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A NOTE ON PARTIAL DERIVATIVES OF CONVEX FUNCTIONS

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Abstract: An elementary construction shows that for any bounded continuous function on R there exists a convex function g on R^2 such that $\frac{\partial g}{\partial y}(x,0) = f(x)$.

Key words: Convex function, partial derivative.

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R.M. Dudley ([1], p. 172) constructed a convex function g on R^2 such that $\frac{\partial g}{\partial y}(x,0)$ is nowhere differentiable with respect to x. His construction is simple but somewhat intricate.

In this note we present a quite elementary construction which gives a sharper result. We show that for any bounded continuous function f on R there exists a convex function g on R^2 such that $\frac{\partial g}{\partial y}(x,0) = f(x)$.

Note that the Dudley's construction has the advantage that it gives a function g which is smooth on R^2.

We present our construction in a more general setting. Let H be a real Hilbert space. If f is a function on H and $x,v \in H$, then we put

$$\partial_v f(x) = \lim_{h \to 0} (f(x+hv) - f(x))h^{-1}$$

and

$$D_v f(x) = \lim_{h \to 0_+} (f(x+hv) - f(x))h^{-1}.$$

Thus $\partial_v f(x)$ is the usual directional derivative and $D_v f(x)$ is the one-sided directional derivative. Clearly $\partial_v f(x)$ exists iff $D_{-v} f(x) = - D_v f(x)$.

Let $X \subset H$ be a closed subspace of codimension 1 and let $u \in X^{\perp}$ be a unit vector.

Proposition. Let f be a bounded upper semicontinuous function on X. Then there exists a continuous convex function g on H such that

(i) $D_u\, g(x) = f(x)$ for $x \in X$

and

(ii) If f is continuous on X, then $\partial_u g(x) = f(x)$ for $x \in X$.

Proof. Let $|f(x)| < M$ for $x \in X$. For $t \in X$, $x \in X$ and $y \in R$ put

$a_t(x+yu) = 2(x,t) - \|t\|^2 + y\, f(t) = \|x\|^2 - \|x-t\|^2 + y\, f(t)$.

The functions a_t are affine on H, $a_t(t) = \|t\|^2$ and $\partial_u a_t(t) = f(t)$. Put $g(z) = g_f(z) = \sup_{t \in X} a_t(z)$.

Since $a_t(x+yu) \leq \|x\|^2 + |y|\, M$,

g is a locally upper bounded convex function and consequently it is continuous. Obviously $g(x) = \|x\|^2$ for $x \in X$ and therefore $D_u g(x) \geq D_u a_x(x) = f(x)$ for $x \in X$. Let $x \in X$ and $\varepsilon > 0$ be fixed. Then there exists a $\delta > 0$ such that $f(t) < f(x) + \varepsilon$ for $t \in X$, $\|t-x\| < \delta$. Since for $\|t-x\| \geq \delta$, $y > 0$, we have $a_t(x + yu) \leq \|x\|^2 - \delta^2 + yM$, we conclude that for all sufficiently small $y > 0$ and all $t \in X$

$$a_t(x+yu) \leq \|x\|^2 + y(f(x) + \varepsilon).$$

Consequently $D_u g(x) \leq f(x)$.

Now suppose that f is continuous and $x \in X$ is fixed. We have

$D_u g_f(x) = f(x)$ and since clearly $g_f(x+yu) = g_{-f}(x-yu)$, we obtain $D_{-u} g_f(x) = D_u g_{-f}(x) = -f(x)$.

Corollary. Let $F \subset R$ be a closed set. Then there exists a convex function h on R^2 such that $\frac{\partial h}{\partial y}(x,0)$ exists iff $x \notin F$.

Proof. Put $H = R^2$, $X = \{(x,0);\ x \in R\}$ and $f(x,0) = C_F(x)$. Let g_f be the function from Proposition. Now it is clearly sufficient to put $h(x,y) = \max (g_f(x,y), x^2)$.

Reference

[1] R.M. DUDLEY: On second derivatives of convex functions, Math. Scand. 41(1977), 159-174.

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