

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Abstract: In this paper we prove that all sets from the universe of sets can be defined from both any arbitrary proper Sd_V class and a suitable monad in any equivalence of indiscernibility. We also show that there is a monad μ such that $Def_{\mu} \neq V$. These facts are further used for proving some interesting assertions concerning similarities; e.g. it is shown here that some special cases of similarities have to be parts of identity.

Key words: Alternative set theory, basic equivalence, monad, definability, similarity, endomorphism.

Classification: Primary 03E70
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There is a question: How large has to be the class of parameters X in order that one can define from it all sets from the universe of sets? More precisely, we shall ask here how to choose X so that $Def_X = V$ and X is as small as possible at the same time.

By means of Peano arithmetic one can prove:

(1) For each proper class $X \in Sd_V$ there is a function $F \in Sd_0$ such that $V = F"X^2$ (and therefore $Def_X = V$). Unfortunately, such a function F depends on X.

Our question is whether it is possible to go "lower". If X is a semiset, then obviously $Def_X \neq V$. That is why such a class X for which $Def_X = F$ has to be very large.

In the first part of this paper (see Theorem 1) we prove that in the alternative set theory, the following statement holds:

There eixsts a function $F \in Sd_0$ such that in any basic equivalence $\overset{\circ}{=}_{\{c\}}$ there is a monad $\mu_{\{c\}}$ for which $V = F'' \mu^2_{\{c\}}$.

This implies $Def_{\mu_{\{c\}}} = V$.

The main difficulty in proving such an assertion in non-standard models of Peano arithmetic lies in the fact that it is not possible to define the relation $\overset{\circ}{=}$ by internal means of the theory.

At the end of the first paragraph we show still one interesting property of the above mentioned monad $\mu_{\{c\}}$, its construction is not at all one-aimed.

In [Č-K 1] we proved that there is no function $F \in Sd_{\{c\}}$ and no monad μ in $\overset{\circ}{=}_{\{c\}}$ such that $V = F''\mu$ (see Theorem 5). We also showed there that it is not possible to immerse any proper class $X \in Sd_V$ into any monad. Therefore, the further mentioned theorem 1 is not a consequence of the assertion (1) since one cannot proceed in such a way that he chooses a monad μ which contains a proper class $X \in Sd_V$ and then creates μ^2.

The first section includes, moreover, a theorem which assures that there is a monad μ for which $Def_{\mu} \neq V$.

In the second paragraph we shall deal with several consequences of the theorem on identity from [Č-K 1]. Results of § 1 will be substantially applied here.

In the book [V], there are investigated similarities, endomorphisms and automorphisms. For proving the existence of these objects there are used strong axioms of the alternative set theory (axiom of choice, axiom of cardinalities). This fact suggests that we have to treat with rather "delicate" objects. This "delicacy" will be more closely specified here. We show namely that such proper classes which are similarities and simultaneously "simpler" have to be identities. For the criteria of simplicity we shall take set-definability, reality, eventually the property to be a π-class.

We also prove here that the condition: the investigated similarities are proper classes, is essential. In other words, we give the example of an infinite set similarity which is not identical in any point.

Still two remarks:

When we speak about ordering on V we bear in mind the natural ordering on the class (see [V], ch. II, § 1).

Further, we remind the notion Def_X (see [V 1]). The set y is said to be definable using parameters from the class X iff there is a set formula $\varphi(z)$ of the language FL_X such that the formulas $(\exists ! z)\varphi(z)$ and $\varphi(y)$ hold. We use the notation Def_X for the class of all sets definable using parameters from the class X.

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§ 1. The following theorem may remind us of the Leibniz's statement "... each created monad represents the whole universe".

Theorem 1. It is possible to define a function $Cd \in Sd_0$ (so-called a coding function) such that

$$(\forall c)(\exists \mu_{\{c\}}) \qquad V = Cd'' \mu^2_{\{c\}}.$$

Firstly we prove the following assertion.

Lemma 1. There is a function f such that for every proper class $X \in Sd_{\{c\}}$ the conditions

(1) $(\forall x)\ (x \in dom(f) \Rightarrow f(x) < x)$;

(2) $Def_{\{c\}} \subseteq f''X$

hold.

Proof. Let us enumerate all classes of $Sd_{\{c\}}$ and let $\{X_i; i \in FN\}$ denote such an enumeration. Let further $Def_{\{c\}} = \{a_i; i \in FN\}$.

At first we shall construct a countable sequence of functions $\{f_i; i \in FN\}$ such that $dom(f_i)$ are mutually disjoint, each f_i satisfies (1) and for each proper class $X \in Sd_{\{c\}}$ the condition $f_i''X = \{a_i\}$ holds.

Functions f_i will be constructed by induction. For creating f_1 we shall at first construct an auxiliary (countable) function F_1; then f_1 will be obtained from F_1 by a convenient prolongation.

The function F_1 is defined as follows:

$F_1 = \{\langle a_1, x_i\rangle;\ x_i$ is the smallest element in $X_i - Seg^{<}(a_1)\}$,

where $X_i \in Sd_{\{c\}}$. Let f_1 be such a prolongation of F_1 for which

$$(\forall x \in dom(f_1)) \qquad f_1(x) = a_1 < x.$$

We construct the function f_{n+1} analogously, only with the difference that we put

$F_{n+1} = \{\langle a_{n+1}, x_i\rangle;\ x_i$ is the smallest element in

$$[(X_i - \bigcup_{k<n+1} dom(f_k)) - Seg^{<}(a_{n+1})]\}.$$

Now we prolong the sequence $\{f_i; i \in FN\}$ in such a way that the condition on disjointness of domains and the property (1) are preserved. Let $\{f_\gamma ; \gamma \in \alpha\}$ denote the prolonged sequence. Put further $f = \bigcup \{f_\gamma ; \gamma \in \alpha\}$. The function f fulfils obviously both conditions (1) and (2).

Proof of Theorem 1. Let us define a function of two variables $Cd \in Sd_0$ in this way: if $y(x) < x$, then $Cd(x,y) = y(x)$; for $y(x) \geq x$ or y is not a function, the function $Cd(x,y)$ is not defined. Let further f be the function from Lemma 1. We shall create the class $\mu^2_{\{c\}}(f)$ and prove that

$$V = Cd'' \mu^2_{\{c\}}(f).$$

For abbreviation, we write $Cd'' \mu^2_{\{c\}}(f) = Y$.

At first we prove that Y is a figure in $\{\overset{\circ}{=}\}$. In [Č-K 1] it is shown (see Theorem 3) that the image of a monad in any $Sd_{\{c\}}$ relation is a closed figure in $\{\overset{\circ}{=}\}$. From this it follows that the image of each figure is a figure (in the investigated relation $\{\overset{\circ}{=}\}$). Put $G(\langle z_1, z_2 \rangle) = Cd(z_1, z_2)$. Then $Y = G'' \mu^2_{\{c\}}(f)$. Therefore it is enough to prove that $\mu^2_{\{c\}}(f)$ is a figure in $\{\overset{\circ}{=}\}$. This assertion is the consequence of Theorem 2 from [Č-K 1] which says that the domain and the range of a monad is a monad, too.

Moreover, the class Y is also a π-class. This fact follows from the following consideration: Each monad is a π-class and its Cartesian square is a π-class, too. In addition, we know that the image of a π-class under a set-definable mapping ($Cd \in Sd_0$) is also a π-class.

Thus Y is a closed class (see [V], ch. III, § 2) containing $Def_{\{c\}}$ which is dense in V (see [V], ch.V, § 1) and therefore $V \subseteq Y$. The inclusion $Y \subseteq V$ is obvious. This completes the proof.

Corollary. In each equivalence of indiscernibility $\ddot{=}$ there is a monad μ such that

$$\mathrm{Def}_{\mu} = V.$$

Proof. In [V 1], it is proved that for every equivalence $\ddot{=}$ there exists c such that $\overset{\circ}{=}_{\{c\}}$ is finer than $\ddot{=}$. Thus it suffices to prove the assertion for $\overset{\circ}{=}_{\{c\}}$. For this we take the monad $\mu_{\{c\}}$ from the proof of Theorem 1 and realize that for each monad μ it is true that $\mathrm{Def}_{\mu} = \mathrm{Def}_{\mu^2}$.

There exists such a monad μ in $\overset{\circ}{=}$ for which $\mathrm{Def}_{\mu} \neq V$. Before proving the assertion we shall recall several facts.

In [S-Ve 1], the notion of the class of indiscernibles is introduced. Let us note here that the definition of "A class X is the class of indiscernibles" mentioned in the above paper is equivalent with the statement : "For each $n \in FN$ the class $P_n(X)$ is the part of a monad in $\overset{\circ}{=}$ ", where $P_{\alpha}(X) = \{x; x \subseteq X \,\&\, x \mathrel{\hat{\approx}} \alpha\}$. If we denote by Ind the class of indiscernibles which is a proper σ-class, then Ind is a monad in $\overset{\circ}{=}$ (for the proof see [Č-K 1]).

Theorem 2. $\mathrm{Def}_{\mathrm{Ind}} \neq V$.

Proof. Let $y \in \mathrm{Def}_{\mathrm{Ind}}$. Then there is a function $F \in Sd_0$ and elements x, k such that $x \in P_k(\mathrm{Ind})$ and $F(x) = y$. Thus each element of $\mathrm{Def}_{\mathrm{Ind}}$ belongs to the image of a class $P_k(\mathrm{Ind})$. Since there are only countably many classes $P_k(\mathrm{Ind})$ and the same is valid for functions of Sd_0 ($F \in Sd_0$) and since for each $k \in FN$ the class $P_k(\mathrm{Ind})$ is a monad ($P_k(\mathrm{Ind})$ is the part of a monad and a σ-class without parameters), we obtain that $\mathrm{Def}_{\mathrm{Ind}}$ consists only from countably many monads. Hence $Def_{Ind} \neq V$.

In the preceding it was proved that for each c there exists a monad μ in $\{\overset{o}{c}\}$ such that $Def_{\mu} = V$. We shall show further that for each proper class $X \in Sd_{\{c\}}$ we have $Def_X = V$. We shall use this assertion substantially in the second paragraph.

Firstly we prove the following statement.

Lemma 2. Let $X \in Sd_{\{c\}}$ be a proper class. Then $c \in Def_X$.

Proof. Since $X \in Sd_{\{c\}}$, we can write $X = \{t;\ \varphi(t,c)\}$. Put $Y = \{\langle t,c\rangle;\ \varphi(t,c)\}$. Then $Y \in Sd_0$ and $X = Y''\{c\}$. Obviously, $Y \subseteq V \times V$ where the natural ordering is defined. Let us define (by induction) a function $F \in Sd_0$ by

$$F(u) = \min(Y''\{u\} - F''Seg^{<}u).$$

If $Y''\{u\}$ is a proper class then there exists $\min(Y''\{u\} - F''Seg^{<}u)$. The function F is a one-one partial function. There exists therefore the function F^{-1} and $F^{-1} \in Sd_0$. Since $F(x) \in X$, we have $c = F^{-1}(F(c))$. Thus $c \in (F^{-1})''X$ and hence $c \in Def_X$.

We shall still prove that in the case that $X \in Sd_{\{c\}}$ is a set, the previous assertion is not generally valid.

Theorem 3. $(\exists a)\ \ a \notin Def_a$.

Proof. From Theorem 2 we know that $Def_{Ind} \neq V$. This implies that there exists x for which $x \notin Def_{Ind}$. Since Ind is a proper class, it contains sets of any great cardinalities. Let a be such a part of Ind for which $card(a) = \alpha$, where α is the number of x in the natural ordering.

Suppose $a \in Def_a$. Then $\alpha \in Def_a$. From $a \subseteq Ind$ it follows $Def_a \subseteq Def_{Ind}$. Thus $\alpha \in Def_{Ind}$. But α is the number of x and therefore also $x \in Def_{Ind}$. This is a contradiction with our assumption $x \notin Def_{Ind}$. Hence $a \notin Def_a$.

Now we come to the theorem promised above.

<u>Theorem 4</u>. Let $X \in Sd_V$ be a proper class, then

$$Def_X = V.$$

<u>Proof</u>. Since $X \in Sd_V$, there exists c such that $X \in Sd_{\{c\}}$ (in the formula which defines V there appears only a finite number of parameters). According to Lemma 2 we have $c \in Def_X$. But then there is a function $F \in Sd_{\{c\}}$ which is a one-one mapping of X onto V. From $F \in Sd_{\{c\}}$ it follows that there is a formula $\varphi(y,x,c)$ which describes F. Thus one can define each $y \in V$ from $x \in X$ and the parameter c. But $c \in Def_X$. Therefore $Def_X = V$.

Further, we shall use the monad $\mu_{\{c\}}(f)$ constructed in the proof of Theorem 1 for solving the problem formulated by A. Vencovská.

At first we recall a notion.

<u>Definition</u>. <u>A class</u> $X \subseteq Y$ <u>is homogeneous for a partition</u> T on $P_2(Y)$ iff there is $Z \in T$ such that $P_2(X) \subseteq Z$. Especially, a <u>class</u> X <u>is homogeneous for</u> an <u>equivalence</u> $\overset{\circ}{=}_{\{c\}}$ iff there is a monad μ in $\overset{\circ}{=}_{\{c\}}$ such that $P_2(X) \subseteq \mu$.

We prove now that there is a monad in $\overset{\circ}{=}_{\{c\}}$ containing a triple of elements which is not homogeneous for this equivalence and that, in addition, one of the elements of the triple can be chosen quite arbitrarily.

<u>Theorem 5</u>. For each c there is a monad $\mu_{\{c\}}$ (in $\overset{\circ}{=}_{\{c\}}$) such that

$$(\forall x \in \mu_{\{c\}})(\exists y \in \mu_{\{c\}} \,\&\, y \neq x)(\forall z \in \mu_{\{c\}} \,\&\, z \neq x \,\&\, z \neq y)$$

$$\{x,y,z\} \text{ is not homogeneous for } \overset{\circ}{=}_{\{c\}} .$$

Proof. Let Cd be the function from Theorem 1 and let $\mu_{\{c\}}$ denote the monad $\mu_{\{c\}}(f)$ from the proof of this theorem.

Define a function H:

$H(\{t,u\}) = Cd(t,u)$ for $t < u$.

Theorem 1 implies that

$(\forall y)(\exists \{t,u\} \in P_2(\mu_{\{c\}}))\quad y = H(\{t,u\})$.

The unordered pair $\{t,u\}$ is called a code of y. For such t, u, y the following inequality

(2) $$(\forall z \in y) \quad z < y < t < u$$

holds since from $z \in y$ we obtain $z < y$ and the remaining inequalities follow from our construction.

Let further x be a non-empty set such that $x \subseteq \mu_{\{c\}}$ and let $\{t_1,u_1\}$ be a code of x; i.e. $H(\{t_1,u_1\}) = x$.

Denote t_2 the smallest element of x and choose u_2 in such a way that $\{t_2,u_2\} \overset{\circ}{=}_{\{c\}} \{t_1,u_1\}$; this choice is possible (see [Č-K 1], Theorem 5). We show that $\{t_2,u_2\}$ is just the element which prevents homogeneity; i.e. we prove that the pair $\{\{t_1,u_1\},\{t_2,u_2\}\}$ cannot be enlarged to a homogeneous triple.

Assume that one can extend the above pair homogeneously into a triple. Then there is an element $\{t_3,u_3\}$ such that the triple $\{\{t_1,u_1\},\{t_2,u_2\},\{t_3,u_3\}\}$ is homogeneous. We should investigate now several cases (from the point of view of the natural ordering).

Let for example $\{t_3,u_3\} < \{t_2,u_2\} < \{t_1,u_1\}$. Then from homogeneity it follows that

$$\langle \{t_3,u_3\},\{t_1,t_1\} \rangle \overset{\circ}{=}_{\{c\}} \langle \{t_2,u_2\},\{t_1,u_1\} \rangle$$

because they belong to the same monad. Since t_2 is the smallest element in $x = H(\{t_1,u_1\})$, the element t_3 is the smallest one in x, too. Therefore $t_3 = t_2$. Similarly (from homogeneity) we have $t_3 = t_1$. But (2) implies that

$$t_1 = t_2 < H(\{t_1,u_1\}) < t_1,$$

which is a contradiction. Analogously, we eliminate all other cases.

Now, putting $\mu_{\{c\}} = \mu_{\{c\}}(\{t_1,u_1\})$; $x = \{t_1,u_1\}$, $y = \{t_2,u_2\}$; $z = \{t_3,u_3\}$ in the text of the proved theorem, the proof is complete.

Remark. The number of monads with properties mentioned in Theorem 5 is not too small. It is not difficult to show that there are uncountably many monads like these. We can even prove tha there is a set-definable function G (without parameters) such that $rng(G) = N - \{0\}$ and for each $\alpha \in rng(G)$ there is a monad ν of the above mentioned properties for which $G''\nu = \mu(\alpha)$. We define the function G as follows:

$$G(v) = card(H(v)),$$

where H is the function from the proof of Theorem 5.

Considering that it is possible to transform the universal class V by a one-one mapping (which is set-definable) onto $N - \{0\}$ (see [V], ch. II, § 1) we proved in fact that we can claim $rng(G) = V$. If we realize now what properties have the monads $\mu_{\{c\}}(f)$ from the proof of Theorem 1 and recall the fact that there is a set-definable function which is a one-one mapping between V and $N - \{0,1\}$, we can improve our result in this sense: For each $a \in V$ there are uncountably many monads with properties described in Theorem 5.

§ 2. In this section we shall formulate at first several assertions which follow from the theorem on identity (see [Č-K 1], Th. 1). For the reader's convenience we shall repeat it here explicitly:

(3) Let $F \in Sd_{\{c\}}$, F be a function. Then

$$(\forall x)[F(x) \overset{\circ}{=}_{\{c\}} x \Rightarrow (\exists X \in Sd_{\{c\}})(F \restriction X = Id \restriction X \;\&\; \mu_{\{c\}}(x) \subseteq X] .$$

<u>Theorem 6</u>. Let F be a similarity, $a \subseteq dom(F)$, $a \in Fin$. Then

$$F''a \subseteq Def_a \Rightarrow F \restriction a = Id \restriction a.$$

<u>Proof</u>. Let card $a = n \in FN$. At first we shall investigate the case $n = 1$. Then there is b such that $a = \{b\}$. Since F is a similarity and $b \in dom(F)$, we have $F(b) \overset{\circ}{=} b$ (see [V], ch. V, § 1). According to the assumption, $F(b) \in Def_{\{b\}}$ holds. Therefore there exists a function $G \in Sd_0$ for which $G(b) = F(b)$. Hence also $G(b) \overset{\circ}{=} b$. From the above mentioned theorem (3), it then follows $G(b) = b$.

Let further $n > 1$. Let us create an ordered n-tuple $\langle b_1,\dots,b_n\rangle$, $b_i \in a$, $i = 1,\dots,n$ according to the natural ordering. The function F will be enlarged on $\langle b_1,\dots,b_n\rangle$ and the first step of this proof will be applied. We perform the enlargement of F in this way: If $x \in Def_{dom(F)}$, then there are a function $G \in Sd_0$ and elements $t_1,\dots,t_n \in dom(F)$ such that $G(t_1,\dots,t_n) = x$. Since F is a similarity, there is a function F_1:

$$F_1(x) = G(F(t_1),\dots,F(t_n)),$$

which enlarges F in the required manner and is a similarity, too (see [Ve 2]). Thus $\langle b_1,\dots,b_n\rangle \in dom(F_1)$. But then

$F_1(\langle b_1,\dots,b_n\rangle) \in Def_{dom(F_1)}$, since according to the assumption, the condition $F(b_i) \in Def_{dom(F)}$, $i = 1,\dots,n$ holds. From the first part of this proof it follows $F_1(\langle b_1,\dots,b_n\rangle) = \langle b_1,\dots,b_n\rangle$ and since F_1 is a similarity, we obtain $F_1(b_i) = b_i$ for each $i = 1,\dots,n$. Hence the function F_1 is identical.

<u>Lemma 3</u>. Let a function F be a monad in $\overset{\circ}{=}_{\{c\}}$. Then there is a function $G \in Sd_{\{c\}}$ such that $F \subseteq G$.

<u>Proof</u>. Since F is a monad, there is a descending sequence of classes $X_n \in Sd_{\{c\}}$ such that $F = \bigcap \{X_n; n \in FN\}$.

We prove that there is $k \in FN$ such that X_k is a function. Assume that for every $n \in FN$ there is $x_n \in dom(X_n)$ such that $X_n'' \{x_n\}$ has at least two elements. We prolong the sequence $\{x_n; n \in FN\}$ by the axiom of prolongation. Let α_i be the largest element such that for each β , $i \leq \beta \leq \alpha_i$ the class $X_i'' \{x_\beta\}$ has at least two elements. Evidently $\alpha_i \supset FN$ for each $i \in FN$. The sequence $\{\alpha_i\}$ is a descending one. Therefore, there exists γ such that for each $i \in FN$ we have $i \in \gamma \in \alpha_i$. By a consequence of the axiom of prolongation, the class $(\bigcap \{X_i;\ i \in FN\})'' \{x_\gamma\}$ has at least two elements, too. At the same time, however, $\bigcap \{X_i; i \in FN\} \subseteq F$ and F is a function - a contradiction.

Put now $G = X_k$; this completes the proof.

<u>Theorem 7</u>. Let F be a similarity which is a monad in $\overset{\circ}{=}_{\{c\}}$ and let $c \in Def_{dom(F)}$. Then $F \subseteq Id$.

<u>Proof</u>. From $c \in Def_{dom(F)}$ it follows that there exists $a \in dom(F)$, $a \in Fin$ such that $c \in Def_a$. Let $G \in Sd_{\{c\}}$ be such a function for which $F \subseteq G$ (the existence of G follows from the previous lemma). Since $G \in Sd_{\{c\}}$ and $c \in Def_a$, we have $G''a \subseteq Def_a$.

Considering further $F"a = G"a$, we obtain $F"a \subseteq Def_a$. Theorem 6 implies then $F \upharpoonright a = Id \upharpoonright a$.

We have to prove that $F \subseteq Id$. Since F is a monad in $\overset{\circ}{=}_{\{c\}}$, dom(F) is also a monad in this equivalence (see [Č-K 1]). Let us denote $T = \{t; G(t) = t\}$; then $T \in Sd_{\{c\}}$ (since $G \in Sd_{\{c\}}$). Further, the result $F \upharpoonright a = G \upharpoonright a$ implies $G \upharpoonright a = Id \upharpoonright a$. Hence for each $u \in a$ we have $G(u) = u$. Thus $u \in T$ and therefore $\mu_{\{c\}}(u) \subseteq T$. Hence $G \upharpoonright \mu_{\{c\}}(u)$ is an identity. But $G \upharpoonright \mu_{\{c\}}(u) = F$ and consequently $F \subseteq Id$.

Corollary. Each similarity which is a π-class without parameters is a part of an identity.

Proof. Let the similarity F satisfy the assumptions. Then F is a figure in $\overset{\circ}{=}$ (since each set-definable without parameters class is a figure in $\overset{\circ}{=}$ and the intersection of such figures is a figure in $\overset{\circ}{=}$, too - see [V]). Let further G be a monad in $\overset{\circ}{=}$, $G \subseteq F$. Then G is also a similarity. Therefore, the assumptions of Theorem 7 are satisfied for $c = 0$. Hence $G \subseteq Id$ and also $F \subseteq Id$.

Theorem 8. Every real endomorphism resp. automorphism is an identity.

Proof. Let F be a real endomorphism or automorphism. Then (from reality) F is a figure in an equivalence $\overset{\circ}{=}_{\{c\}}$ (see [Č-V]).

In the first section we proved that there is a monad μ for which $Def_\mu = V$. Take such a monad μ and put $G = F \upharpoonright \mu$. Then G is a similarity (since F is a similarity). Further $c \in Def_\mu$. Applying now Theorem 7 to the function G, we obtain $G \subseteq Id$. Since on $Def_{dom(G)}$ the function G enlarges canonically (see [Ve 2]) and since $\mathrm{Def}_{\mathrm{dom}(G)} = V$, we obtain from here that $F = \mathrm{Id}$.

<u>Theorem 9</u>. Each similarity which is a proper Sd_V class is an identity.

<u>Proof</u>. Let F be a similarity which is a proper Sd_V class. Then also $\mathrm{dom}(F)$ is a proper class from Sd_V. By Theorem 4 we know that $\mathrm{Def}_{\mathrm{dom}(F)} = V$. Let us enlarge F canonically on $\mathrm{Def}_{\mathrm{dom}(F)}$; denote F_1 such an enlargement. From [Ve 2] it follows that F_1 is a similarity which is a σ-class. Therefore F_1 is a real class (see [Č-V]). Further, since $\mathrm{dom}(F_1) = V$, the similarity F_1 is a real endomorphism and from Theorem 8 it follows that F_1 is an identity.

<u>Corollary.</u> Each endomorphism resp. automorphism which contains a proper Sd_V class is an identity.

<u>Proof</u>. The assertion follows directly from Theorem 9.

Finally, we show that the condition "being a proper class" - claimed in the previous theorems - is essential.

<u>Theorem 10</u>. There is an infinite similarity f such that $\mathrm{dom}(f) \cap \mathrm{rng}(f) = \emptyset$.

<u>Proof</u>. It is shown in [V], ch. 5, § 2 that there is an endomorphic universe which is a semiset - let us denote it A.

Let F be such an endomorphism that $F: V \longleftrightarrow A$. Let further X be a countable class for which $X \subseteq V - A$. Then $F_1 = F \restriction X$ is a countable function which is a similarity (since F is a similarity) and $\mathrm{dom}(F_1) \cap \mathrm{rng}(F_1) = \emptyset$.

Let us enumerate F_1 by numbers from FN; denote the enumeration by G. Then $G: FN \longleftrightarrow F_1$. Since there is only a countable amount of all formulas of the language FL, we can enumerate them by FN, too.

From the definition of similarity it follows:

$$(\forall x_1,\dots,x_k \in \mathrm{dom}(F_1))\ \varphi_i(x_1,\dots,x_k) \equiv \varphi_i(F_1(x_1),\dots,F_1(x_k))$$

for each $\varphi_i \in FL$.

Let g be a prolongation of G and let $\alpha \in N - FN$ be such an element that $\mathrm{dom}(g''\alpha) \cap \mathrm{rng}(g''\alpha) = \emptyset$.

Denote

$$\psi_i H \equiv [(\forall x_1,\dots,x_k \in \mathrm{dom}(H))\ \varphi_i(x_1,\dots,x_k) \equiv \varphi_i(H(x_1),\dots$$
$$\dots,H(x_k))].$$

Put further $\alpha_1 = \max\{\beta \in \alpha;\ \psi_1(g''\beta)\}$. Then since for each $n \in FN$ the formula $\psi_1(g''n)$ holds ($g''n$ is a similarity), we obtain that $\alpha_1 \notin FN$. If we put

$$\alpha_{i+1} = \max\{\beta \in \alpha_i + 1;\ \psi_{i+1}(g''\beta)\},$$

then also $\alpha_{i+1} \notin FN$.

Hence $\{\alpha_i\}_{i \in FN}$ is a descending sequence of elements which do not belong to FN. This implies that there is $\bar{\alpha}$ such that $\bar{\alpha} \notin FN$ and $\bar{\alpha} \leq \alpha_i$, $i \in FN$.

Put now $f = g''\bar{\alpha}$. The function f is obviously the required similarity.

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