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# THE EXISTENCE AND SOME PROPERTIES OF SOLUTIONS OF A DIFFERENTIAL EQUATION WITH DEVIATED ARGUMENT

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Abstract: The paper contains a theorem on existence and asymptotic behavior of solutions for some differential equation with deviated argument and with implicit derivative. Considerations are based on the notion of measure of noncompactness and the fixed point theorem of Darbo type.

Key words: Differential equation with deviated argument, measure of noncompactness, fixed point theorem of Darbo type.

Classification: 34K25, 34K99, 47H09

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1. Introduction. The theory of differential equations with deviated argument was developed by several mathematicians (cf. the well known monograph [11]). At most there have been studied the functional differential equations of the form

$$x' = f(t, x(\varphi(t))), \tag{0}$$

where the unknown function $x(t)$ must additionally satisfy some initial condition (see e.g. [3,4,7,8,9,10,13]). Moreover, there have been examined also solutions of differential equations with deviated argument in the Myškis's sense (see [8, 11,14], for example).

This paper is devoted to the study of existence and asymptotic behavior of some differential equation with deviated argument more general than the equation (0), i.e. such that the right hand side of this equation depends on the delayed derivative.

We will use the method of a fixed point theorem of the Darbo type [6] which is based on the notion of a so-called measure of noncompactness. This notion was intensively studied in the last years by several authors. The most expository papers on this topic are those of Daneš [5], Sadovskii [12] and Banaś and Goebel [2].

In this paper we will apply measures of noncompactness defined in the axiomatic way in the work [2].

2. <u>Notations and basic definitions.</u> Denote by $E$ a fixed Banach space with the zero element $\theta$ and with norm $\|\ \|$. Further, let us denote:

$\mathfrak{M}_E$ - the family of all nonempty and bounded subsets of the space $E$,

$\mathfrak{N}_E$ - the family of all nonempty and relatively compact subsets of $E$.

If we have some nonempty family $\mathfrak{X}$ of subsets of $E$ then we will denote by $\mathfrak{X}^c$ its subfamily consisting of all closed sets.

Moreover, we will use standard notations (cf. [2]), for example $K(x,r)$ will denote the ball centered at $x$ and with radius $r$, the symbol $X$ denotes the closure of the set $\bar{X}$, the symbol $\mathrm{Conv}\, X$ denotes the closed convex closure of a set $X$, and so on.

Now we recall the definition of a measure of noncompactness from [2] (cf. also [1]).

Definition 1. A function $\mu : \mathfrak{M}_E \to \langle 0,+\infty)$ will be called a measure of noncompactness if it satisfies the following conditions:

$1^o$ the family $\mathcal{P} = [X \in \mathfrak{M}_E : \mu(X) = 0]$ is nonempty and $\mathcal{P} \subset \mathfrak{N}_E$,

$2^o$ $X \subset Y \Longrightarrow \mu(X) \leq \mu(Y)$,

$3^o$ $\mu(\overline{X}) = \mu(X)$,

$4^o$ $\mu(\mathrm{Conv}\ X) = \mu(X)$,

$5^o$ $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\, \mu(X) + (1-\lambda)\, \mu(Y)$, for all $\lambda \in \langle 0,1 \rangle$,

$6^o$ if $X_n \in \mathfrak{M}_E^c$, $X_{n+1} \subset X_n$, for $n = 1,2,\dots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\mathcal{P}$ described in $1^o$ is said to be a kernel of the measure $\mu$ and will be denoted by $\ker \mu$ . It may be shown that $(\ker \mu)^c$ forms a closed subspace of the space $\mathfrak{M}_E^c$ in topology generated by the Hausdorff distance [2]. For other properties of measures of noncompactness (in the above sense) and their examples see [2].
Let us notice that the set $X_\infty$ in axiom $6^o$ is a member of $\ker \mu$ [2]. This fact is used in the fixed point theorem of the Darbo type which will be given below.

First we recall the following definition. We assume that $\mu$ is a given measure of noncompactness on the space E.

Definition 2 [2]. Let $M \subset E$ be a given nonempty set and let $T : M \to E$ be a continuous transformation such that $TX \in \mathfrak{M}_E$ for any $X \in \mathfrak{M}_E$. A transformation T will be called $\mu$-contraction if there exists a constant $k \in \langle 0,1)$ such that $\mu(TX) \leq k\, \mu(X)$ for each set $X \in \mathfrak{M}_E$.

Theorem 1 [1]. Let C be a nonempty, closed, convex and bounded subset of the space E and let $T:C \rightarrow C$ be a $\mu$-contraction. Then the set $\text{Fix } T = [x \in C : Tx = x]$ is nonempty and $\text{Fix } T \in \ker \mu$.

For the details of the proof we refer to [1,2]. Let us only mention that the information that the set Fix T of fixed points of transformation T belongs to $\ker \mu$ plays an important role in the characterization of solutions of some functional equations (cf. [2]).

3. The space $C(\langle 0,+\infty), p(t))$. Let p(t) be a given function defined and continuous on the interval $\langle 0,+\infty)$ and taking real positive values. Denote by $C(\langle 0,+\infty), p(t)) = C_p$ the set of all real continuous functions x(t), defined on the interval $\langle 0,+\infty)$ and such that

$$\sup [|x(t)|p(t) : t \geq 0] < +\infty .$$

It is easy to check that $C_p$ forms a real Banach space with respect to the norm

$$\|x\| = \sup [|x(t)|p(t) : t \geq 0]$$

(cf. [13]).

Next, for an arbitrary $x \in C_p$, $X \in \mathfrak{M}_{C_p}$, $T > 0$ and $\varepsilon > 0$ let us denote:

$$\omega^T(x, \varepsilon) = \sup [|x(t)p(t) - x(s)p(s)| : t, s \in \langle 0, T\rangle, \\ |t-s| \leq \varepsilon],$$

$$\omega^T(X, \varepsilon) = \sup [\omega^T(x, \varepsilon) : x \in X],$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),$$

$$\omega_0(X) = \lim_{T \to \infty} \omega_0^T(X),$$

$$a(X) = \lim_{T\to\infty} \sup_{x \in X} \{ \sup [|x(t)|p(t): t \geq T] \},$$

$$\mu(X) = \omega_0(X) + a(X).$$

The function $\mu(X)$, defined by the last formula, is the measure of noncompactness in the space $C_p$ [2]. Its kernel is the family of all bounded sets consisting of functions which are equicontinuous on each compact interval and such that $\lim_{t\to\infty} x(t)p(t) = 0$ uniformly with respect to $x \in X$ [2]. For other properties of the measure $\mu$ see [2].

We will still use the following notation. If $x \in C_p$ then $\nu^T(x, \varepsilon)$ will denote the usual modulus of continuity of $x$ on the interval $\langle 0,T\rangle$, i.e.

$$\nu^T(X, \varepsilon) = \sup [|x(t) - x(s)| : |t-s| \leq \varepsilon, t,s \in \langle 0,T\rangle].$$

4. Differential equation with deviated argument and its solutions. Consider now the following differential equation

$$x'(t) = f(t, x(H(t)), x'(h(t))), \quad t \geq 0 \tag{1}$$

with the initial condition

$$x(0) = 0, \tag{2}$$

where $x(t)$ is an unknown function.

We will seek continuously differentiable solutions of the problem (1) - (2).

Let us assume that

(i) the functions $h, H: \langle 0,+\infty) \to \langle 0,+\infty)$ are continuous,

(ii) $f: \langle 0,+\infty) \times R \times R \to R$ is a continuous function.

Under the above hypotheses we may put

$$x'(t) = y(t)$$

and transform the equation (1) with condition (2) into the following functional-integral equation

$$(3) \qquad y(t) = f(t, \int_0^{H(t)} y(s)ds, y(h(t))), \ t \geq 0.$$

In the sequel we will examine the equation (3). Apart from the assumptions (i),(ii) we will additionally assume that:

(iii) the function $f(t,x,y)$ satisfies the Lipschitz condition with respect to the last variable i.e.

$$|f(t,x,y_1) - f(t,x,y_2)| \leq k_1|y_1 - y_2|, \ k_1 \geq 0,$$

(iv) $|f(t,x,0)| \leq L_0(t) + e^{L_1(t)}|x|$, where $L_0: \langle 0,+\infty) \to \langle 0,+\infty)$ is a continuous function such that $\lim_{t\to\infty} L_0(t)\exp(-\int_0^t L_0(s)ds) = 0$ and $L_1: \langle 0,+\infty) \to R$ is a continuous decreasing function such that $\lim_{t\to\infty} te^{L_1(t)} = 0$,

(v) $H(t) \geq t$, $\lim_{t\to\infty}(H(t) - t) = 0$ and $H(t)$ is such that

$$\sup [\int_t^{H(t)} L_0(s)ds : t \geq 0] = K < +\infty,$$

(vi) $h(t) \leq t$ and $\lim_{t\to\infty} (t - h(t)) = 0$.

In what follows let us define

$$L(t) = \int_0^t (L_0(s) + e^{L_1(s)})ds.$$

Denote by $C_L$ the space $C(\langle 0,+\infty), e^{-ML(t)})$, where $M$ is some arbitrarily fixed constant, $M > 1$.

Notice that in view of (iv) and (v) the number

$$k_2 = \sup [(e^{L_1(t)} \int_t^{H(t)} e^{ML(s)} ds)e^{-ML(t)} : t \geq 0]$$

is finite.

Now we may formulate the main theorem of our paper.

<u>Theorem 2</u>. Under the assumptions (i)-(vi), if in addition $k = k_1 + k_2 + \frac{1}{M} < 1$ and if the function $f(t,x,y)$ satisfies

the condition (vii) $|f(t,x_1,y) - f(t,x_2,y)| = o(e^{ML(t)})$ if $t$ tends to infinity, uniformly with respect to $x_1, x_2, y \in R$, then the equation (3) has at least one solution $y = y(t)$ which belongs to the space $C_L$ and such that $y(t) = o(e^{ML(t)})$ if $t$ tends to infinity.

Proof. Consider the transformation $F$ defined on the space $C_L$ by the formula

$$(Fy)(t) = f(t, \int_0^{H(t)} y(s)ds, y(h(t))), \quad t \geq 0.$$

Actually, for any $y \in C_L$ the function $(Fy)(t)$ is continuous. Moreover, using our assumptions we have

$$|(Fy)(t)|e^{-ML(t)} \leq |f(t, \int_0^{H(t)} y(s)ds, y(h(t))) -$$
$$- f(t, \int_0^{H(t)} y(s)ds, 0)| e^{-ML(t)} + |f(t, \int_0^{H(t)} y(s)ds, 0)| e^{-ML(t)} \leq$$
$$\leq k_1 |y(h(t))| e^{-ML(t)} + [L_0(t) + e^{L_1(t)} \int_0^{H(t)} |y(s)| ds] e^{-ML(t)} \leq$$
$$\leq k_1 |y(h(t))| e^{-ML(h(t))} e^{M(L(h(t))-L(t))} +$$
$$+ [L_0(t) + \|y\| e^{L_1(t)} \int_0^{H(t)} e^{ML(s)} ds] e^{-ML(t)} \leq k_1 \|y\| +$$
$$+ L_0(t)e^{-ML(t)} + \|y\| [e^{L_1(t)} \int_0^t e^{ML(s)} ds +$$
$$+ e^{L_1(t)} \int_t^{H(t)} e^{ML(s)} ds] e^{-ML(t)}.$$

Hence, denoting $g(t) = L_0(t)e^{-ML(t)}$, we get

$$|(Fy)(t)|e^{-ML(t)} \leq k_1 \|y\| + g(t) + \frac{1}{M} \|y\| ( \int_0^t M(L_0(s) +$$
$$+ e^{L_1(s)}) e^{ML(s)} ds) e^{-ML(t)} + k_2 \|y\| \leq g(t) + (k_1 + k_2 + \frac{1}{M}) \|y\| .$$

Thus

$$\|Fy\| \leq \sup [g(t) : t \geq 0] + k \|y\|$$

so that the transformation $F$ maps the space $C_L$ into itself.

Moreover, from the obtained evaluation we conclude that for $r = (1-k)^{-1}\sup[g(t):t \geq 0]$ the operator $F$ maps the ball $K(\theta,r)$ into itself.

Now we prove continuity of $F$ on the ball $K(\theta,r)$. Let $y, y_n \in K(\theta,r)$ and let $y_n$ converge to $y$ in the space $C_L$. Keeping our assumptions in mind, we have

$$|f(t, \int_0^{H(t)} y_n(s)ds, y_n(h(t)))e^{-ML(t)} - f(t, \int_0^{H(t)} y(s)ds, y(h(t)))$$

$$e^{-ML(t)}| \leq | f(t, \int_0^{H(t)} y_n(s)ds, y_n(h(t))) -$$

$$- f(t, \int_0^{H(t)} y_n(s)ds, y(h(t)))| e^{-ML(t)} +$$

$$+ |f(t, \int_0^{H(t)} y_n(s)ds, y(h(t))) - f(t, \int_0^{H(t)} y(s)ds, y(h(t)))| e^{-ML(t)} \leq$$

$$\leq k_1 |y_n(h(t)) - y(h(t))| e^{-ML(h(t))} + |f(t, \int_0^{H(t)} y_n(s)ds, y(h(t))) -$$

$$- f(t, \int_0^{H(t)} y(s)\, ds, y(h(t)))| e^{-ML(t)}.$$

From the above inequality it follows that it suffices to prove that the term

$$|f(t, \int_0^{H(t)} y_n(s)ds, y(h(t))) - f(t, \int_0^{H(t)} y(s)ds, y(h(t)))| e^{-ML(t)}$$

tends to 0 as n tends to infinity.

To do it let us fix $T > 0$ and let $\varepsilon > 0$ be arbitrarily small. Taking into account the uniform continuity of the function $f(t,x,y)$ on the compact set, for $t \in \langle 0,T \rangle$ we obtain

$$|f(t, \int_0^{H(t)} y_n(s)ds, y(h(t))) - f(t, \int_0^{H(t)} y(s)ds, y(h(t)))| \leq \varepsilon$$

for n sufficiently large.

On the other hand, choosing T suitably large and using the assumption (vii), for $t \geq T$ we get

$$|f(t,\int_0^{H(t)} y_n(s)ds, y(h(t))) - f(t,\int_0^{H(t)} y(s)ds, y(h(t)))|\, e^{-ML(t)} \leq \varepsilon,$$

which finally gives the desired continuity.

Further, let us fix $T>0$, $Y \in K(\theta, r)$ and $y \in Y$. In virtue of our assumptions, for an arbitrary $t \geq T$ we get

$$|(Fy)(t)|\, e^{-ML(t)} \leq k_1 |y(h(t))|\, e^{-ML(h(t))} e^{M[L(h(t))-L(t)]} +$$
$$+ e^{L_1(t)} [\int_0^T |y(s)|ds + \int_T^t |y(s)|ds + \int_t^{H(t)} |y(s)|ds] e^{-ML(t)} \leq$$
$$\leq k_1 |y(h(t))|\, e^{-ML(h(t))} + g(t) + Te^{L_1(T)} r +$$
$$+ (e^{L_1(t)} \int_t^{H(t)} re^{ML(s)} ds) e^{-ML(t)} +$$
$$+ (e^{L_1(t)} \int_T^t |y(s)|\, e^{-ML(s)} e^{ML(s)} ds) e^{-ML(t)} \leq$$
$$\leq k_1 |y(h(t))|\, e^{-ML(h(t))} + g(t) + rTe^{L_1(T)} +$$
$$+ re^{L_1(t)} e^{M[L(H(t))-L(t)]} (H(t)-t) +$$
$$+ (\sup_{t \geq T} |y(t)|\, e^{-ML(t)}) [\frac{1}{M} \int_T^t (L_0(s) +$$
$$+ e^{L_1(s)}) Me^{ML(s)} ds] e^{-ML(t)} \leq k_1 |y(h(t))|\, e^{-ML(h(t))} + g(t) +$$
$$+ Tre^{L_1(T)} + r(H(t) - t) e^{M[L(H(t)-L(t)]} e^{L_1(t)} +$$
$$+ \frac{1}{M} \sup_{t \geq T} |y(t)|\, e^{-ML(t)} \leq (\frac{1}{M} + k_1)(\sup_{t \geq h(T)} |y(t)|\, e^{-ML(t)}) + g(t) +$$
$$+ rTe^{L_1(T)} + r(H(t)) - t) e^{M[L(H(t))-L(t)]} e^{L_1(t)}.$$

Hence, owing to the fact that

$$\lim_{t \to \infty} e^{L_1(t)} (H(t) - t) e^{M[L(H(t))-L(t)]} = \lim_{t \to \infty} g(t) =$$
$= \lim_{t \to \infty} te^{L_1(t)} = 0$ and taking into account that $\lim_{t \to \infty} h(t) = \infty$ we have

(4) $$a(FY) \le (k_1 + \frac{1}{M})a(Y) \le ka(Y).$$

On the other hand, fixing an arbitrary $\varepsilon > 0$, $T>0$ and taking $t,s \in \langle 0,T\rangle$ such that $|t-s| \le \varepsilon$, we may calculate the following sequence of inequalities

$$|(Fy)(t)e^{-ML(t)} - (Fy)(s)e^{-ML(s)}| \le |(Fy)(t)e^{-ML(t)} -$$
$$- (Fy)(t)e^{-ML(s)}| + |(Fy)(t)e^{-ML(s)} - (Fy)(s)e^{-ML(s)}| \le$$
$$\le |e^{-ML(t)} - e^{-ML(s)}| [|f(t,\int_0^{H(t)} y(s)ds, y(h(t)) -$$
$$- f(t,\int_0^{H(t)} y(s)ds,0)| + f(t,\int_0^{H(t)} y(s)ds,0)|] +$$
$$+ |f(t,\int_0^{H(t)} y(\tau)d\tau, y(h(t))) - f(s,\int_0^{H(s)} y(\tau)d\tau, y(h(s)))| \le$$
$$\le \nu^T(e^{-ML(t)},\varepsilon)(k_1 r e^{ML(T)} + \sup[L_0(t): t \le T] +$$
$$+ r\sup[e^{L_1(t)} e^{ML(H(t))} H(t): t \le T]) + \tilde{\nu}_r^T(f,\varepsilon),$$

where we have denoted

$$\tilde{\nu}_r^T(f,\varepsilon) = \sup[|f(t,x_1,y_1) - f(s,x_2,y_2)| : t,s \in \langle 0,T\rangle,$$
$$|t-s| \le \varepsilon, \ |x_1 - x_2| \le r\,\nu^T(H(t),\varepsilon)e^{ML[H(T)]}, \ |y_1 - y_2| \le$$
$$\le 2re^{ML(T)}].$$

Thus, by means of the above estimation we deduce that

$$\omega_0^T(FY) = 0$$

and consequently

(5) $$\omega_0(FY) = 0.$$

Finally, combining (4) and (5) we obtain

$$\mu(FY) \le k\,\mu(Y)$$

so that $F$ is $\mu$-contraction.

Applying now Theorem 1 we complete the proof.

Remark. From the Theorem 1 it follows that all solutions of the equation (3) have the property mentioned in the thesis of Theorem 2.

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