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BIPARTITE INTERSECTION GRAPHS

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ABSTRACT

The well-known interval graphs are intersection graphs of a finite set of distinct intervals. A corresponding bi-interval graph $G = (V,E)$ is formed by taking two families of intervals, R and S, and defining $V = R \cup S$ and $E = \{xy: x \in R, y \in S, x \cap y \neq \emptyset\}$. The characterizations of interval graphs by Lekkerkerker and Boland are modified to obtain two criteria for bi-interval graphs. We observe that every bipartite graph can be represented as a bipartite intersection graph of some star.

Key words: Interval graph, bipartite intersection graph, bi-interval graph, bi-subtree graph.

AMS (MOS) subject classification: 05C75.

1. INTRODUCTION.

Our purpose is to introduce a bipartite version of the notion of intersection graphs. Some expected results are derived together with an unexpected one. All sets will be finite and the graph theoretic terminology of [4] is followed.

The idea of using the intersections of a family of sets to define the adjacencies of a graph is so natural that it arose independently in a number of areas in connection with both pure and applied problems (see Roberts [8]). Formally, if S is a set and $\mathcal{F} = \{F_i\}$ is a family of distinct, nonempty subsets of S, the *intersection graph* $\Omega(\mathcal{F})$ is the graph $G = (V,E)$ with point set $V = \mathcal{F}$ and $F_iF_j \in E$ if and only if $F_i \cap F_j \neq \emptyset$ and $i \neq j$. If G is a graph such that $G \cong \Omega(\mathcal{F})$, then $\mathcal{F}$ is called a *representation* of G. It is easy to show (Marczewski [6]) that every graph has such a representation.

Since the class of intersection graphs is so broad, interest has focused on cases in which restrictions are placed on the nature of the set S or the family $\mathcal{F}$. We now recall some of the basic definitions and results on two types of intersection graphs. If S is the real line and each $F_i \in \mathcal{F}$ is an interval, then $\Omega(\mathcal{F})$ is called an *interval graph*. There are several characterizations of interval graphs. The one we will generalize is due to Lekkerkerker and Boland [5]. First we require some definitions. A *chord* of a cycle is a line joining two points which are not adjacent along the cycle. A graph in which every cycle of length greater than 3 has a chord is called *chordal*. Three points u,v,w in a graph G form an *asteroidal triple* if each pair of them is joined by a path which contains no neighbors of the third point.

THEOREM A (Lekkerkerker and Boland). *A graph G is an interval graph if and only if it is chordal and contains no asteroidal triples.*

An interval graph may be alternatively defined as an intersection graph of a family of subgraphs of a path. Viewed in this way the natural generalization is to consider the intersection graph of a family of subtrees as a tree, called a *subtree graph*. They have been characterized independently by Buneman [1], Gavril [2], and Walter [9].

THEOREM B (Buneman, Gavril, Walter). *A graph is a subtree graph if and only if it is chordal.*

We now introduce a bipartite analog to the above. Given a set S and a family $\mathcal{F}$ of distinct subsets of S, partition $\mathcal{F}$ into two subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$. The *bipartite intersection* graph of $\mathcal{F}$ with respect to the given partition, written $\Omega(\mathcal{F}_1,\mathcal{F}_2)$, is the graph $G = (V,E)$ with $V = \mathcal{F}$ and $F_iF_j \in E$ if $F_i \in \mathcal{F}_1$, $F_j \in \mathcal{F}_2$ and $F_i \cap F_j \neq \emptyset$. That is, $\Omega(\mathcal{F}_1,\mathcal{F}_2)$ is that graph obtained from $\Omega(\mathcal{F})$ by removing those edges between points in $\mathcal{F}_1$ and between points in $\mathcal{F}_2$. Since every graph is an intersection graph, it is obvious that every bipartite graph is a bipartite intersection graph.

As a natural example of a bipartite intersection graph, consider the subdivision graph SG of a graph G. It is readily apparent that $SG \cong \Omega(V,E)$. For another example, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two partitions of a set into distinct parts. Clearly $\Omega(\mathcal{F}_1 \cup \mathcal{F}_2)$ is a bipartite intersection graph and one might ask if all bipartite graphs arise this way. It is easy to see that a bipartite graph G is the intersection graph of the parts of two partitions of some set if and only if G has no isolated points.

2. BI-INTERVAL GRAPHS.

Since the most intensively studied intersection graphs are the interval graphs, it is natural to investigate the bipartite version. If $\mathcal{F}$ is a family of intervals, partitioned into subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\Omega(\mathcal{F}_1,\mathcal{F}_2)$ will be called a *bi-interval graph*. These will be characterized by a result analogous to Theorem A, but to do so we must define modifications of the notions of chordal graph, asteroidal triple, and simplicial point.

A bipartite graph will be called *bi-chordal* if it has no induced cycle of length greater than or equal to six. A *bi-asteroidal triple* is a set of points $\{u,v,w\}$ of a bipartite graph such that between any pair of them, there exists a path which is not adjacent to any point in the neighborhood of the third point. A somewhat unconventional notion of deleting an edge is the following: if $e = \{u,v\}$, then consider $G - \{u,v\}$. Thus when an edge is deleted, all edges adjacent to it are deleted as well. By link(e), we shall mean the subgraph induced by $N(u) \cup N(v) - \{u,v\}$. An edge for which link(e) is complete bipartite is now called a *simplicial edge*. We will need to distinguish two types of simplicial edges, which are analogous to the strongly and weakly simplicial points of Lekkerkerker and Boland [5]. A *strongly simplicial edge* e has G - link(e) connected. The remaining simplicial edges are *weakly simplicial*. Two edges are *apart* if the subgraph induced by their points is $2K_2$.

We can now state the characterization theorem.

THEOREM 1. *A bipartite graph is a bi-interval graph if and only if it is bi-chordal and contains no bi-asteroidal triples.*

Proof: It is convenient to consider the partition of the family of intervals to be defined by coloring each interval black or blue.

The necessity of both conditions is readily established. Suppose G contains a cycle of length 6 or more, $u_1w_1u_2w_2\ldots u_kw_ku_1$. Let U_i and W_i be the intervals corresponding to u_i and w_i respectively. Then the black interval U_1 and the blue interval W_2 must be disjoint, and likewise W_1 and U_2. But U_3 is joined to U_1 by the pairwise overlapping chain of intervals $W_3\ldots U_kW_k$, so it must be the case that in this chain, either a black interval overlaps W_2, or a blue interval overlaps U_2, in either case giving a chord in the cycle. A similar argument suffices for the other condition.

For the converse, we first need to make some observations. Clearly, under the hypotheses, G cannot contain three mutually apart strongly simplicial edges, since they would necessarily give rise to a bi-asteroidal triple. A result of Golumbic and Goss [3], however, guarantees the existence of some simplicial edges. Specifically they prove:
(1) In a connected, bi-chordal graph with no two apart edges, every point is incident with a simplicial edge; (2) If G is a connected, bi-chordal graph containing two apart edges and if S is a minimal separating set of points for which at least two components of G - S are nontrivial, then each nontrivial component of G - S contains a simplicial edge.

The demonstration of the sufficiency can now be accomplished by modifying the proof of Theorem A as in [5], replacing simplicial point by simplicial edge throughout, and similarly for the other corresponding concepts, remembering that an edge is represented by a pair of intervals, one black and one blue. ■

The determination of forbidden subgraphs for bi-interval graphs is again exactly parallel to the corresponding derivation of Lekkerkerker and Boland for interval graphs. It differs only in that now only one infinite family is needed.

COROLLARY 1. *A bipartite graph G is a bi-interval graph if and only if it does not contain as an induced subgraph any of the four graphs of Figure 1 or any cycle* C_n, $n \geq 6$.

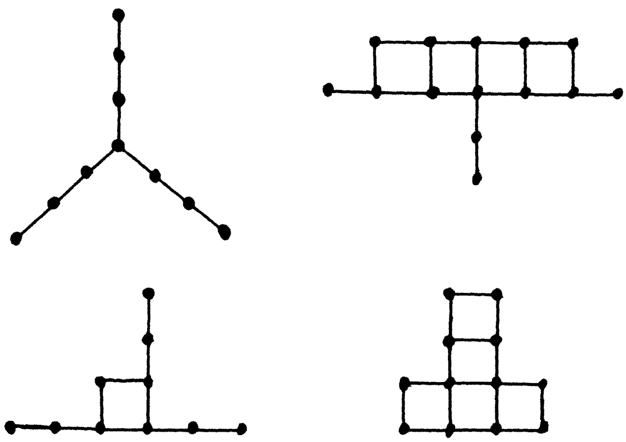

Figure 1. Four Forbidden Subgraphs for Bi-interval Graphs.

3. BI-SUBTREE GRAPHS.

A *bi-subtree graph* is the bipartite intersection graph of subtrees of some tree. Looking at Theorem B, and keeping in mind the ease in which the bipartite version of Theorem A was proved, one might well conjecture that bi-chordal graphs are precisely the bi-subtree graphs. Surprisingly this is not even close to being true.

EXAMPLE. Let $G = K_{1,3}$ and with the endpoints labeled 1,2,3. Let $T_i = V(G) - \{i\}$ for $i = 1,2,3$ and let $\mathfrak{T}_1 = \{\{1\},\{2\},\{3\}\}$, $\mathfrak{T}_2 = \{T_1,T_2,T_3\}$. So $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are sets of subtrees of G, but clearly $\Omega(\mathfrak{T}_1,\mathfrak{T}_2)$ is not bi-chordal.•

THEOREM 2. *Every bipartite graph is a bi-subtree graph of some star* $K_{1,n}$.

Proof: Let G be a bipartite graph with bipartition X,Y. Form the graph H by adding to G an edge between every pair of points in Y. McMorris and Shier [7] showed that such graphs, called split graphs, are characterized by having representations as intersection graphs of subtrees of some star $K_{1,n}$. Let $\mathfrak{T}_1$ be the set of subtrees corresponding to points in X and $\mathfrak{T}_2$ the set of subtrees corresponding to points in Y. Clearly $G \cong \Omega(\mathfrak{T}_1,\mathfrak{T}_2)$. ■

Obviously the converse of Theorem 2 holds as every bipartite intersection graph is bipartite.

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