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Maciej M. Sysło

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ADJACENCY MATRIX EQUATIONS AND RELATED PROBLEMS: RESEARCH NOTES

MACIEJ M. SYSŁO

Abstract: The purpose of this paper is to discuss some generalizations of the notion of regularity in graphs which are derived from matrix equations involving adjacency matrices.

Key words and phrases: Adjacency matrix of a graph, degree sequence, Γ-degree sequence, matrix equation, regular and semi-regular graphs, spectral technique.

Classification: 05C50

Let $\mathcal{G}_k$ $(k \geq 1)$ denote the class of all graphs such that $G \in \mathcal{G}_k$ if there exists an integer constant r for which $A^k \mathbb{1} = r\mathbb{1}$ holds, where A is the adjacency matrix of G and $\mathbb{1}$ denotes the vector of 1's. Evidently, $G \in \mathcal{G}_1$ if and only if G is regular and $\mathcal{G}_2$ consists of regular and semi-regular bipartite graphs. For $k \geq 3$, we have $\mathcal{G}_k = \mathcal{G}_2$ if k is even and $\mathcal{G}_k = \mathcal{G}_1$ if k is odd.

In the class of directed graphs, only some partial solutions of the equation $A^k \mathbb{1} = r\mathbb{1}$ are known.

Some other matrix equations are also studied.

Let $G = (V,E)$ be a graph and $d_\Gamma(v) = \sum_{u \in \Gamma v} d(u)$, where Γv is the set of neighbours of v; $d_\Gamma(v)$ is called a Γ-degree of v. Evidently, there exists an integer r such that $d_\Gamma(v) = r$ for every vertex v of G if and only if $G \in \mathcal{G}_2$. In the last section, we consider a problem of finding when a sequence of n integers forms a Γ-degree sequence of some graph on n vertices.

1. Equation $A^k \mathbb{1} = r\mathbb{1}$ in the class of simple graphs

Let G be a simple graph (i.e., a symmetric graph without loops and multiple edges) and A denote its adjacency matrix. If $\mathbb{1}^T = (1,1,\dots,1)$ and $d^T = (d_1,d_2,\dots,d_n)$ denote respectively the vector of n ones and the degree sequence of G then we have evidently $A\mathbb{1} = d$.

A graph G is regular if $d(v) = r$ for every vertex v of G, that is if $A\mathbb{1} = r\mathbb{1}$. A graph G is called semiregular of degrees p, q if G is bipartite, each vertex has degree p or q, and each edge connects a vertex of degree p with a vertex of degree q.

Płonka [6] and Rama Chandran [8] have considered the following generalization of the regularity in graphs. Let

$$d_\Gamma(v) = \sum_{w \in \Gamma v} d(w),$$

where Γv denotes the set of neighbours of v. Then a graph G is Γ-regular (Płonka) or NDS regular (Rama Chandran) if there exists a positive integer r such that $d_\Gamma(v) = r$ for every vertex v of G. They proved the following theorem

Theorem 1.1 ([6] and [8]). A graph G is Γ-regular of degree r if and only if every connected component of G is either $\sqrt{r}$-regular or semiregular of degrees p, q, where $p \cdot q = r$. □

One can easily notice that G is Γ-regular of degree r if and only if $A^2\mathbb{1} = r\mathbb{1}$, since $A \cdot A\mathbb{1} = Ad$.

The main purpose of this section is to characterize simple graphs which satisfy

(1) $A^k\mathbb{1} = r\mathbb{1}$ for some positive integer r, where $k \geq 3$.

Let $\mathcal{G}_k$ $(k \geq 1)$ denote the class of all graphs which satisfy (1). Hence, $\mathcal{G}_1$ is the class of all regular graphs and $\mathcal{G}_2$ consists of Γ-regular graphs. In terms of graphs, G belongs to $\mathcal{G}_k$ if the number of paths of length k outgoing from a vertex v is the same for all vertices of G . One can easily check that $\mathcal{G}_2 \not\subset \mathcal{G}_3$ since $K_{2,3} \in \mathcal{G}_2$ and $K_{2,3} \notin \mathcal{G}_3$. Some relations between the classes $\mathcal{G}_k$, $k \geq 1$ can be obtained by simple matrix calculations.

<u>Proposition 1.1</u>. For every $k \geq 1$, $G \in \mathcal{G}_k \cap \mathcal{G}_{k+1}$ if and only if G is regular.

<u>Proof</u>. Let G be s-regular, i.e., $A\mathbb{1} = s\mathbb{1}$. Therefore $A^k\mathbb{1} = A^{k-1}A\mathbb{1} = A^{k-1}s\mathbb{1} = sA^{k-1}\mathbb{1} = \ldots = s^k\mathbb{1}$ and $A^{k+1}\mathbb{1} = s^{k+1}\mathbb{1}$. Otherwise, if $G \in \mathcal{G}_k \cap \mathcal{G}_{k+1}$ then there exist two positive integers r_1 and r_2 such that $A^k\mathbb{1} = r_1\mathbb{1}$ and $A^{k+1}\mathbb{1} = r_2\mathbb{1}$. Hence, $A^{k+1}\mathbb{1} = A \cdot A^k\mathbb{1} = A \cdot r_1\mathbb{1} = r_1A\mathbb{1}$ and $r_2\mathbb{1} = r_1A\mathbb{1}$. Since A is an integer matrix, we obtain $A\mathbb{1} = \frac{r_2}{r_1}\mathbb{1}$, where $\frac{r_2}{r_1}$ is a positive integer. Therefore $G \in \mathcal{G}_1$. $\square$

By similar calculations one can prove the following properties of $\mathcal{G}_k$ $(k \geq 1)$.

<u>Proposition 1.2</u>. If $G \in \mathcal{G}_k \cap \mathcal{G}_\ell$ then $G \in \mathcal{G}_{k+\ell}$. $\square$

<u>Proposition 1.3</u>. $\mathcal{G}_\ell \subset \mathcal{G}_{k\ell}$ for every $k = 2,3,\ldots$ and $\ell = 2,3,\ldots$. $\square$

<u>Proposition 1.4</u>. $G \in \mathcal{G}_k \cap \mathcal{G}_\ell$ if and only if $G \in \mathcal{G}_m$,

where $m = GCD(k,\ell)$.

Proof. If $G \in \mathcal{G}_k \cap \mathcal{G}_\ell$ then let us assume that $\ell \geq k$ and apply the Euclid algorithm to ℓ and k ($= k_1$):

$$\ell = m_1 k_1 + k_2, \text{ where } k_2 < k_1,$$
$$k_1 = m_2 k_2 + k_3, \text{ where } k_3 < k_2,$$
$$\vdots$$
$$k_p = m_{p+1} k_{p+1},$$

$k_{p+1} = GCD(k, \ell)$.

From the first equality, it follows that

$A^{\ell}\mathbb{1} = A^{m_1 k_1 + k_2}\mathbb{1} = A^{k_2}.A^{m_1 k_1}\mathbb{1} = r_{k_1}^{m_1} A^{k_2}\mathbb{1}$, where r_{k_1} is a constant such that $A^{k_1}\mathbb{1} = r_{k_1}\mathbb{1}$. Hence, $A^{k_2}\mathbb{1} = r/r_{k_1}^{m_1}\mathbb{1}$ and r/r_{k_1} is the integer where $A^{\ell}\mathbb{1} = r\mathbb{1}$. Therefore $G \in \mathcal{G}_{k_2}$. Iterating this process we obtain $G \in \mathcal{G}_{k_2}$, $G \in \mathcal{G}_{k_3}, \ldots$, $G \in$ $\in \mathcal{G}_{k_{p+1}} = \mathcal{G}_{GCD(k,\ell)}$.

Otherwise, if $G \in \mathcal{G}_{GCD(k,\ell)}$, then, by Proposition 1.3, $G \in \mathcal{G}_k$ and $G \in \mathcal{G}_\ell$, since there exist integers k' and ℓ' such that $k = GCD(k,\ell) \cdot k'$ and $\ell = GCD(k,\ell) \cdot \ell'$. $\square$

Corollary 1.1. $\mathcal{G}_k \subset \mathcal{G}_\ell$ if and only if $\mathcal{G}_k =$ $= \mathcal{G}_{GCD(k,\ell)}$. $\square$

Notice that the corollary is not equivalent to say that $k = GCD(k,\ell)$.

Let $\ell = 6$, and $k = 2,3$. The above propositions show that $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \subset \mathcal{G}_6$ and $\mathcal{G}_2 \cap \mathcal{G}_3 = \mathcal{G}_1$. We know also that $\mathcal{G}_3 \neq \mathcal{G}_6$, since otherwise $\mathcal{G}_2 \subset \mathcal{G}_3$ contradicting the observation that $K_{2,3} \in \mathcal{G}_2 - \mathcal{G}_3$. It suggests therefore that maybe $\mathcal{G}_6 = \mathcal{G}_2$ and, in general, $\mathcal{G}_{2\ell} = \mathcal{G}_2$, for $\ell \geq 2$.

Notice that if it holds, then, by the corollary, $\mathcal{G}_k = \mathcal{G}_1$ for every odd integer k, since $\mathcal{G}_k \subset \mathcal{G}_{2k} = \mathcal{G}_2$.

This conjecture was proved by Cvetković and Doob [1] by using spectral technique [2]. Notice that if for a symmetric 0-1 matrix A, there exist integer constants k and r such that $A^k \mathbb{1} = r\mathbb{1}$ then r is an eigenvalue of A^k and $\mathbb{1}$ is an eigenvector of A^k belonging to the eigenvalue r.

<u>Theorem 1.2</u>. For $k \geq 1$, $\mathcal{G}_k = \mathcal{G}_1$ if k is odd, and $\mathcal{G}_k = \mathcal{G}_2$ if k is even. □

It would be interesting to prove this theorem without using spectral technique, either by matrix calculations or by applying some graph-theoretic results.

2. <u>Other matrix equations</u>. Lam [3] has considered the following equation

$$A^k = \alpha I + \beta J, \tag{2}$$

where A is a 0,1-matrix, I denotes the identity matrix, J denotes the matrix with all entries equal to 1 and k, α, β are integers. He proved that if A satisfies (2) then it has constant row and column sums. Hence if A is the adjacency matrix of a simple graph G and satisfies (2), then G is regular. Therefore, the equation (2) with a symmetric matrix A produces only trivial solutions of (1).

Let $d^+_\Gamma(v) = d_\Gamma(v) + d(v)$ and $d^-_\Gamma(v) = d_\Gamma(v) - d(v)$. A graph G is <u>$\Gamma^+$-regular</u> (<u>$\Gamma^-$-regular</u>) of degree r if $d^+_\Gamma(v) = r$ ($d^-_\Gamma(v) = r$, respectively) for every vertex v of G. Płonka [7] has proved that G is Γ^+-regular if and only if G is regular. However, for Γ^--regular graphs, only some partial characterizations have been obtained [7] and degree sequences of such graphs have been studied in [4].

Notice that G is Γ^+-regular if there exists an integer r such that

$$(3) \qquad (A^2 + A)\mathbb{1} = r\mathbb{1}$$

holds, and G is Γ^--regular if there exists an integer s such that

$$(4) \qquad (A^2 - A)\mathbb{1} = s\mathbb{1}$$

holds.

One may again generalize Γ^+-regular graphs and ask for graphs which satisfy the equation

$$(5) \qquad (A^k + A^{k-1} + \dots + A)\mathbb{1} = r\mathbb{1}$$

for some integer r. In terms of elements of graphs, G satisfies (5) if and only if the number of paths of length not greater than k outgoing from a vertex of G is the same for all vertices of G. We conjectured that only regular graphs satisfy (5) for every $k \geq 1$, and it has been settled in the affirmative by B. McKay [5] using again spectral technique.

3. <u>Equation</u> $A^k\mathbb{1} = r\mathbb{1}$ <u>in the class of digraphs</u>. Let D = = (V,E) be a digraph, where $E \subset V \times V$. One may now ask for the solution of the equations, (1), (3), (4) and (5) when A is an adjacency matrix of a digraph, that is, when A is an arbitrary 0-1 matrix.

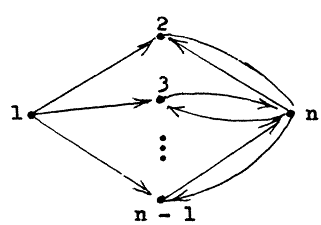
Let $\mathcal{D}_k$ denote the class of all digraphs such that $D \in \mathcal{D}_k$ if its adjacency matrix satisfies (1). There are many different types of digraphs in $\mathfrak{D}_k$ and a complete characterization of $\mathfrak{D}_k$ seems to be very difficult.

B. McKay [5] found the complete solution of (1) in the class of strongly connected digraphs. A digraph D is <u>regular cyclically r-partite</u> if V can be decomposed $V = V_1 \cup V_2 \cup \dots \cup V_r$, where $V_i \cap V_j = \emptyset$, $V_i \neq \emptyset$ $(i, j = 1,2,\dots,r)$, and there exist integers $m_1, m_2, \dots, m_r$ such that for every $v \in V_i$ and $i = 1,2,\dots,r$ we have

$$|E \cap (\{v\} \times V_j)| = \begin{cases} m_i, & \text{if } j \equiv i+1 \pmod r \\ 0, & \text{otherwise.} \end{cases}$$

Every regular digraph is regular cyclically 1-partite and regular cyclically 2-partite digraphs are exactly semiregular bipartite.

<u>Theorem 3.1</u> (B. McKay). If D is strongly connected then $D \in \mathfrak{D}_k$ if and only if D is regular cyclically r-partite for some $r | k$.



D_n

Fig. 1

Figure 1 shows a digraph D_n which is not strongly connected and $A^2\mathbb{1} = (n-2)\mathbb{1}$ for every $n \geq 3$. Every strongly connected member of $\mathfrak{D}_k$ can be used for a similar construction.

4. **Γ-graphical sequences.** It is one of the classical problems in the graph theory to ask when a sequence of n integers $\mathcal{d} = (d_1, d_2, \ldots, d_n)$ is graphical, i.e., constitutes the sequence of vertex degrees of a graph, which is usually assumed to be from a restricted family of graphs.

In a similar way, we say that $\mathcal{d}' = (d_1', d_2', \ldots, d_n')$ is Γ-graphical if there exists a simple graph G such that $\mathcal{d}'$ is the sequence of Γ-degrees of G. The problem of finding when a sequence of n integers is Γ-graphical seems to be very hard. We present here only some comments.

First notice that if $\mathcal{d}'$ is Γ-graphical then

$$\sum_{i=1}^{n} d_i' = \sum_{i=1}^{n} d_i^2 = \Big(\sum_{i=1}^{n} d_i\Big)^2 - 2\sum_{i<j} d_i d_j,$$

where $\mathcal{d} = (d_1, d_2, \ldots, d_n)$ is the degree sequence of a graph having Γ-degree sequence $\mathcal{d}'$.

Therefore, if $\mathcal{d}'$ is Γ-graphical then it has the following properties:

(a) $R = \sum_{i=1}^{n} d_i'$ is even,

(b) R can be decomposed into n squares of integers d_1, $d_2, \ldots, d_n$,

(c) $\mathcal{d} = (d_1, d_2, \ldots, d_n)$ is graphical.

It is not difficult to find a non Γ-graphical sequence $\mathcal{d}'$ which satisfies (a) - (c). For instance, let $\mathcal{d}' = (4,2,2,2)$. In this case $R = 10$ and there is only one decomposition of 10 into 4 squares, namely $10 = 2^2 + 2^2 + 1^2 + 1^2$. The sequence (2,2,1,1) is the degree sequence of P_4 which however has the Γ-degree sequence (3,3,2,2), different from the sequence we started with. Notice that in the class of multigraphs with loops, the sequence (2,2,1,1) has a realization (consisting of P_3 and a loop) with the given Γ-degree sequence.

Therefore we have to add the following requirement

(d) the sequence $\mathfrak{d}$ has a realization with Γ-degree sequence $\mathfrak{d}'$.

Notice that for every even integer R there exists a Γ-graphical sequence $\mathfrak{d}' = (d_1', d_2', \ldots, d_n')$ such that $\sum_{i=1}^{n} d_i' = R$. It is sufficient to take $n = R$ and $d_i' = 1$ for every $i = 1,2, \ldots, n$; $\mathfrak{d}'$ corresponds to the graph on n vertices and n/2 separate edges. One may ask now which even integers have connected realizations. For instance if $R = 8$ then we have

$R = 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2 + 1^2,$

$R = 2^2 + 2^2,$

$R = 2^2 + 1^2 + 1^2 + 1^2 + 1^2,$

and only the first decomposition corresponds to a graph. Therefore, 8 has no connected realization.

It can be easily proved that if $\mathfrak{d}'$ is Γ-graphical then the number of 1's in $\mathfrak{d}'$ is even and they can be removed from $\mathfrak{d}'$, since $\mathfrak{d}'$ is Γ-graphical if and only if $\mathfrak{d}'$ without an even number of 1's is Γ-graphical.

Elements of $\mathfrak{d}'$ equal 2 may occur in two configurations shown in Fig. 2, which however cannot be recognized from a Γ-degree sequence. If G is assumed to be connected and has at least four vertices, then only 2(b) may occur, that is, every vertex of Γ-degree 2 is pendant and adjacent to a vertex w such that $d(w) = 2$ and $d'(w) \geq 3$.

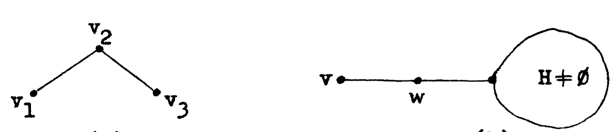


Fig. 2

We have been able to enumerate all graphs with Γ-degree sequence $\mathbb{d}' = (d_1', d_2', \ldots, d_n')$ satisfying $d_i' \leq 5$ $(i = 1,2, \ldots, n)$. For instance, if $d_i' \leq 4$ $(i = 1,2,\ldots,n)$ then every component of a graph with Γ-degree sequence $\mathbb{d}'$ is one of the graphs: a simple path, a simple cycle, $K_{1,4}$, and P_4 with an edge attached to a non-pendant vertex of P_4.

Let us return to the conditions (a) - (d). The first of them can be trivially checked. The second one, however, is not trivial at all. First, some integers, usually of small value, cannot be decomposed into a fixed number of squares of integers. For instance, no sequence of five integers with the sum equal to 18 is Γ-graphical since 18 cannot be decomposed into 5 squares of integers. There are several such results in the number theory (see for instance [9]), which can be used to eliminate some sequences of integers as non Γ-graphical. Secondly, if the sum R is decomposable into squares then the number of decompositions can be very large. In this case we have to find whether among the decompositions there exists at least one which gives a graphical sequence compatible with the given Γ-degree sequence. However, neither in the number theory nor in the theory of graphs there exists a result dealing simultaneously with the decomposition of an even integer into squares of integers and graphical realizations of the sequence of these integers.

We have not succeeded (even restricting our attention to trees) in finding for Γ-degree sequences counterparts of d-invariant operations which can be applied to degree sequences or to graphs, for instance, the removal of a vertex of maximum degree (as in the theorem of Havel and Hakimi) and switching two mutually non-adjacent edges (see for instance [10]).

Figure 3 shows that non-isomorphic trees can have the same Γ-degree sequence.

Fig. 3.

It is known that if $\mathfrak{d} = (d_1, d_2, \ldots, d_n)$, $d_i \geq 1$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} d_i = 2(n-1)$ then every realization of $\mathfrak{d}$ in the class of connected simple graphs is a tree. It does not hold however for Γ-degree sequences. For instance, the sequence of five $4'$s corresponds to $K_{1,4}$ and to C_5. We conjecture that if a Γ-degree sequence $\mathfrak{d}'$ has a tree realization then each such a realization of $\mathfrak{d}'$ has the same number of pendant vertices. It is easy to check that this is not true for disconnected realizations of Γ-degree sequences.

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Institute of Computer Science, University of Wrocław,
ul. Przemyckiego 20, 51151 Wrocław, Poland

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