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INVERSE LIMITS OF SMOOTH CONTINUA

Wlodzimierz J. CHARATONIK

Abstract: It is proved that (1) smoothness of continua in the sense of Mačkowiak is preserved under the inverse limit operation for sequences with bonding mappings being monotone relatively to points which form a thread; and (2) the property of Kelley is preserved under the inverse limit operation for sequences of continua with confluent bonding mappings.

Key words: Smooth continuum, inverse limit, monotone mapping.

Classification: Primary 54F15
Secondary 54B25, 54C10

The aim of this note is to prove that smoothness of continua in the sense introduced by Mačkowiak ([5],p. 81) is preserved by the inverse limit operation if the bonding mappings are monotone relative to points which form a thread. This is an answer to Problem 2 asked in [1]. It is also proved that the property of Kelley (see [7], p. 291; cf. [6], p. 538) is preserved under the inverse limit operation with confluent bonding mappings.

All spaces considered in this paper are assumed to be metric continua. The following notation will be used. The hyperspace of subcontinua of a continuum X (with Hausdorff metric) is denoted by $C(X)$, and we put $C^2(X)$ for $C(C(X))$. Given a continuous mapping $f:X \longrightarrow Y$, we denote by $f^*:C(X) \longrightarrow C(Y)$ the induced mapping defined by $f^*(K) = f(K)$, and analogously $f^{**}:C^2(X) \longrightarrow C^2(Y)$. Further, we use the lower and upper limits and the limit of a sequence A_n of subsets of a continuum X (in symbols $\mathrm{Li}\, A_n$, $\mathrm{Ls}\, A_n$ and $\mathrm{Lim}\, A_n$ respectively) in the sense of [2], § 29, p. 335-340. Similarly, the notion of upper (lower) semi-continuity of a set-valued mapping will be used in the sense of [2], § 18, p. 173 (cf. [3], § 43, II, Theorems 1 and 2, p. 61 and 62). The symbol $\{X^i, f^i\}_{i=1}^{\infty}$ denotes the inverse sequence of continua X^i with continuous bonding mappings $f^i:X^{i+1} \longrightarrow X^i$; we denote by $X = \varprojlim \{X^i, f^i\}$ the inverse limit space, and by $\pi^i:X \longrightarrow X^i$ the projection from X into the i-th factor space X^i. Given two inverse sequences $\{X^i, f^i\}_{i=1}^{\infty}$ and $\{Y^i, g^i\}_{i=1}^{\infty}$, and a mapping $\{h^i\}_{i=1}^{\infty}$ between the two sequences, we denote the limit mapping by $\varprojlim h^i: \varprojlim \{X^i, f^i\} \longrightarrow \varprojlim \{Y^i, g^i\}$ (cf. [2], p. 28-30).

Finally recall that a continuous mapping $f:X \longrightarrow Y$ is said to be (see [4], p. 720):

- confluent, if for every subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q,
- monotone relative to a point $p \in X$, if for each subcontinuum Q of Y such that $f(p) \in Q$ the inverse image $f^{-1}(Q)$ is connected.

We say that a continuum X is smooth at the point $p \in X$ if for each convergent sequence $\{x_n\}$ of points of X and for each subcontinuum K ~~of X~~ such that $p, x \in K$, where $x = \lim x_n$, there exists a sequence $\{K_n\}$ of subcontinua of X such that $p, x_n \in K_n$ for each $n = 1,2,\ldots$ and $\mathrm{Lim}\, K_n = K$ (see [5], p. 81).

I. The mapping F. Fix a point p of a continuum X and consider a mapping $F[X,p]:X \longrightarrow C^2(X)$ which assigns to a point $x \in X$ the family of all subcontinua K of X containing both p and x, i.e.,

$$F[X,p](x) = \{K \in C(X) : p, x \in K\}.$$

Note that, for each $x \in X$, this is a compact and arcwise connected subset of $C(X)$, whence this is really an element of $C^2(X)$. In this section the considered continuum X and the point p are assumed to be fixed, so we will write F instead of $F[X,p]$.

<u>Proposition 1</u>. The mapping F is upper semi-continuous.

Indeed, let $x_n \in X$ and $x_n \longrightarrow x$. We have to prove that $\mathrm{Ls}\, F(x_n) \subset F(x)$. Let a continuum K be in $\mathrm{Ls}\, F(x_n)$. Then there exist a subsequence $\{n_k\}$ of natural numbers and a sequence of points of $F(x_{n_k})$ that converges to K. Each of these points is a continuum in X containing p and x_{n_k}, whence K contains p and x, i.e., $K \in F(x)$.

<u>Proposition 2</u>. The mapping F is continuous if and only if the continuum X is smooth at the point p.

Proof. Assume F is continuous. Let a point $x \in X$, a continuum $K \in F(x)$ and a sequence of points $x_n \in X$ convergent to x be given. By continuity of F we have $\mathrm{Lim}\, F(x_n) = F(x)$, so there exist points K_n of $F(x_n)$ tending to K. Since the continua K_n contain both p and x_n, we are done by the definition of smoothness.

Assume X is smooth at p. By Proposition 1 we have only to show that F is lower semi-continuous, i.e., that

$F(x) \subset \mathrm{Li}\, F(x_n)$ for any sequence $x_n \longrightarrow x$. Let $K \in F(x)$. By smoothness of X at p there is a sequence of continua K_n, with $p, x_n \in K_n$, converging to K. Thus $K_n \in F(x_n)$, and the conclusion follows by the definition of the lower limit.

<u>Proposition 3</u>. Let a continuous surjection $f: X \longrightarrow Y$ and points $p \in X$ and $q \in Y$ with $q = f(p)$ be given. If $F_1 = F[X,p]$ and $F_2 = F[Y,q]$, then the diagram

$$\begin{array}{ccc} Y & \xleftarrow{\;f\;} & X \\ {\scriptstyle F_2}\downarrow & & \downarrow{\scriptstyle F_1} \\ C^2(Y) & \xleftarrow[f^{**}]{} & C^2(X) \end{array}$$

commutes if and only if f is monotone relative to p.

Proof. Assume that the diagram commutes, i.e., that $f^{**}(F_1(x)) = F_2(f(x))$ for each $x \in X$, which means that

(1) $\{f^*(K): K \in C(X) \text{ and } p, x \in K\} = \{L \subset C(Y): q, f(x) \in L\}$ for each $x \in X$.

Let $Q \subset Y$ be a continuum containing the point q. Suppose that $f^{-1}(Q)$ is not connected, and pick up a point x in another component of $f^{-1}(Q)$ than that to which the point p belongs. Then Q is in the right member of (1), while it is not in the left one.

Conversely, assume that f is monotone relative to p. We have to show that (1) holds. Take an arbitrary x in X and note that the left member of (1) is obviously a subset of the right. To prove the inverse inclusion take a continuum L in the right member of (1), i.e., such that $q, f(x) \in L$. Since f is monotone relative to p we conclude that $K = f^{-1}(L)$ is a continuum, so $L = f^*(K)$ belongs to the left member of (1).

Corollary. Let a continuum X be smooth at a point $p \in X$, and let a mapping $f: X \to Y$ from X onto a continuum Y be monotone relative to p. Then Y is smooth at $f(p)$.

Proof. By Proposition 2 we ought to show that the mapping $F_2: Y \to C^2(Y)$ defined as in Proposition 3 is continuous. Take a sequence of points $y_n \in Y$ which converges to a point $y \in Y$. We have to show that $F_2(y_n)$ tend to $F_2(y)$. Choose $x_n \in f^{-1}(y_n)$ and $x \in f^{-1}(y)$ such that $x_n \to x$ (take a proper subsequence if necessary). Now $F_2(y_n) = F_2(f(x_n)) = f^{**}(F_1(x_n))$ by Proposition 3, and similarly we have $F_2(y) = F_2(f(x)) = f^{**}(F_1(x))$. Since F_1 is continuous by Proposition 2 and f^{**} is continuous by its definition, we conclude that $f^{**}(F_1(x_n))$ converge to $f^{**}(F_1(x))$, i.e., $F_2(y_n)$ converge to $F_2(y)$; thus the proof is finished.

II. Smoothness of inverse limits. Now we are ready to prove the following

Theorem 1. Let $\{X^i, f^i\}_{i=1}^{\infty}$ be an inverse sequence such that for each $i = 1,2,\ldots$ (a) the continuum X^i is smooth at a point p^i; (b) $f^i(p^{i+1}) = p^i$; (c) f^i is monotone relative to p^{i+1}. Then the inverse limit continuum $X = \varprojlim \{X^i, f^i\}$ is smooth at the thread $p = \{p^i\}_{i=1}^{\infty}$.

Proof. Put $F^i = F[X^i, p^i]$ for $i = 1,2,\ldots$ and consider the mapping $\{F^i\}_{i=1}^{\infty}$ between the inverse sequences $\{X^i, f^i\}_{i=1}^{\infty}$ and $\{C^2(X^i), f^{i**}\}_{i=1}^{\infty}$:

$$\begin{array}{ccccccc} X^1 & \xleftarrow{f^1} & X^2 & \xleftarrow{f^2} & \cdots & & X \\ {\scriptstyle F^1}\downarrow & & \downarrow{\scriptstyle F^2} & & & & \downarrow{\scriptstyle F^{\infty}} \\ C^2(X^1) & \xleftarrow[f^{1**}]{} & C^2(X^2) & \xleftarrow[f^{2**}]{} & \cdots & & C^2_{\infty}(X) \end{array}$$

Since for each $i = 1,2,\ldots$ the diagram

$$\begin{array}{ccc} X^i & \xleftarrow{f^i} & X^{i+1} \\ {\scriptstyle F^i}\downarrow & & \downarrow{\scriptstyle F^{i+1}} \\ C^2(X^i) & \xleftarrow[f^{i**}]{} & C^2(X^{i+1}) \end{array}$$

commutes by Proposition 3, and since all mappings F^i are continuous by Proposition 2, hence the limit mapping $F^{\infty} = \varprojlim F^i$ is continuous. Note that the inverse limit $C^2_{\infty}(X) = \varprojlim \{C^2(X^i), f^{i**}\}$ is homeomorphic to $C^2(X)$. Indeed, by [6], Theorem (1.169), p. 171, $C_{\infty}(X) = \varprojlim \{C(X^i), f^{i*}\}$ is homeomorphic to $C(X)$ under a homeomorphism $h: C_{\infty}(X) \longrightarrow C(X)$ defined by $h(A) = \varprojlim \{A^i, f^i | A^{i+1}\}$, where $A = \{A^i\}_{i=1}^{\infty} \in C_{\infty}(X)$ (see [6],[5], p. 172). Using the same result once more we see that $C^2_{\infty}(X) = \varprojlim \{C^2(X^i), f^{i**}\}$ is homeomorphic to $C(C_{\infty}(X))$ under a homeomorphism $g: C^2_{\infty}(X) \longrightarrow C(C_{\infty}(X))$ defined by

$$g(B) = \varprojlim \{B^i, f^{i*} \mid B^{i+1}\}, \tag{2}$$

where $B = \{B^i\}_{i=1}^{\infty} \in C^2_{\infty}(X)$. The composite of g and h^* is the required homeomorphism from $C^2_{\infty}(X)$ to $C^2(X)$.

Now let us consider the following diagram in which $F = F_{X,p}$

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle F^{\infty}}\swarrow & & \searrow{\scriptstyle F} & \\ C^2_{\infty}(X) & \xrightarrow[g]{} & C(C_{\infty}(X)) & \xrightarrow[h^*]{} & C^2(X), \end{array}$$

and note that its commutativity implies continuity of $\mathcal{F}$, which is equivalent by Proposition 2 to the conclusion of the theorem. To prove that the above diagram commutes, take a point $x = \{x^i\}_{i=1}^{\infty} \in X$. We ought to show that

$$(3) \qquad h^*(g(\mathcal{F}^{\infty}(x))) = \mathcal{F}(x).$$

Applying the definition of $\mathcal{F}^{\infty}$ and (2) we have

$$(4) \quad g(\mathcal{F}^{\infty}(x)) = g(\{\mathcal{F}^i(x^i)\}_{i=1}^{\infty}) = \varprojlim \{\mathcal{F}^i(x^i), f^{i*}|\mathcal{F}^{i+1}(x^{i+1})\},$$

whence $h^*(g(\mathcal{F}^{\infty}(x))) = h^*(\varprojlim \{\mathcal{F}^i(x^i), f^{i*}|\mathcal{F}^{i+1}(x^{i+1})\})$. Take an element K in $h^*(g(\mathcal{F}^{\infty}(x)))$. Thus there exists a thread $\{K^i\}_{i=1}^{\infty}$ such that

$$(5) \qquad \{K^i\}_{i=1}^{\infty} \in \varprojlim \{\mathcal{F}^i(x^i), f^{i*}|\mathcal{F}^{i+1}(x^{i+1})\}$$

with $K = h(\{K^i\}_{i=1}^{\infty})$, i.e., $K = \varprojlim \{K^i, f^i|K^{i+1}\}$. Note that (5) implies that p^i, $x^i \in K^i$ for each $i = 1,2,\ldots$, whence p, $x \in K$, i.e., $K \in \mathcal{F}(x)$. So one inclusion in (3) is proved.

To show the other one, take $L \in \mathcal{F}(x)$. Thus, $p, x \in L$. Putting $L^i = \pi^i(L)$ we have p^i, $x^i \in L^i$ for each $i = 1,2,\ldots$, whence $L^i \in \mathcal{F}^i(x^i)$, and therefore the thread $\{L^i\}_{i=1}^{\infty}$ is in the right member of (5), so it is in $g(\mathcal{F}^{\infty}(x))$ by (4). Thus we conclude that $L = \varprojlim \{L^i, f^i|L^{i+1}\}$ is in the left member of (3). Hence (3) is shown and so the proof is complete.

III. The property of Kelley. Let d denote a metric on a continuum X. The continuum X is said to have the property of Kelley ([7], II, p. 291 and 292; cf. [6], (16.10), p. 538) provided that given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $a, b \in X$, $d(a,b) < \delta$ and $a \in A \in C(X)$, then there exists

$B \in C(X)$ such that $b \in B$ and $H(A,B) < \varepsilon$, where H denotes the Hausdorff metric in $C(X)$.

Define a mapping $\alpha[X]: X \longrightarrow C^2(X)$ by $\alpha[X](x) = \{K \in C(X): x \in K\}$ (see [7], p. 292; cf. [6], p. 551). The following two statements are known ([7], Theorem 2.2, p. 292 and Theorem 4.2, p. 296).

A. The mapping $\alpha[X]$ is continuous if and only if X has the property of Kelley.

B. The diagram

$$\begin{array}{ccc} Y & \xleftarrow{\quad f \quad} & X \\ {\scriptstyle \alpha[Y]}\downarrow & & \downarrow{\scriptstyle \alpha[X]} \\ C^2(Y) & \xleftarrow[\quad f^{**} \quad]{} & C^2(X) \end{array}$$

commutes if and only if f is confluent.

Using the same methods as in the proof of the previous theorem, we will prove

<u>Theorem 2</u>. Let $\{X^i, f^i\}_{i=1}^{\infty}$ be an inverse sequence such that for each $i = 1,2,\ldots$ (a) the continuum X^i has the property of Kelley, and (b) the mapping $f^i: X^{i+1} \to X^i$ is confluent. Then the inverse limit continuum $X = \varprojlim \{X^i, f^i\}$ has the property of Kelley.

In fact, to prove the theorem it is enough to replace in the proof of Theorem 1 the mapping $F[X^i, p^i]$ by $\alpha[X^i]$ for $i = 1,2,\ldots$ and to delete the points p^i and p from the considerations. Then the role of Propositions 2 and 3 is performed by the statements A and B respectively.

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