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Milan Studený

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ON THE DIFFERENTIATION THEOREM IN METRIC GROUPS

M. STUDENÝ

Abstract: Davies' example can be embedded isometrically into a compact metric group with a translation invariant metric. On a metric space with an almost uniform measure a weak type of Differentiation theorem holds for each measure.

Key words: Compact metric group with a translation invariant metric, almost uniform measure.

Classification: 28A15

In this note we deal with compact metric spaces on which there are two different measures agreeing on balls. We show that these measures can always be chosen mutually singular and that some of such spaces can be isometrically embedded into a compact metric group with a translation invariant metric. This also gives an example that the Differentiation theorem in convergence in measure need not hold in a compact metric group. Further, we prove that on metric spaces which admit an almost uniform measure, the weak Differentiation theorem of Christensen's type holds for each measure. Finally, we show the uniqueness of almost uniform measures.

We shall consider a separable metric space (M,ρ). Unless otherwise specified, all the measures considered will be positive, finite, countably additive Borel measures. If μ and ν

are measures on M then we denote by $d\nu_a/\mu$ the Radon-Nikodym derivative of the absolutely continuous part ν_a of ν with respect to μ. We shall denote the closed ball with centre at x and radius r by U(x,r). Further, we denote the set of all natural numbers by $\mathbb{N}$, the boundary of the set A by ∂A and the distance from the point x to the set A by $\rho(x,A)$. Finally, $U(G,\varepsilon) = \{x \in M, \rho(x,G) \leq \varepsilon\}$ where G is a subset of M.

1. Remarks to R.O. Davies example. R.O. Davies [4] has constructed a well-known example that two different measures on a compact metric space may agree on balls. Now we shall show that each such metric space admits also a pair of such measures which are mutually singular (or mutually absolutely continuous).

Proposition 1. Let $\mathcal{F}$ be some collection of Borel subsets of M such that there is a pair of different measures on M which agree on $\mathcal{F}$.

A) Then there is also a pair of mutually singular measures agreeing on $\mathcal{F}$.

B) Then there is also a pair of different and mutually absolutely continuous measures agreeing on $\mathcal{F}$.

Proof: Let $[\mu, \nu]$ be the original pair. We put $\lambda = \mu - \nu$; obviously λ is a bounded signed measure and according to the Hahn decomposition theorem (see [6]) $\lambda = \lambda^+ - \lambda^-$ where λ^+ and λ^- are mutually singular. Obviously $\lambda^+ B = \lambda^- B$ if $\mu B = \nu B$, hence $[\lambda^+, \lambda^-]$ has the properties from A). For the proof of B) we put $u = \mu + 2\nu$ and $v = \nu + 2\mu$.

It follows from the results of J.P.R. Christensen [3] that no compact Abelian group with a translation invariant metric admits a pair of different measures which agree on balls. Nevertheless, we show that the original example of R. O. Davies can be isometrically embedded into such group. In this connection we note that the following question is open. Can a separable metric space which admits a pair of different measures agreeing on balls be isometrically embedded into a Hilbert space?

We consider compact metric spaces M with the following special property.

For each $\varepsilon > 0$ there is a finite disjoint partition $\{A_i\}_{i\in k}$ of M such that

(1) a) $A_i \neq \emptyset$

b) $\operatorname{diam} A_i < \varepsilon$

c) if $i \neq j$ then $\rho(x,y) = \rho(z,y)$ whenever $x,z \in A_i$ and $y \in A_j$.

<u>Lemma 1</u>. Let M be a compact metric space with property (1). Then there is a sequence of real numbers $a_n \geq 0$, for which $\lim_{n\to\infty} a_n = 0$, such that there is an isometry of M into the space $Y = \prod_{n\in\mathbb{N}} [0,a_n]$ metrized by $d(x,y) = \sup\{|x_n - y_n|, n \in \mathbb{N}\}$.

Proof: It is easy to find a sequence of finite, disjoint partitions $\mathcal{A}_n$ such that a)-c) hold for $\varepsilon = n^{-1}$ and $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$ (i.e. if $B \in \mathcal{A}_{n+1}$ then there is $C \in \mathcal{A}_n$ such that $B \subset C$).

Let $n \in \mathbb{N}$; we choose $L \in \mathcal{A}_n$ and consider $B \in \mathcal{A}_{n+1}$ such that $B \subset L$. We put $f_{B,L}(y) = \min\{\rho(y,B), \operatorname{diam} L\}$. Clearly

$|f_{B,L}(y) - f_{B,L}(x)| \leq \rho(x,y)$ for $x,y \in M$. Conversely, if $x,y \in M$ then there is $[B,L]$ such that $\rho(x,y) = |f_{B,L}(x) - f_{B,L}(y)|$ (it is enough to find $n \in \mathbb{N}$ such that there are $B,C \in \mathcal{A}_{n+1}$, $B \cap C = \emptyset$, $x \in B$, $y \in C$ and $x,y \in L \in \mathcal{A}_n$).
We take a sequence of all pairs $[B_n,L_n]$, evidently $a_n = \operatorname{diam} L_n$ tends to zero. Obviously the mapping $x \longmapsto \{f_{B_n,L_n}(x)\}_{n \in \mathbb{N}}$ is an isometry.

Proposition 2. Let M be a compact metric space with property (1).
Then there is an isometry of M into a compact Abelian group with a translation invariant metric.

Proof: According to Lemma 1 it suffices to embed the space $\prod_{n \in \mathbb{N}} [0,a_n]$. Let C be the unit circle with the metric $k(x,y)$ equal to the angle between x and y. We embed isometrically every segment $[0,a_n]$ into the space $C_n = C$ metrized by $k_n = a_n \cdot \pi^{-1} k$. Since $a_n = \operatorname{diam} C_n$ tends to zero, $\prod_{n \in \mathbb{N}} C_n$ is a compact Abelian group with a translation invariant metric $h(x,y) = \sup\{k_n(x_n,y_n), n \in \mathbb{N}\}$.

The Davies' example has property (1), hence we can apply Proposition 2. We transfer a pair $[\mu, \nu]$ of mutually singular measures which agree on balls by an isometry into the mentioned group. Now we put $\mathcal{T} = \mu + \nu$ and get the following

Corollary 1. There is a compact metric Abelian group X with a translation invariant metric such that there is a pair of measures $[\mu, \mathcal{T}]$ on X for which

1) $\mu \leq \mathcal{T}$
2) $f(x) = \lim_{r \to 0+} [\mathcal{T} U(x,r)]^{-1} \cdot \mu U(x,r)$ exists for $\mathcal{T}$-almost all $x \in X$

3) $f(x) \neq d\mu/\sigma(x)$ for σ-almost all $x \in X$.

2. Weak differentiation. Recall that a locally finite non-zero measure on a separable metric space is called uniform if the measure of a ball depends only on its radius. According to the results of J.P.R. Christensen (see [2],[3]), there is no pair of different measures on M agreeing on balls, if M admits a uniform measure. Even the following weak form of the Differentiation theorem holds.

C) Whenever μ, ν are measures on M, $r > 0$ and $\mu B \leq \nu B$ for every ball B with diam $B < r$, then $\mu \leq \nu$.

To illustrate the connection of C) with the Differentiation theorem we remark that the validity of the Differentiation theorem for all measures on M is equivalent to the following property.

Whenever μ, ν are measures on M such that for each $x \in M$ there is a sequence of real numbers $r_n(x) > 0$ tending to zero for which $\mu U(x, r_n(x)) \leq \nu U(x, r_n(x))$, then $\mu \leq \nu$.

This leads naturally to the question whether on a metric space with a uniform measure the Differentiation theorem holds for each measure. This was answered by P. Mattila [5] who constructed a compact metric group with a translation invariant metric such that the Differentiation theorem does not hold for its Haar measure. But for Haar measures a weaker form of the Differentiation theorem holds (see [1]); this was generalized by P. Mattila [5] for almost uniform measures.

Definition 1. A locally finite Borel measure m on M is called almost uniform if there is $c \in (0,1]$ and a nondecreasing function $h: (0,r_o) \longmapsto (0,\infty)$ such that

(2) $c \cdot h(r) \leq mU(x,r) \leq h(r)$ whenever $x \in M$ and $r < r_o$.

We note that uniform measures are a special case of almost uniform measures but one can construct a compact metric space with an almost uniform measure on which there is no uniform measure. P. Mattila [5] proved that for an almost uniform measure m the Differentiation theorem holds in convergence in measure, i.e.

for each measure ν

(3) $[mU(x,r)]^{-1} \cdot \nu U(x,r)$ tends to $d\nu_a/m(x)$ in m-measure on every set of a finite measure.

The assumption that m is almost uniform is essential since

Proposition 3. There is a compact metric space admitting a uniform measure and measures μ, ν such that $[\nu U(x,r)]^{-1} \cdot \mu U(x,r)$ does not converge to $d\mu_a/\nu\,(x)$ in ν-measure.

Proof: A stronger result is given by Corollary 1.

We show that a weaker assumption of the existence of an almost uniform measure implies the weak Differentiation theorem C).

Proposition 4. If M admits an almost uniform measure then C) holds.

Proof: Assume that μ and ν are measures on M such that whenever $x \in M$ and $r \in (0,r_1]$, then

(4) $\mu U(x,r) \leq \nu U(x,r)$.

Let m be an almost uniform measure on M. Let $c \in (0,1]$ and let $h:(0,r_0) \longmapsto (0,\infty)$ be a nondecreasing function such that (2) holds. We put $\mathcal{T} = \nu - \mu$; evidently $\mathcal{T}$ is a bounded signed measure and according to the Hahn decomposition theorem (see [6]), $\mathcal{T} = w - u$ where w and u are mutually singular. We want to show $\mu \leq \nu$, i.e. $u = 0$. It is enough to prove that u is absolutely continuous with respect to w, we show that even $c^2 \cdot u \leq w$.

Since each open set is approximable by open sets G with $(\mu + \nu)(\partial G) = 0$, it is enough to prove this inequality only for such sets G.

Consider $r < \min\{r_0, r_1\}$. From (4) it follows

$$(5) \quad A^-(r) = \int_G [mU(y,r)]^{-1} \cdot uU(y,r)\,dm(y) \leq$$
$$\leq \int_G [mU(y,r)]^{-1} \cdot wU(y,r)\,dm(y) = A^+(r).$$

We define a function g_r on $M \times M$ by

$$(6) \quad g_r(x,y) = 1 \text{ whenever } \rho(x,y) \leq r$$
$$g_r(x,y) = 0 \text{ otherwise.}$$

By Fubini theorem and inequality (2) we get

$$A^+(r) = \int_M \int_G [mU(y,r)]^{-1} \cdot g_r(x,y)\,dm(y)\,dw(x) \leq$$
$$\leq c^{-1} \cdot \int_M [mU(x,r)]^{-1} \cdot m\{U(x\ ,r) \cap G\}\,dw(x).$$

Similarly we estimate $A^-(r)$ from below and by (5) get

$$(7) \quad c^2 \cdot \int_M [mU(x,r)]^{-1} \cdot m\{U(x,r) \cap G\}\,du(x) \leq$$
$$\leq \int_M [mU(x,r)]^{-1} \cdot m\{U(x,r) \cap G\}\,dw(x).$$

Letting r tend to zero, by Lebesgue´s dominated convergence theorem we get the desired inequality.

A uniform measure is unique up to a multiple by a positive constant (see [3] and also Corollary 3). If m is an almost uniform measure on M and $f:M \longmapsto [K_1, K_2]$ is a Borel measurable function (where K_1 and K_2 are positive constants) then the new measure $\nu B = \int_B f \, dm$ (for each Borel set B) is almost uniform, too. We show that almost uniform measures are unique within the framework of this equivalence.

<u>Lemma 2</u>. Let ν and μ be locally finite Borel measures on M. Assume that there are real numbers $c, d \in (0,1]$ and non-decreasing functions $h, s: (0, r_0) \longmapsto (0, \infty)$ such that for each $x \in M$ the following inequalities hold:

(8) $c \cdot h(r) \leq \mu U(x,r) \leq h(r)$ for $r < r_0$

(9) $c \cdot s(r) \leq \nu U(x,r) \leq s(r)$ for $r < r_0$

Then there is a positive K such that

(10) $cd \cdot K \cdot \mu \leq \nu \leq [cd]^{-1} \cdot K \cdot \mu$.

Proof: Denote by $\mathcal{F}$ the family of all open sets G such that $(\mu+\nu)(\partial G) = 0$ and there is $\epsilon > 0$ with $(\mu+\nu)U(G,\epsilon)$ finite. For such set G we put $r_1(G) = \min\{\epsilon, r_0\}$ and $v_G(r) = \int_M [\nu\{U(x,r)]^{-1} \cdot \nu U(x,r) \cap G\} \, d\mu(x)$ for $r < r_1(G)$. By the Lebesgue s dominated convergence theorem

(11) $\lim_{r \to 0_+} v_G(r) = \mu G$ for each $G \in \mathcal{F}$.

The rest of the proof will be divided into 3 parts.

A) For $G \in \mathcal{F}$ and $r < r_1(G)$

(12) $c \cdot [s(r)]^{-1} \cdot \nu G \leq [h(r)]^{-1} \cdot v_G(r) \leq [d \cdot s(r)]^{-1} \cdot \nu G$.

We take $t_r(x,y) = [s(r) \cdot h(r)]^{-1} \cdot g_r(x,y)$ (see (6)) and put $I = \int_{M \times G} t_r(x,y) d(\mu \times \nu)(x,y)$.

By Fubini theorem and inequality (8) we get

$c \cdot [s(r)]^{-1} \cdot \nu G \leq I \leq [s(r)]^{-1} \cdot \nu G$.

Similarly from (9) it follows

$I \leq [h(r)]^{-1} \cdot v_G(r) \leq d^{-1} \cdot I$.

B) If $G \in \mathcal{F}$ and $v_G(r) > 0$ for all sufficiently small r, then for these r

(13) $c\,[v_G(r)]^{-1}\cdot\nu G \leq [h(r)]^{-1}\cdot s(r) \leq [d\cdot v_G(r)]^{-1}\cdot \nu G.$

C) There is a positive number K such that (10) holds for each set $G \in \mathcal{F}$.

We choose $H \in \mathcal{F}$ such that $\mu H > 0$ and put $K = [\mu H]^{-1}\cdot \nu H$.

If r tend to zero, we see from (13) that

(14) $cK \leq \underline{\lim}_{r\to 0+} [h(r)]^{-1}\cdot s(r) \leq \overline{\lim}_{r\to 0+} [h(r)]^{-1}\cdot s(r) \leq d^{-1}\cdot K.$

Now we use (13) for general $G \in \mathcal{F}$ and get (10). If the assumptions in B) do not hold, it is $\nu G = 0$ by (12) and by (11) $\mu G = 0$. Now (10) follows because each open set of finite measure is approximable by sets from $\mathcal{F}$.

Corollary 2. If μ and ν are almost uniform measures on M then there are constants $K_1, K_2 > 0$ and a Borel measurable function $f: M \mapsto [K_1, K_2]$ such that $f = d\mu/\nu$.

Corollary 3. Let μ and ν be uniform measures on M. Then there is a positive number K such that $\nu = K\cdot\mu$.

R e f e r e n c e s

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Matematicko-fyzikální fakulta, Karlova universita, Sokolovská 83, 18600 Praha 8, Czechoslovakia

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