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THE NUMBER OF MINIMAL VARIETIES OF IDEMPOTENT GROUPOIDS

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Abstract: It is proved that there are uncountably many minimal varieties of commutative idempotent groupoids.

Key words: Minimal variety, commutative idempotent groupoid.

Classification: 08B05, 08B15

Kalicki [2] proved that there are uncountably many minimal varieties of commutative groupoids. Although this result was strengthened and generalized in various ways (see e.g. [1],[3],[4],[5]), there seems to be no mention of idempotency in the literature in this connection. The purpose of this paper is to prove the following

Theorem. There are $2^{\aleph_0}$ minimal varieties of commutative idempotent groupoids.

The proof will be divided into several lemmas. It will be convenient to work in the free commutative groupoid G over $\{x,y\}$ (x,y are two different elements). The binary operation of G will be denoted multiplicatively. If $a,b,c,d \in G$ then $ab = cd$ takes place iff either $a=c$ & $b=d$ or $a=d$ & $b=c$. G is a cancellation groupoid. There exists a unique mapping λ of G into the set of positive integers such that $\lambda(x) = \lambda(y) = 1$

and $\lambda(ab) = \lambda(a) + \lambda(b)$ for all $a,b \in G$; the number $\lambda(a)$ is called the length of an element $a \in G$. An element $a \in G$ is said to be a subterm of an element $b \in G$ if $b = (((ac_1)c_2 \ldots \ldots)c_k$ for some $k \geq 0$ and some elements $c_1, c_2, \ldots, c_k \in G$; if $k \geq 1$, a is said to be a proper subterm of b. Evidently, an element $a \in G$ is a proper subterm of $b_1 b_2$ iff it is a subterm of either b_1 or b_2.

If $n \geq 0$ and $a,b \in G$, we define an element $[a,b]^n \in G$ as follows: $[a,b]^0 = a$; $[a,b]^{n+1} = [a,b]^n b$. Hence $[a,b]^n = ((((ab)b)\ldots)b$ with n appearances of b.

Put $N = \{2,3,4,\ldots\}$. Denote by E the set of all finite sequences $(e_1,\ldots,e_k)$ such that $k \geq 1$, $e_1 \in N$ and $e_i \in N \times \{1,2\}$ for all $i \in \{2,\ldots,k\}$.

In the following let M be an arbitrary subset of N.

For every $e \in E$ define three elements R_e, S_e, T_e of G as follows:

(1) Let $e=(n)$, $n \in N$. Then $R_e = [x,y]^n x$, $S_e = [x,y]^{2n} x$, $T_e = x$ if $n \in M$ and $T_e = y$ if $n \notin M$.

(2) Let $e=(f,(n,1))$, $f \in E$, $n \in N$. Then $R_e = [T_f,S_f]^{n-1} R_f$, $S_e = [T_f,S_f]^{2n-1} R_f$, $T_e = R_f$ if $n \in M$ and $T_e = S_f$ if $n \notin M$.

(3) Let $e=(f,(n,2))$, $f \in E$, $n \in N$. Then $R_e = [T_f,R_f]^{n-1} S_f$, $S_e = [T_f,R_f]^{2n-1} S_f$, $T_e = S_f$ if $n \in M$ and $T_e = R_f$ if $n \notin M$.

<u>Lemma 1</u>. Let $e \in E$ and let p be an endomorphism of G. Then $p(R_e)$ is shorter than $p(S_e)$; $p(T_e)$ is a proper subterm of both $p(R_e)$ and $p(S_e)$.

Proof. It is obvious.

<u>Lemma 2</u>. Let $n,m \geq 2$ and let $a,b,c,d \in G$ be such that $[a,b]^{n-1} = [c,d]^{m-1}$ and $[a,b]^{2n-1} = [c,d]^{2m-1}$. Then $n=m$, $a=c$ and

b=d.

Proof. It is enough to consider the case $n \leq m$. We have b=d, since otherwise $b=[c,d]^{m-2}=[c,d]^{2m-2}$, which is impossible. From this we get by cancellation $a=[c,b]^{m-n}$ and $a = =[c,b]^{2m-2n}$; hence $m-n=2m-2n$, i.e. m=n; we get a=c as a consequence.

Lemma 3. Let $e,f\in E$ and let p,q be two endomorphisms of G such that $p(R_eS_e)=q(R_fS_f)$. Then e=f and p=q.

Proof. By induction on the sum of the lengths of e and f. If e,f are both one-termed, it is evident. Suppose e=(m) and f=(g,(n,1)). We have $p([x,y]^m x)=q([T_g,S_g]^{n-1}R_g)$ and $p([x,y]^{2m}x)=q([T_g,S_g]^{2n-1}R_g)$. Evidently $p(x)=q(R_g)$, $p([xy,y]^{m-1})=q([T_g,S_g]^{n-1})$ and $p([xy,y]^{2m-1})=q([T_g,S_g]^{2n-1})$. By Lemma 2 we get n=m and $p(xy)=q(T_g)$, so that $q(T_g)$ is longer than $p(x)=q(R_g)$, which is impossible by Lemma 1. Quite similarly, we cannot have e=(m) and f=(g,(n,2)).

Let e=(g,(n,1)) and f=(j,(m,1)). We have $p([T_g,S_g]^{n-1}R_g)= q([T_h,S_h]^{m-1}R_h)$ and $p([T_g,S_g]^{2n-1}R_g)=q([T_h,S_h]^{2m-1}R_h)$. Evidently $p(R_g)=q(R_h)$, $p([T_g,S_g]^{n-1})=q([T_h,S_h]^{m-1})$ and $p([T_g,S_g]^{2n-1})=q([T_h,S_h]^{2m-1})$. By Lemma 2, n=m and $p(S_g) = =q(S_h)$. By the induction assumption, g=h and p=q; since n=m, we get e=f.

If e=(g,(n,2)) and f=(h,(m,2)), the proof is quite analogous.

Suppose e=(g,(n,1)) and f=(h,(m,2)). Similarly as above we get $p(R_g)=q(S_h)$ and $p(S_g)=q(R_h)$. However, this is a contradiction by Lemma 1.

Denote by A the set of all $a \in G$ such that whenever $e \in \mathbf{E}$ and p is an endomorphism of G then neither $p(xx)$ nor $p(R_e S_e)$ is a subterm of a. Define a binary operation $\circ$ on A as follows:

(1) if $a, b \in A$ and $ab \in A$, put $a \circ b = ab$;

(2) if $a \in A$, put $a \circ a = a$;

(3) if $a, b \in A$ and $ab = p(R_e S_e)$ for some $e \in \mathbf{E}$ and some endomorphism p of G, put $a \circ b = p(T_e)$.

The correctness of this definition follows from Lemmas 1 and 3. Evidently $A(\circ)$ is a commutative idempotent groupoid.

Lemma 4. Let $a, b \in A$ and $ab \notin A$. Then either $a=b$ or there are elements $R, S, T \in G$ with $R \neq S$ and a number $m \geq 2$ such that $ab = ([T,S]^{m-1}R)([T,S]^{2m-1}R)$.

Proof. It is easy.

Lemma 5. Let $u, v \in A$ and let u be a proper subterm of v. Then $uv \in A$.

Proof. There are an integer $k \geq 1$ and elements $w_1, \ldots$ $\ldots, w_k \in G$ with $v = (((uw_1)w_2)\ldots)w_k$. Suppose $uv \notin A$. It follows from Lemma 4 that we can write $u = [T,S]^{m-1}R$ and $v = [T,S]^{2m-1}R$ for some R, S, T, m with $R \neq S$ and $m \geq 2$. Let us prove by induction on $j = 1, \ldots, k$ that $2m-j > 0$ and $(((uw_1)w_2)\ldots)w_{k-j} = [T,S]^{2m-j}$. For $j=1$ it follows from $(((uw_1)w_2)\ldots)w_k = [T,S]^{2m-1}R$, since we cannot have $(((uw_1)w_2)\ldots)w_{k-1} = R$. Assume that the two assertions are proved for some $j < k$. If it were $(((uw_1)w_2)\ldots$ $\ldots)w_{k-j-1} = S$ then we would have $\lambda(u) \leq \lambda(S)$; but u is longer than S, a contradiction. Thus there remains only one possibility: $(((uw_1)w_2)\ldots)w_{k-j-1} = [T,S]^{2m-j-1}$. If it were $2m-j-1=0$

then we would have $\lambda(u) \leq \lambda(T)$; but u is longer than T, a contradiction. Hence $2m-j-1>0$. The induction is thus finished. Especially, for $j=k$ we get: there is an $i>0$ with $u = [T,S]^i$. Hence $[T,S]^{m-1}R = [T,S]^i$. We cannot have $S=[T,S]^{m-1}$ and so we get $S=R$, a contradiction.

Lemma 6. Let $a,b \in A$, $ab \notin A$ and $a \neq b$; let $i \geq 1$. Then $[a \circ b, b]^i a \in A$.

Proof. Since $a \circ b$ is a proper subterm of b, several applications of Lemma 5 give $[a \circ b, b]^i \in A$. Suppose $[a \circ b, b]^i a \notin A$. If it were $[a \circ b, b]^i = a$ then b would be a proper subterm of a, so that $ab \in A$ by Lemma 5, a contradiction. By Lemma 4 we get $[a \circ b, b]^i a = ([T,S]^{m-1}R)([T,S]^{2m-1}R)$ for some R,S,T,m with $R \neq S$ and $m \geq 2$. If it were $[a \circ b, b]^i = [T,S]^{m-1}R$ and $a=[T,S]^{2m-1}R$, then we would have either $b=R$ or $b=[T,S]^{m-1}$, so that b would be a proper subterm of a and so $ab \in A$ by Lemma 5, a contradiction. Hence $[a \circ b, b]^i = [T,S]^{2m-1}R$ and $a = [T,S]^{m-1}R$. Since $b=R$ is impossible, we get $b=[T,S]^{2m-1}$. By Lemma 4 there are $r,s,t \in G$ and a $k \geq 2$ such that $ab=([t,s]^{k-1}r)([t,s]^{2k-1}r)$. There are two possible cases.

Case 1: $a=[T,S]^{m-1}R=[t,s]^{k-1}r$ and $b=[T,S]^{2m-1} = [t,s]^{2k-1}r$. Since either $r=[T,S]^{2m-2}$ or $r=S$, we cannot have $r=[T,S]^{m-1}$. Hence $r=R$. Since $R \neq S$, we get $[t,s]^{2k-1}=S$ and $[t,s]^{k-1}=[T,S]^{m-1}$, evidently a contradiction.

Case 2: $a=[T,S]^{m-1}R=[t,s]^{2k-1}r$ and $b=[T,S]^{2m-1} = [t,s]^{k-1}r$. Similarly as in the previous case we get $[t,s]^{k-1}=S$ and $[t,s]^{2k-1}=[T,S]^{m-1}$; we have either $S=s$ or $S=[t,s]^{2k-2}$, evidently a contradiction.

Lemma 7. Let $n \in N$. Then the groupoid $A(\circ)$ satisfies the identity $R_{(n)}S_{(n)}=T_{(n)}$.

Proof. Let φ be any homomorphism of G into $A(\circ)$; we must prove $\varphi(R_{(n)}S_{(n)})=\varphi(T_{(n)})$. Put $a=\varphi(x)$ and $b=\varphi(y)$. If $a=b$, everything is clear. If $ab \in A$ then by Lemma 5, $\varphi(R_{(n)}S_{(n)})=[a,b]^n a \circ [a,b]^{2n} a=\varphi(T_{(n)})$. It remains to consider the case when $ab=p(R_e S_e)$ for some $e \in E$ and some endomorphism p of G; we have $a \circ b=p(T_e)$. There are two possible cases.

Case 1: $a=p(R_e)$ and $b=p(S_e)$. By Lemma 6 we have $\varphi(R_{(n)}S_{(n)})=[a \circ b,b]^{n-1}a \circ [a \circ b,b]^{2n-1}a=p(R_{(e,(n,1))}) \circ$ $\circ\, p(S_{(e,(n,1))})=p(T_{(e,(n,1))})=\varphi(T_{(n)})$.

Case 2: $a=p(S_e)$ and $b=p(R_e)$. By Lemma 6 we have $\varphi(R_{(n)}S_{(n)})=[a \circ b,b]^{n-1}a \circ [a \circ b,b]^{2n-1}a=p(R_{(e,(n,2))}) \circ$ $\circ\, p(S_{(e,(n,2))})=p(T_{(e,(n,2))})=\varphi(T_{(n)})$.

The proof of the Theorem can now be completed in the following way. For any subset M of N denote by V_M the variety of commutative idempotent groupoids determined by the identities $([x,y]^n x)([x,y]^{2n}x)=x$ for any $n \in M$ and $([x,y]^n x)([x,y]^{2n}x)=y$ for any $n \in N \setminus M$. It follows from Lemma 7 that V_M is non-trivial, so that it contains a minimal subvariety U_M. If M_1, M_2 are two different subsets of N, then evidently $V_{M_1} \cap V_{M_2}$ is trivial and so $U_{M_1} \neq U_{M_2}$. Hence the number of minimal varieties of commutative idempotent groupoids cannot be smaller than the number of subsets of N, i.e. than $2^{\aleph_0}$. On the other hand, it cannot be larger than $2^{\aleph_0}$, since there are only $2^{\aleph_0}$ varieties of groupoids.

R e f e r e n c e s

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