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FUNCTIONAL TIGHTNESS, Q-SPACES AND τ-EMBEDDINGS

A. V. ARHANGEL'SKII

Abstract: A mapping $f:X \longrightarrow Y$ is called strictly τ-continuous if for every set $A \subset X$ such that $|A| \leq \tau$ there exists a continuous mapping $g:X \longrightarrow Y$ such that $f(x) = g(x)$ for all $x \in A$. The weak functional tightness $t_m(X)$ of X is countable if every strictly $\aleph_0$-continuous real-valued function on X is continuous. A particular case of Theorem 4: $t_m(X)$ is countable if and only if the space $C_p(X)$ of all continuous real-valued functions on X in the topology of pointwise convergence is realcompact. It follows that if $C_p(X)$ and $C_p(Y)$ are homeomorphic and $t_m(X)$ is countable then $t_m(Y)$ is also countable. Other corollaries and related results are obtained.

Key words: Tightness, functional tightness, realcompactness, pointwise convergence, τ-continuity, strict τ-continuity, type G_τ, type G_δ, density.

Classification: 54A25, 54C40, 54D60

Notations and terminology. In this article the symbols X, Y, Z denote topological spaces, τ, λ denote infinite cardinals, $Cl_X(A)$ (or $\overline{A}$) is the closure of a set A in a space X, $|A|$ is the cardinality of the set A, $d(X) = \min\{|A| : A \subset X \text{ and } \overline{A} = X\}$ is the density of the space X, βX denotes the Čech-Stone compactification of a completely regular Hausdorff space X. The restriction of a mapping $f:X \longrightarrow Y$ to $A \subset X$ is denoted by $f|_A$. A mapping $f:X \longrightarrow Y$ is called τ-continuous if for every set $A \subset X$ such that $|A| \leq \tau$ the mapping

$f|_A: A \longrightarrow Y$ is continuous. A space is called functionally closed (or realcompact or a -space) if it is homeomorphic to a closed subspace of the space R^{τ} where R is the usual space of real numbers and τ is a cardinal. The space of all continuous real-valued functions on X with the topology of pointwise convergence is denoted by $C_p(X)$. All spaces under consideration are assumed to be completely regular and Hausdorff. Bicompact spaces are called compact spaces.

The tightness t(X) of a space X is the smallest infinite cardinal τ such that the following condition is satisfied: if $x \in X$, $A \subset X$ and $x \in \overline{A}$, then there exists a set $B \subset A$ for which $|B| \leq \tau$ and $x \in \overline{B}$.

1. <u>Functional tightness</u>

<u>Definition 1</u> ([1]). The functional tightness $t_o(X)$ of a space X is the smallest infinite cardinal number τ such that every τ-continuous real-valued function on X is continuous.

<u>Definition 2</u>. The weak tightness $t_c(X)$ of a space X is the smallest infinite cardinal number τ such that the following condition is satisfied: if a set $A \subset X$ is not closed in X, then there exist a point $x \in \overline{A} \setminus A$, a set $B \subset A$ and a set $C \subset X$ such that $x \in \overline{B}$, $\overline{B} \subset \overline{C}$ and $|C| \leq \tau$.

<u>Proposition 1</u>. For every space X we have the following: (a) $t_o(X) \leq t_c(X)$; (b) $t_c(X) \leq t(X)$; (c) $t_c(X) \leq d(X)$.

Proof. The last two assertions are obvious.

(d) We put $\tau = t_c(X)$. Let f be any τ-continuous real-valued function on X and let P be a subset of X. We have to show that $f(\overline{P}) \subset \overline{f(P)}$. For the set $A = \{x \in \overline{P} : f(x) \in \overline{f(P)}\}$ we have the following: $P \subset A \subset \overline{P} = \overline{A}$. If $A = \overline{P}$ then we are done. Suppose that $A \neq \overline{P}$. Then $\overline{A} \neq A$. Since $t_c(X) = \tau$, there exist $x \in \overline{A} \setminus A$, $B \subset A$ and $C \subset X$ such that $x \in \overline{B}$, $B \subset \overline{C}$ and $|C| \leq \tau$. Take any open set U in R such that $f(B) \subset U$ and put $C_0 = \{c \in C : f(c) \in U\}$, $C_1 = C \setminus C_0$. Suppose that $b \in \overline{C_1}$. Then since f is τ-continuous and $|C_1| \leq \tau$, one has $f(b) \in \overline{f(C_1)}$. But $f(C_1) \cap U = \emptyset$ and U is open in R. Hence $f(b) \notin U$. From $f(B) \subset U$ it follows that $b \notin B$. Thus $\overline{C_1} \cap B = \emptyset$. Since $B \subset \overline{C} = \overline{C_0} \cup \overline{C_1}$ we have $B \subset \overline{C_0}$ and $x \in \overline{B} \subset \overline{C_0}$. Since f is τ-continuous, from $x \in \overline{C_0}$ and $|C_0| \leq \tau$ it follows that $f(x) \in \overline{f(C_0)}$. But $f(C_0) \subset U$. Therefore $f(x) \in \overline{U}$. By regularity of R, $\overline{f(B)} = \cap \{\overline{U} : U$ is open in R and $U \supset f(B)\}$. It follows that $f(x) \in \overline{f(B)} \subset \overline{f(A)}$. Now, by definition of the set A, $f(A) \subset \overline{f(P)}$. Hence $f(x) \in \overline{f(P)}$. Note that $x \in \overline{A} = \overline{P}$. Applying again the definition of A we conclude that $x \in A$ which is in contradiction with the choice of x. Thus $A = \overline{P}$, which means that the function f is continuous and that $t_0(X) \leq \tau$.

<u>Corollary 1</u>. For every space X, $t_0(X) \leq d(X)$.

<u>Example 1</u>. The weak tightness (and the functional tightness) does not always coincide with the tightness.

Indeed, the tightness of the space R^c is equal to $c = 2^{\aleph_0}$. At the same time, R^c is separable (see [4], ch. 2, № 380) and hence $t_0(R^c) = t_c(R^c) = \aleph_0$.

<u>Example 2</u>. Let $T(\omega_1) = \{\alpha : \alpha \leq \omega_1\}$ be the space of all ordinal numbers not exceeding the first uncountable ordinal number ω_1 in the order topology. Put $f(\alpha) = 0$ for all $\alpha < \omega_1$ and $f(\omega_1) = 1$. The function $f : T(\omega_1) \to R$ thus defined is obviously $\aleph_0$-continuous and not continuous. It follows that the functional tightness of the space $T(\omega_1)$ is

equal to $\aleph_1$. Hence $t_c(T(\omega_1)) = \aleph_1$. The space $T(\omega_1)$ is homeomorphic to a closed subspace of the separable space R^c. We have:

$$t_o(T(\omega_1)) = \aleph_1 > \aleph_o = t_o(R^c)$$

and

$$t_c(T(\omega_1)) = \aleph_1 > \aleph_o = t_c(R^c).$$

This means that neither the functional tightness, nor the weak tightness are monotonic with respect to closed subspaces. This is in a sharp contrast with the behaviour of tightness which is monotonic with respect to arbitrary subspaces.

It may seem that the concepts of τ-continuity and of functional tightness express a certain idea in the most appropriate way. However, it is possible to modify these two concepts in a rather curious and quite useful way.

<u>Definition 3</u>. A mapping $f:X \longrightarrow Y$ will be called strictly τ-continuous if for every set $A \subset X$ such that $|A| \leq \tau$ there exists a continuous mapping $g:X \longrightarrow Y$ such that $g|_A = f|_A$ i.e. $f(x) = g(x)$ for all $x \in A$.

<u>Definition 4</u>. The weak functional tightness (or minitightness) $t_m(X)$ of a space X is the smallest infinite cardinal number τ such that every strictly τ-continuous real-valued function on X is continuous.

Clearly, every strictly τ-continuous function is τ-continuous. Hence the definitions 1 and 4 imply the following assertion.

<u>Proposition 2</u>. For any space X, $t_m(X) \leq t_o(X)$.

In connection with the non-monotonicity of the functional tightness observed in the example 2, the following result is of interest.

Theorem 1. The following conditions are equivalent to each other for arbitrary space X:

(a) $t_m(Y) \leq \tau$ for every $Y \subset X$;

(b) $t_o(Y) \leq \tau$ for every $Y \subset X$;

(c) $t(X) \leq \tau$.

Proof. Clearly, from (c) follows (b) and from (b) follows (a). Let us derive (c) from (a).

Take any $A \subset X$. We shall show that the set $P = \bigcup \{ \overline{B} : B \subset A$ and $|B| \leq \tau \}$ is closed in X. Assume the contrary, fix $y \in \overline{A} \setminus P$ and put $Y = P \cup \{y\}$. We are going to verify that the function $f: Y \longrightarrow R$ defined by: $f(x) = 0$ for every $x \in P$ and $f(y) = 1$, - is strictly τ-continuous. Let $C \subset Y$ and $|C| \leq \tau$. Put $C_o =$ $= C \cap P$. From $|C_o| \leq \tau$ and $C_o \subset P$ it follows easily (see for example [4], ch. 2, №106) that $y \notin \overline{C_o}$. Thus there exists a continuous real-valued function g on Y such that $g(y) = 1$ and $g(x) = 0$ for all $x \in C_o$. Obviously $f|_C = g|_C$ which implies that the function f is strictly τ-continuous. But $t_m(Y) \leq \tau$. Hence the function f is continuous - which is in contradiction with $y \in \overline{P}$, $f(y) = 1$ and $f(P) = \{0\}$.

In the case of compact Hausdorff spaces Theorem 1 can be considerably strengthened.

Proposition 3. Let $f: X \longrightarrow Y$ be a factor mapping and $f(X) = Y$. Then:

(a) $t_o(Y) \leq t_o(X)$;

(b) $t_m(Y) \leq t_m(X)$.

Proof. Let g be a τ-continuous (strictly τ-continuous) function on Y. Then $h = g \circ f$ is τ-continuous (correspondingly, strictly τ-continuous) function on X. Letting $\tau = t_o(X)$ (correspondingly, $\tau = t_m(X)$) we conclude in both cases that h is continuous. From $h = g \circ f$ and the fact that f is a factor mapping, it follows that g is continuous. Hence $t_o(Y) \leq \tau = t_o(X)$ (correspondingly, $t_m(Y) \leq \tau = t_m(X)$).

We shall need also the following

Proposition 4. Let X be an infinite set, $<$ - a well-ordering on X such that $|\{y \in X : y < x\}| = |X|$ for some $x \in X$ and let X be given the topology generated by this well-ordering. Then $t_m(X) = |X|$ (moreover, $t_o(X) = |X|$).

Proof. Clearly, $t_m(X) \leq |X|$. Put $X_x = \{y \in X : y < x\}$ for $x \in X$ and let $\tau = |X|$. By our assumptions the set $P = \{x \in X : |X_x| = \tau\}$ is not empty. Let $a = \min P$.

Case I. τ is regular. Put $f_a(x) = 0$ for $x < a$ and $f_a(x) = 1$ for $a \leq x$. It is easy to verify that the function $f_a : X \to R$ is strictly λ-continuous for every $\lambda < \tau$ (as $cf(\tau) = \tau$) and that f_a is not continuous (indeed, $a \in \overline{X_a}$, $f(X_a) = \{0\}$ and $f(a) = 1$). Hence $\lambda < t_m(X)$ for every $\lambda < \tau$, i.e. $\tau \leq t_m(X)$.

Case 2. τ is not isolated. For any $\lambda < \tau$ there exists $b \in X$ such that $|X_b| = \lambda^+$ (where λ^+ is the first cardinal greater than λ) and $|X_x| \leq \lambda$ for each $x < b$. Put $f_b(x) = 0$ for $x < b$ and $f_b(x) = 1$ for $x \geq b$. Obviously the function $f_b : X \to R$ is strictly λ-continuous and not continuous. Hence $\lambda < t_m(X)$ i.e. $\tau \leq t_m(X)$.

Theorem 2. For every compact Hausdorff space X the following two conditions are equivalent:

(a) $t(X) \leq \tau$;

(b) $t_m(Y) \leq \tau$ for each closed subspace Y of the space X.

Proof. Let us deduce (a) from (b). Assume that $t(X) > \tau$. As X is compact, there exists then a free sequence $\{x_\alpha : \alpha < \tau^+\}$ in X of the length τ^+ (see [1] or [4], ch. III, 142). It was shown in [6] that the closed subspace $F = \{x_\alpha : \alpha < \tau^+\}$ of the space X can be mapped continuously onto the space $T(\tau^+) = \{\alpha : \alpha \leq \tau^+\}$. This mapping is automatically perfect. Then by Proposition 3 $t_m(F) \geq t_m(T(\tau^+))$ and by Proposition 4 $t_m(T(\tau^+)) = \tau^+$. Since F is closed in X, $t_m(F) \leq \tau$. Hence we have:

$$\tau \geq t_m(F) \geq t_m(T(\tau^+)) = \tau^+,$$

- a contradiction.

Remark 1. Theorems 1 and 2 can be reformulated in the following manner:

(a) $t(X) = \sup\{t_m(Y) : Y \subset X\}$ for every X;

(b) $t(X) = \sup\{t_m(F) : F \subset X$ and F is closed in X$\}$, for every compact Hausdorff space X.

Problem 1 (unsolved). Find a space for which weak functional tightness and functional tightness are not equal.

In connection with this problem the following result is of some interest.

Theorem 3. Let X be a normal space and τ - an infinite cardinal. Then:

(a) every τ-continuous real-valued function f on X is strictly τ-continuous, and

(b) $t_m(X) = t_o(X)$.

Proof. Obviously, (a) implies (b). Let us prove (a). Let $A \subset X$ and $|A| \leq \tau$. By Corollary 1, $t_o(\bar{A}) \leq d(A) \leq |A| \leq \tau$. The function $f|_{\bar{A}}$ is τ-continuous on the space $\bar{A}$. Hence $f|_{\bar{A}}$ is continuous on $\bar{A}$. As X is normal, there exists a continuous function $g: X \to R$ such that $g|_{\bar{A}} = f|_{\bar{A}}$. Then $f|_A = g|_A$. Thus we have proved that the function f is strictly τ-continuous.

Remark 2. This argument shows that if $t_m(X) \leq \tau$ and every closed subspace F of X such that $d(F) \leq \tau$ is C-embedded in X, then $t_o(X) = t_m(X)$.

2. τ-embeddings, the Hewitt number and m_τ-spaces. We shall say that a set A is of type G_τ in X if there exists a family γ of open sets in X such that $\bigcap \gamma = A$ and $|\gamma| \leq \tau$. A subset $A \subset X$ is said to be τ-embedded in X, if for each $x \in X \setminus A$ there exists a set $P \subset X$ of type G_τ in X such that $x \in P \subset X \setminus A$.

Proposition 5. If $X \subset Y \subset Z$ where X is τ-embedded in Y and Y is τ-embedded in Z then X is τ-embedded in Z.

The proof is obvious.

Proposition 6. If a space X is τ-embedded in some compact Hausdorff extension bX of X, then X is also τ-embedded in βX - the Čech-Stone compactification of the space X.

For the proof it is sufficient to recall that βX can be mapped continuously onto bX in such a way that $f^{-1}(bX \setminus X) = \beta X \setminus X$.

Definition 5. Put $q(X) = \min\{\tau \geq \aleph_0 : X$ is τ-embedded in $\beta X\}$. We call the cardinal $q(X)$ the Hewitt number of the space X.

It is well known (for example, see [4], ch. IV, №82) that the inequality $q(X) \leq \aleph_0$ just means that X is a Q-space in the sense of E. Hewitt (is realcompact in other terminology).

Proposition 7. If $X \subset Y$ and X is closed in Y, then $q(X) \leq q(Y)$.

This assertion can be easily deduced from Proposition 6.

We recall that a network of a space X (in a space X) is any family S of subsets of X such that every open subset U of X can be represented as the union of some subfamily of the family S (this concept was introduced in (8),(9)).

The following generalization of the concept of network plays a role in the argument to follow.

Definition 6. A family $\mathcal{P}$ of sets is called network of a family Q of sets, if every $A \in Q$ is the union of a subfamily of the family $\mathcal{P}$ i.e. if for each $U \in Q$ and each $x \in U$ there exist $V \in \mathcal{P}$ such that $x \in V \subset U$.

Thus a network of a space is just a network of the topology of this space. We remind that a subset $F \subset X$ is called canonical closed subset of the space X if $F = \overline{U}$ for some open set U in X.

Definition 7. A space X is called an m_τ-space if the family of all sets of type G_τ in X is a network of the family of all canonical closed subsets of X.

Clearly, X is an m_τ-space if $\tau \geq |X|$. Hence the following definition makes sense.

Definition 8. Put $m(X) = \min\{\tau \geq \aleph_0 : X$ is an m_τ-space$\}$. We call the cardinal $m(X)$ modality of X. If $m(X) \leq \aleph_0$ we shall say that X is a moscow space.

Remark 3. By the upper $\aleph_0$-modification of the space X we understand the space for which the family of all sets of type G_δ (i.e. of type $G_{\aleph_0}$) in X serves as a base. Obviously, X is a moscow space if and only if the closure of every open set in X is open with respect to the upper $\aleph_0$-modification of X.

Example 3. If in the space X every canonical closed subset is of type G_δ then X is a moscow space. Hence the space R^A of all mappings of a set A in R (in the topology of pointwise convergence - i.e. in the topology of product) is a moscow space. If L is any real linear space, then the space L' of all real linear functionals on L in the topology of pointwise convergence is also a moscow space. Indeed, L' is (canonically) homeomorphic to the space R^A where A is any Hamel's basis of L. Observe that among moscow spaces, there are all dyadic compacts and all perfectly $\varkappa$-normal spaces in the sense of Ščepin [5]. It follows (see [5]) that the product of any family of metric spaces is a moscow space. Besides, $C_p(X)$ is always a moscow space (as $\overline{C_p(X)} = R^X$ or by perfect $\varkappa$-normality of $C_p(X)$).

The next assertion generalizes a result of A.Ch. Chigogidze [3].

Proposition 8. Let $X \subset Y$, $q(X) \leq \tau$, $\overline{X} = Y$ and $m(Y) \leq \tau$.

Then X is τ-embedded in Y.

Proof. Let B be any compact Hausdorff space such that $Y \subset B$. There exists a continuous mapping $f: \beta X \to B$ such that $f(x) = x$ for every $x \in X$ (see [4], ch. IV, №18). Let $y \in Y \setminus X$. As $\overline{X} = Y$ and the compact space $f(\beta X)$ is closed in the Hausdorff space B, we have: $f(\beta X) \supset Y$. Hence $|f^{-1}(y)| \geq 1$.

Case I. Let $|f^{-1}(y)| = 1$, i.e. $f^{-1}(y) = \{z\}$, for some $z \in \beta X$. Since $f^{-1}(X) = X$ and $y \notin X$, we obtain: $z \in \beta X \setminus X$. The condition $q(X) \leq \tau$ permits us to fix a family γ of open sets in βX such that $\bigcap \gamma = \{z\}$ and $|\gamma| \leq \tau$. Then $f^{-1}(y) \subset \bigcap \gamma$ which implies that $y \in \bigcap \lambda$, where $\lambda = \{B \setminus f(\beta X \setminus U) : U \in \gamma\}$, $|\lambda| \leq |\gamma| \leq \tau$ and each set $V \in \lambda$ is open in B.

Clearly, $\bigcap \lambda = B \setminus \bigcup \{f(\beta X \setminus U) : U \in \gamma\} = B \setminus f(\bigcup \{\beta X \setminus U : U \in \gamma\}) \subset B \setminus f(X) = B \setminus X$, as $f(X) = X$ and $(\bigcap \gamma) \cap X = \emptyset$. We put $P = (\bigcap \lambda) \cap Y$. Then $y \in P \subset Y \setminus X$ and the set P is of type G_τ in Y.

Case II. If $|f^{-1}(y)| \geq q$. Fix $z_1, z_2 \in f^{-1}(y)$, $z_1 \neq z_2$. Take neighborhoods Oz_1, Oz_2 of the points z_1, z_2 in βX such that $C\ell_{\beta X}(Oz_1) \cap C\ell_{\beta X}(Oz_2) = \emptyset$ and let $V_i = Oz_i \cap X$, $F_i = C\ell_Y(V_i)$, $i = 1,2$.

Clearly $z_i \in C\ell_{\beta X}(V_i)$. Since f is continuous and $f(V_i) = V_i$, $y = f(z_i)$, $i = 1,2$, we have: $y \in F_1 \cap F_2$.

Also, $F_1 \cap F_2 \cap X \subset C\ell_{\beta X}(Oz_1) \cap C\ell_{\beta X}(Oz_2) = \emptyset$, which means that $F_1 \cap F_2 \subset Y \setminus X$.

Take an open set W_i in Y such that $W_i \cap X = V_i$, $i = 1,2$. From $\overline{X} = Y$ it follows that $F_i = C\ell_Y(W_i)$, hence F_i is a canonical closed set in Y. By $m(Y) \leq \tau$ there exists a family γ_i of open sets in Y such that $y \in \bigcap \gamma_i \subset F_i$ and $|\gamma_i| \leq \tau$,

$i = 1,2$. Then for $\gamma = \gamma_1 \cup \gamma_2$ we have: $|\gamma| \leq \tau$ and $y \in \bigcap \gamma \subset (\bigcap \gamma_1) \cap (\bigcap \gamma_2) \subset F_1 \cap F_2 \subset Y \setminus X$.

3. Main results: theorems on duality

Proposition 9. If $t_m(X) \leq \tau$ then $C_p(X)$ is τ-embedded in R^X.

Proof. Let $g \in R^X \setminus C_p(X)$. From $t_m(X) \leq \tau$ it follows that there exists a set $A \subset X$ such that $|A| \leq \tau$ and $g|_A \neq f|_A$ for every $f \in C_p(X)$. We consider the restriction mapping $\pi : R^X \to R^A$, - i.e. $\pi(h) = h|_A$ for all $h \in R^X$. The set $Y = \pi(C_p(X))$ is τ-embedded in the space R^A since $|A| \leq \tau$ (all one-point sets in R^A are of type G_τ). From $\pi(g) = g|_A \notin Y$ it follows that there exists a set P_o of type G_τ in R^A such that $\pi(g) \in P_o \subset R^A \setminus Y$. Then the set $P = \pi^{-1}(P_o)$ is of type G_τ in R^X and $g \in P \subset R^X \setminus C_p(X)$.

Proposition 10. If $C_p(X)$ is τ-embedded in R^X, then $t_m(X) \leq \tau$.

Proof. Let g be a strictly τ-continuous function on X and P a set of type G_τ in R^X such that $g \in P$. Evidently one can find a set $A \subset X$ such that $|A| \leq \tau$ and $g \in \{f \in R^X : f|_A = g|_A\} \subset P$. Since g is strictly τ-continuous there exists $h \in C_p(X)$ such that $h|_A = g|_A$. Then $h \in P$. Thus $P \cap C_p(X) \neq \emptyset$ for every set P of type G_τ in R^X such that $g \in P$. Because $C_p(X)$ is τ-embedded in R^X we have $g \in C_p(X)$ - i.e. the function g is continuous. Hence $t_m(X) \leq \tau$.

Theorem 4. For any space X, $t_m(X) = q(C_p(X))$.

Proof. Put $\tau = t_m(X)$. By Proposition 9, $C_p(X)$ is τ-

embedded in R^X. As the space R^X is realcompact [4], R^X is $\aleph_0$-embedded in $\beta(R^X)$. It follows (by Proposition 5) that $C_p(X)$ is τ-embedded in $\beta(R^X)$. This implies, by Proposition 6 that $C_p(X)$ is τ-embedded in $\beta(C_p(X))$. Hence $q(C_p(X)) \leq \tau$ i.e. $q(C_p(X)) \leq t_m(X)$. Put $\lambda = q(C_p(X))$. We have: $m(R^X) = \aleph_0 \leq \lambda$ and $\widetilde{C_p(X)} = R^X$. By Proposition 8, $C_p(X)$ is λ-embedded in R^X. We now apply Proposition 10 and arrive at the conclusion: $t_m(X) \leq \lambda$. Thus $t_m(X) \leq q(C_p(X))$ and, finally, $q(C_p(X)) = t_m(X)$.

Corollary 2. The weak functional tightness of the space X is countable if and only if the space $C_p(X)$ is realcompact.

Corollary 3. If the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic then $t_m(X) = t_m(Y)$.

Corollary 4. If a space X is normal then $t_0(X) = q(C_p(X))$.

This follows from Theorems 3 and 4. A particular case of Theorem 4 is the following assertion: the functional tightness of a normal space X is countable if and only if the space $C_p(X)$ is realcompact. This assertion is a combination of some results of A.V. Arhangel'skii and A.Ch. Chigogidze published in [2, 4] and [3].

Corollary 5. If X and Y are normal spaces and $C_p(X)$ is homeomorphic to $C_p(Y)$ then $t_0(X) = t_0(Y)$.

Problem 2. Let $C_p(X)$ and $C_p(Y)$ be homeomorphic. Is it true then that $t(X) = t(Y)$? Is this true under the additiona assumption that the spaces X and Y are normal?

Corollary 6. It is always true that $q(X) \leq t_m(C_p(X))$.

Proof. By Theorem 4, $t_m(C_p(X)) = q(C_p(C_p(X)))$. But X is homeomorphic to a closed subspace of the space $C_p(C_p(X))$ (see [7]). Hence (by Proposition 7)

$q(X) \leq q(C_p(C_p(X))) = t_m(C_p(X))$.

Corollary 7. If the weak functional tightness of $C_p(X)$ is countable then X is realcompact.

After the concepts and results contained in this article were exposed in my course on the topology of function spaces at the Moscow University and in answer to my question V.V. Uspenskij has shown that $t_o(C_p(X)) \leq q(X)$ for every space X (see [10]). From this beautiful result and Corollary 6 it follows that always $t_m(C_p(X)) = q(X)$. Thus we have

Corollary 8. Let $C_p(X)$ and $C_p(Y)$ be homeomorphic. Then $q(X) = q(Y)$. In particular, if X is realcompact, then Y is realcompact.

4. The case of mappings into any space

It is easy to prove the following assertions (compare with [4], ch. IV, №128).

Proposition 11. Let $X = \prod\{X_\alpha : \alpha \in A\}$ and $Y = \prod\{Y_\alpha : \alpha \in A\}$ be topological products, $X_\alpha \subset Y_\alpha$ for all $\alpha \in A$ and let τ be an infinite cardinal. Then:

a) if X_α is τ-embedded in Y_α for each $\alpha \in A$ then X is τ-embedded in Y;

b) if $q(X_\alpha) \leq \tau$ for all $\alpha \in A$ then $q(X) \leq \tau$.

Proposition 12. If every τ-continuous (strictly τ-continuous) real-valued function on X is continuous then each

τ-continuous (strictly τ-continuous) mapping f of the space X into arbitrary space Y is continuous.

For the proof it suffices to represent Y as a subspace of the space R^Y and to consider the compositions of the mapping f with projections $R^Y \longrightarrow R$.

With the help of Proposition 12 we easily get the following generalization of Proposition 9:

<u>Proposition 13</u>. If $t_m(X) \leq \tau$ then $C_p(X,Y)$ is τ-embedded in Y^X.

Here, as usually, $C_p(X,Y)$ denotes the space of all continuous mappings of X into Y endowed with the topology of pointwise convergence.

From Propositions 13, 11, 5 and 6 we get the following result:

<u>Theorem 5</u>. If $t_m(X) \leq \tau$ and $q(Y) \leq \tau$ then $q(C_p(X,Y)) \leq \tau$.

Theorems 4 and 5 imply

<u>Corollary 9</u>. For any X and Y, $q(C_p(X,Y)) \leq \max\{q(Y), q(C_p(X))\}$.

<u>Corollary 10</u>. If $t_m(X) \leq \aleph_0$ and Y is realcompact then $C_p(X,Y)$ is realcompact.

<u>Corollary 11</u>. If Y and $C_p(X)$ are realcompact then $C_p(X,Y)$ is also realcompact.

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