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ON VON NEUMANN REGULAR RINGS, VII

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Abstract: Generalizations of quasi-injective modules and p-injective modules over continuous rings, noted pQ(fQ)-injectivity, PQF-injectivity, MUP-injectivity, are introduced to study von Neumann regular rings. Sufficient conditions for von Neumann regularity are given. Left continuous regular, left Noetherian, quasi-Frobeniusean and semi-simple Artinian rings are characterized.

Key words: p-injective, f-injective, pQ-injective, fQ-injective, PQF-injective, MUP-injective, von Neumann regular, continuous regular, V-rings, CS-rings, Noetherian, quasi-Frobeniusean, semi-simple Artinian.

Classification: 16A12, 1A230, 16A32, 16A36, 16A40, 16A52.

Throughout, A represents an associative ring with identity and A-modules are unitary. Z, J will denote respectively the left singular ideal and the Jacobson radical of A. A von Neumann regular ring A may be characterized by any one of the following conditions: (a) every left A-module is flat; (b) every left A-module is p-injective (f-injective). Note that if I is a p-injective left ideal of A, then A/I is a flat left A-module. As usual, an ideal of A will always mean a two-sided ideal. A is called fully idempotent (resp. fully left idempotent) if every ideal (resp. left ideal) of A is idempotent. Following [3], A is called a left V-ring if every simple left

A-module is injective. It is now well-known that there is no inclusion between the classes of arbitrary von Neumann regular rings and V-rings [5]. However, they coincide in the commutative case (I. Kaplansky). Y. Utumi introduced left continuous rings as a generalization of left self-injective rings (cf. [9]). The notion of continuity was extended to modules and studied by various authors. Quasi-injective modules, intermediate between injective and continuous modules, have been extensively studied (cf. for example, the bibliography of [3], [4],[6]). Recall that (1) A left A-module M is quasi-injective iff any left A-homomorphism of every left submodule of M into M extends to an endomorphism of ${}_AM$; (2) ${}_AM$ is continuous iff (a) every complement left submodule of M is a direct summand of ${}_AM$ and (b) every left submodule which is isomorphic to a direct summand of ${}_AM$ is a direct summand of ${}_AM$; (3) ${}_AM$ is p-injective (f-injective) iff, for any principal (resp. finitely generated) left ideal I of A, any left A-homomorphism $g:I \longrightarrow M$, there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. It is easy to see that there is no inclusion between the classes of quasi-injective left modules and p-injective (f-injective) left modules.

We now introduce the following definitions.

Definition 1. A left A-module M is called MUP-injective if, for any complement left ideal C of A, $a \in A$, any left A-monomorphism $g:Ca \longrightarrow M$, there exists $y \in M$ such that $g(ca) = cay$ for all $c \in C$.

Definition 2. A left A-module M is called pQ-injective (resp. (1) fQ-injective ; (2) PQF-injective) if, for any p-injective (resp. (1) f-injective; (2) projective) left submodule

N of M, any left A-homomorphism of N into M may be extended to an endomorphism of ${}_AM$.

MUP-injective left modules generalize injective left modules and p-injective modules over left continuous rings. PQF-injectivity, pQ-injectivity and FQ-injectivity generalize quasi-injectivity.

A is called a left MUP (resp. pQ, fQ, PQF)-injective ring if ${}_AA$ is MUP (resp. pQ, fQ, PQF)-injective.

Since a finitely generated p-injective left ideal of A is a direct summand of ${}_AA$, then any left Noetherian ring is left pQ and fQ-injective.

<u>Proposition 1</u>. <u>If</u> A <u>is semi-prime, then any simple left</u> A<u>-module is MUP-injective. Consequently, MUP-injectivity does not imply</u> p<u>-injectivity</u>.

<u>Proof</u>. Let M be a simple left A-module, C a complement left ideal of A, $a \in A$, $g: Ca \longrightarrow M$ a non-zero left A-monomorphism. Then g is an isomorphism (since ${}_AM$ is simple) which implies that Ca is a minimal left ideal of A. Since A is semi-prime, $Ca = Ae$, where $e = e^2 \in A$. If $y = g(e) \in M$, then for any $a \in A$, $g(ae) = aeg(e) = aey$ which proves that ${}_AM$ is MUP-injective. Since simple modules over arbitrary semi-prime rings need not be p-injective (even in the commutative case), then the last assertion of Proposition 1 follows.

We see from Proposition 1 that MUP-injectivity does not imply Tp-injectivity considered in [17]. The converse is not true either (otherwise, by [17, Theorem 3] and Theorem 3 below, any simple regular ring would be left self-injective !). MUP-injective left modules need not be left continous (cf. Theorems

3 and 10). Recall that a left A-module is non-singular iff its singular submodule is zero.

Lemma 2. Let A be a left MUP-injective ring. Then

(1) Any non-singular finitely generated left ideal of A is a direct summand of ${}_A A$;

(2) A/Z is von Neumann regular and Z = J;

(3) For any $a \in A$, $\ell(a) = o$ iff a is right invertible;

(4) Every left or right A-module is divisible.

Proof. (1) Let $P = Ab$, $o \neq b \in A$, be a non-singular principal left ideal. If C is a non-zero complement left ideal such that $L = C \oplus \ell(b)$ is an essential left ideal, $g: Cb \longrightarrow A$ the left A-homomorphism defined by $g(cb) = c$ for all $c \in C$, then g is a monomorphism, whence there exists $y \in A$ such that $c = g(cb) = cby$ for all $c \in C$. Then $C \subseteq \ell(b-byb)$ and therefore $L \subseteq \ell(b-byb)$ which yields $b = byb$ (since ${}_A P$ is non-singular). This proves that Ab is generated by an idempotent. Now if $F = Ab + Aa$ is non-singular, $b, a \in A$, then $Ab = Ae$, where $e = e^2 \in F$ and $F = Ae + Aa(1-e) = Ae + Aw$, where $Aa(1-e) = Aw$, $w = w^2 \in F$. If $v = (1-e)w$, then $v \in F$, $wv = w$, $v^2 = v$ and $Aw = Av$ yielding $F = Ae + Av = A(e+v)$ which is again generated by an idempotent. (1) then follows by induction on the number of generators.

(2) The proof of (1) shows that A/Z is a von Neumann regular ring. Now if $z \in Z$, $a \in A$. $L = \ell(za)$, then with $w = 1-za$, $L \cap \ell(w) = o$ implies $\ell(w) = o$. If $g: Aw \longrightarrow A$ is the left A-homomorphism defined by $g(bw) = b$ for all $b \in A$, then there exists $v \in A$ such that $b = g(bw) = bwv$ for all $b \in A$. In particular, $1 = wv = (1-za)v$ which implies that $z \in J$. Then J/Z is contained in the Jacobson radical of A/Z which is zero, whence Z = J.

(3) If $a \in A$ such that $ac = 1$ for some $c \in A$, then $\mathcal{l}(a) = o$. Conversely, let $\mathcal{l}(a) = o$. If $f: Aa \longrightarrow A$ is the map $ba \longrightarrow b (b \in A)$, then there exists $d \in A$ such that $1 = f(a) = ad$.

(4) If c is a non-zero-divisor of A, $cd = 1$ for some $d \in A$ by (3), and $c = cdc$ implies $dc = 1$. For any left A-module M, $M = cdM \subseteq cM \subseteq M$ which implies $M = cM$ and similarly, any right A-module is divisible.

Lemma 2 enables us to have a nice characterization of division rings. As usual, a left (right) ideal of A is called reduced if it contains no non-zero nilpotent element.

Corollary 2.1. The following conditions are equivalent:

(1) A *is a division ring*;

(2) A *is a prime left self-injective ring containing a non-zero reduced left ideal*;

(3) A *is a prime left MUP-injective ring containing a non-zero reduced left ideal*.

Proof. Obviously, (1) implies (2) and (2) implies (3).

Assume (3). Let I be a non-zero reduced left ideal, $o \neq b \in I$. Then ${}_A Ab$ is non-singular. By Lemma 2(1), Ab is a direct summand of ${}_A A$ which implies that A is an integral domain [13, Proposition 6]. Now by Lemma 2(3), any non-zero element of A is right invertible which proves that (3) implies (1).

Theorem 3. *The following conditions are equivalent*:

(1) A *is left continuous regular*;

(2) *Every left* A-*module is MUP-injective*;

(3) *Every essential left ideal of* A *is MUP-injective*;

(4) *Any ideal or complement left ideal of* A *is a MUP-injective left* A-*module*;

(5) A *is a left non-singular left MUP-injective ring whose complement left ideals are finitely generated*.

Proof. Obviously, (1) implies (2) which, in turn, implies (3).

Assume (3). Let I be either a principal or a complement left ideal of A, K a complement left ideal such that $L = I \oplus K$ is an essential left ideal. If $f: I \longrightarrow L$ is the natural injection, there exists $u \in L$ such that $f(b) = bu$ for all $b \in I$. If $u = c + k$, $c \in I$, $k \in K$, then $b = b(c+k)$ implies $b = bc$, whence $c = c^2$. Since $I \subseteq Ac$, then $I = Ac$ which proves A left continuous regular and (3) implies (4).

Assume (4). If $z \in Z$, $i: Az \longrightarrow Z$ the canonical injection, then $z = i(z) = zw$ for some $w \in Z$. Now $Az \cap \ell(w) = 0$ implies $z = 0$, whence $Z = 0$. By Lemma 2(2), A is von Neumann regular. If C is a complement left ideal, $j: C \longrightarrow C$ the identity map, there exists an idempotent $e \in C$ such that $C = Ae$. Thus (4) implies (5).

(5) implies (1) by Lemma 2(2).

The next result shows that arbitrary p-injective modules need not be pQ-injective.

Theorem 4. *The following conditions are equivalent*:

(1) A *is a left Noetherian ring whose* p*-injective left modules are injective*;

(2) *Every* p*-injective left* A*-module is* pQ*-injective*.

Proof. Obviously, (1) implies (2).

Assume (2). Let M be a p-injective left A-module, ${}_AH$ the injective hull of ${}_AM$. Then $B = {}_AM \oplus {}_AH$ is p-injective and therefore pQ-injective. If $i: M \longrightarrow B$ is the natural injection,

$j: M \to H$ and $K: H \quad B$ the inclusion maps, since ${}_AB$ is pQ-injective, there exists a left A-homomorphism $g: B \to B$ such that $gkj = i$. If $p: B \quad M$ is the natural projection, then $pgkj = pi$ = identity map on M. Thus $f = pgk$ is a left A-homomorphism of H into M such that fj = identity map on M. This proves that ${}_AM$ is a direct summand of ${}_AH$, whence $M = H$ is injective. Since we know that any direct sum of injective left A-modules is p-injective and hence injective, which proves that A is a left Noetherian [4, Theorem 20.1]. Thus (2) Implies (1).

It is well-known that A is left hereditary iff the sum of any two injective left A-modules is injective. Since a commutative ring A is regular iff every simple A-module is p-injective, the next corollary then follows.

Corollary 4.1. Let A *be a commutative ring. Then*

(1) A *is Noetherian hereditary iff the sum of any two* p-*injective* A-*modules is* p-*injective and* pQ-*injective*;

(2) A *is semi-simple Artinian iff* p-*injective* A-*modules coincide with* pQ-*injective* A-*modules*.

The next theorem may be similarly proved.

Theorem 5. *The following conditions are equivalent*:

(1) A *is left Noetherian*;

(2) *Every* f-*injective left* A-*module is injective*;

(3) *Every* f-*injective left* A-*module is* fQ-*injective*.

Corollary 5.1. *If* A *is a left MUP-injective ring such that every* f-*injective left* A-*module is* fQ-*injective, then* A *is left Artinian*.

Proof. By Lemma 2 (2) and Theorem 5, A/J is semi-simple Artinian. Since A is left Noetherian, then J = Z is nilpotent which implies A semi-primary. Therefore A is left Artinian by [4, Proposition 9.12].

We are now in a position to give some new characteristic properties of quasi-Frobeniusean rings . Since it is well-known that A is semi-simple Artinian iff every left A-module is quasi-injective, then the next result shows that PQF-injectivity effectively generalizes quasi-injectivity.

Theorem 6. *The following conditions are equivalent*:

(1) A *is quasi-Frobeniusean*;

(2) *Every left* A*-module is PQF-injective*;

(3) *The direct sum of any projective left* A*-module and any injective left A-module is quasi-injective*.

(4) *The direct sum of any projective left* A*-module and any injective left* A*-module is PQF-injective*;

(5) A *is a left* f*-injective ring such that every* f*-injective left* A*-module is fQ-injective*.

Proof. Since A is quasi-Frobeniusean iff every projective left A-module is injective [4, Theorem 24.20], then (1) implies (2) and (3).

Obviously, either (2) or (3) impliès (4).

Assume (4). The proof of Theorem 4 shows that any projective left A-module is injective. Then (4) implies (5) by [4, Theorem 24.20].

(5) implies (1) by [4, Theorem 24.20] and Theorem 5.

Left WP-rings (weak p-injective) and left CPP-rings are considered in [14] and [12] respectively.

Applying [14, Lemma 1.1] and [12, Theorem 2], we get

Proposition 7. *The following conditions are equivalent* for a left PQF-injective ring A:

(1) A is von Neumann regular;

(2) A is a left WP-ring;

(3) A is a left CPP-ring.

It may be noted that PQF-injectivity does not imply p-injectivity and the converse is not true either.

Lemma 2(2) implies the next result.

Proposition 8. (1) If A is a left PQF-injective ring such that for any complement left ideal C of A, $a \in A$, ${}_ACa$ is projective, then A is a left MUP-injective regular ring.

(2) A primitive left MUP-injective ring is von Neumann regular.

(3) A MUP-injective left or right V-ring is von Neumann regular (cf. [5, Query (b)]).

(4) If A is left Tp-injective such that every complement left ideal is an ideal, then A/Z is von Neumann regular (this extends [17, Theorem 9]).

At this point, we may note that there is no inclusion between the class of p-injective modules and any of the four classes of modules which we have introduced at the beginning of this note. We now consider a generalization of continuous modules. Following [2], a left A-module M is called CS if every complement left submodule is a direct summand of ${}_AM$. Indeed, CS-modules generalize even quasi-continuous modules effectively (cf. [1]). If ${}_AM$ is CS, then for complement left submodule N, ${}_AN$ is CS [2, Proposition 2.2] and every left A-homomorphism of N into M extends to an endomorphism of ${}_AM$. But if ${}_AM$ is CS and D a left submodule of M which is CS, then D is not necessarily a complement submodule of ${}_AM$. This motivates the following class of rings: Write "A satisfies (*)" if, for any left A-module M, any CS left submodule N, every left A-homomorphism of N into M may be extended to an endomorphism of ${}_AM$.

Lemma 9. The following conditions are equivalent:

(1) A is a left Noetherian left V-ring whose CS left modules are injective;

(2) A satisfies (*).

Proof. (1) implies (2) evidently.

Assume (2). If M is a CS left A-module, ${}_AH$ the injective hull of ${}_AM$, $B = {}_AM \oplus {}_AH$, then any left A-homomorphism of M into B extends to an endomorphism of ${}_AB$. The proof of Theorem 4 shows that M = H is injective. Then A is a left Noetherian, left V-ring by [4, Corollary 20.3E].

ALD and left CM-rings are studied in [15] and [16]. Semi-prime rings whose left ideals are left annihilators must be semi-simple Artinian [10]. But semi-prime rings whose ideals are left annihilators are not necessarily Noetherian or regular. Left CM, left V-rings need not be regular (the Cozzen's domains [4] are here relevant). Recall that A is of bounded index iff the supremum of the indices of the nilpotent elements of A is finite.

Theorem 10. The following conditions are equivalent:

(1) A is semi-simple Artinian;

(2) A is a left MUP-injective ring satisfying (*);

(3) A is a right MUP-injective ring satisfying (*);

(4) A is a left p-injective ring satisfying (*);

(5) A <u>is a right p-injective ring satisfying (*)</u>;

(6) A <u>is an ALD ring satisfying (*)</u>;

(7) A <u>is a left CM, left PQF-injective ring such that for any ideal</u> T, <u>any</u> $a \in A$, Ta <u>is a projective left</u> A<u>-module</u>;

(8) A <u>is a semi-prime ring such that the direct sum of any non-zero projective left</u> A<u>-module and any non-zero injective left</u> A<u>-module is PQF-injective</u>.

(9) <u>The direct sum of a non-zero projective and a non-zero PQF-injective</u> A<u>-modules is quasi-injective</u>;

(10) A <u>is a semi-prime left CM, left MUP-injective ring whose ideals are left annihilators</u>;

(11) A <u>is a regular ring of bounded index such that every ideal is a left annihilator</u>;

(12) A <u>is a semi-prime ring whose ideals are left annihilators and whose simple factor rings are Artinian</u>;

(13) A <u>is a semi-prime left p-injective ring such that every</u> p<u>-injective left</u> A<u>-module is pQ-injective</u>;

(14) A <u>is a left PQF-injective ring whose cyclic left modules are either injective or projective</u>.

<u>Proof</u>. Obviously, (1) implies (2) and (3).

Either (2) or (3) implies both (4) and (5) by Lemma 2(2) and Lemma 9.

Since semi-prime left p-injective rings with maximum condition on left annihilators are semi-simple Artinian, then either (4) or (5) implies (6) by Lemma 9.

(6) implies (7) by [15, Theorem 1.3].

Assume (7). Since every principal left ideal of A is projective, then Z = c and since A is left CM, then A is either reduced or semi-simple Artinian. Suppose that A is reduced.

Since A is a left PQF-injective ring, then A is left p-injective whose principal left ideals are projective which implies A regular and hence strongly regular. Therefore every left ideal of A is an ideal and is therefore a projective left A-module which implies A left hereditary, whence A is left self-injective. By a well-known result of B. Osofsky, A is semi-simple Artinian and (7) implies (8).

Since a semi-prime quasi-Frobeniusean ring is semi-simple Artinian, then (8) implies (9) by Theorem 6.

Since a simple left A-module is PQF-injective [4, Theorem 24.20] and the proof of Theorem 4 shows that (9) implies (10).

Assume (10). Since A is semi-prime whose ideals are left annihilators, then every ideal of A is generated by a central idempotent. This implies Z = o which yields A regular by Lemma 2(2). Since A is left CM, then A is either strongly regular or semi-simple Artinian. In either case, A has bounded index and (10) implies (11).

(11) implies (12) by [7, Corollary 7.10].

Assume (12). Then every ideal of A is generated by a central idempotent, whence A is a fully idempotent ring whose prime factor rings are simple. A is therefore regular by [7, Corollary 1.18]. Then any essential left ideal L of A contains an essential left ideal E which is an ideal of A [7, Lemma 6.20]. Now E is generated by a non-zero central idempotent which implies E = L = A. This proves A semi-simple Artinian and therefore (12) implies (13).

(13) implies (14) by Theorem 4.

Finally, (14) implies (1) by [8, Corollary 10].

We now mention, without proof, an analogue for certain PQF-injective modules of a well-known theorem of C. Faith - Y. Utumi [6, Theorem 2.16] concerning quasi-injective modules.

Theorem 11. Let M be a PQF-injective left A-module such that every complement left submodule is projective. If $E = \mathrm{End}({}_A M)$, then $E/J(E)$ is von Neumann regular, where $J(E) = \{f \in E / \ker f$ is essential in ${}_A M\}$ is the Jacobson radical of E.

Let us add a last result on MUP-injective rings.

Proposition 12. If A is a prime left MUP-injective ring, then the centre of A is a field.

Proof. Let D denote the centre of A. If $0 \neq d \in D$, the proof of Lemma 2(1) shows that there exist an essential left ideal L of A and an element y of A such that $L \subseteq \ell(d-dyd)$. Since $d \in D$, $AdL(1-yd) = L(d-dyd) = 0$ implies $L \subseteq \ell(1-yd)$, whence $1-yd \in Z = J$ by Lemma 2(2). It follows that there exists $u \in A$ such that $uyd = 1$. If $v = uy$, since $d \in D$, then $1 = vd = v^3 d^3 = (d^2v^3)d$ and $vd^2 = d^2v = dvd = d$ which yields $d(d^2v^3)d = d$. It is therefore sufficient to show that $d^2v^3 \in D$ and then D will be a field. For any $a \in A$, $v(d^2a) = da = ad = a(d^2v) = (d^2a)v$ which yields $(d^2v^3)a = v^3(d^2a) = (d^2a)v^3 = a(d^2v^3)$. This proves that $d^2v^3 \in D$.

Simple rings, semi-simple Artinian rings and strongly regular rings are all biregular. It is known that (1) Von Neumann regular rings need not be biregular; (2) Biregular rings (even if reduced) need not be regular; (3) Simple Noetherian rings need not be Artinian. We know that biregular rings are fully left and right idempotent. We conclude with a few remarks, the first two being a sequel to [17].

Remark 1. If A is a biregular ring which is of left finite Goldie dimension, then (1) Every ideal of A is generated by a central idempotent; (2) A is a finite direct sum of simple rings; (3) A is the only essential left (or right) ideal which is an ideal of A.

Remark 2. If A is left non-singular left CM-ring whose ideals are left annihilators, then A is a Baer, Dedekind finite left and right CS, biregular ring.

Remark 3. If A is a left self-injective ring whose essential left ideals are left annihilators, then every left ideal of A is a left annihilator. In that case, each semi-prime factor ring of A is semi-simple Artinian.

Remark 4. (1) If A is a right MUP-injective ring, then (a) Any reduced finitely generated right ideal is generated by an idempotent. Consequently, [1, Theorem 12] holds for right MUP-injective rings whose complement right ideals are finitely generated; (b) If every maximal essential right ideal of A is an ideal, then A is von Neumann regular iff every simple left A-module is flat (this extends [11, Proposition 2.2(IV)]); (2) A is strongly regular iff A is a semi-prime left duo ring whose MUP-injective left modules are p-injective (apply [15, Theorem 1.3] and Proposition 1).

Remark 5. Let A be a left PQF-injective ring. Then (1) Every left and right A-module is divisible; (2) If A is left uniform, then A/Z is von Neumann regular and Z = J; (3) A is left self-injective regular iff A is left non-singular such that every finitely generated non-singular left A-module is projective (apply [6, Theorem 3.12]); (4) A is simple Artinian iff A is prime left non-singular left CM.

Remark 6. A is a left principal ideal ring iff every finitely generated left ideal of A is principal and every p-injective left A-module is pQ-injective (cf. Theorem 4).

In view of [15, Theorem 1.7] and [18, Theorem 2.21], we raise the following question: Is A strongly regular if every maximal left ideal of A is an ideal and every simple left A-module is flat?

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