Olga Hadžić

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ON MULTIVALUED MAPPINGS IN PARANORMED SPACES

Olga HADŽIĆ

Abstract: In Theorem 1 a sufficient condition for multivalued mapping $F:K \longrightarrow 2^K$ ($K \subseteq E$ and E is a paranormed space) is given such that F has the finite approximation property [3] and in Theorem 2 that F has the almost continuous selection property, where K satisfies Zima's condition [6]. Also some corollaries in the fixed point theory are given.

Key words: Multivalued mappings, paranormed space, fixed point.

Classification: 47H10

Let E be a linear space over the real or complex number field. The function $\| \ \|^*: E \longrightarrow [0,\infty)$ will be called a paranorm iff:

1. $\| x \|^* = 0 \Longleftrightarrow x=0$
2. $\| -x \|^* = \| x \|^*$, for every $x \in E$
3. $\| x + y \|^* \le \| x \|^* + \| y \|^*$, for every $x,y \in E$
4. If $\| x_n - x_o \|^* \longrightarrow 0$, $\lambda_n \longrightarrow \lambda_o$ then $\| \lambda_n x_n - \lambda_o x_o \|^* \longrightarrow 0$

Then we say that $(E, \| \ \|^*)$ is a paranormed space. E is also a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\{U_\varepsilon\}_{\varepsilon > 0}$ where $U_\varepsilon = \{x \mid x \in E, \ \| x \|^* < \varepsilon\}$.

In [6] the following theorem is proved:

Let K *be a bounded, closed and convex subset of* E *and* $T:K \longrightarrow K$ *be a completely continuous operator on* K. *If there exists a number* $C>0$ *such that:*

(1) $\quad \|\lambda x\|^* \leq C\lambda \|x\|^*$, *for every* $0 \leq \lambda \leq 1$ *and* $x \in K-K$

then there exists an element $p \in K$ *such that* $Tp=p$.

Zima has given in [6] an example of the space E and of the set K such that the condition (1) is satisfied.

Definition. *Let* $(E, \|\ \|^*)$ *be a paranormed space and* K *be a non-empty subset of* E. *If there exists* $C>0$ *such that:*

$$\|\lambda x\|^* \leq C\lambda \|x\|^*, \text{ for every } 0 \leq \lambda \leq 1 \text{ and } x \in K-K$$

we say that K *satisfies the Zima condition.*

In the next text we shall use the following notation. By 2^K $(K \subset E)$ we shall denote the family of all non-empty subsets of the set K and by $\mathcal{R}(K)$ the family of all non-empty convex and closed subsets of the set K.

Now, we shall prove a theorem on the finite approximation property.

Theorem 1. *Let* $(E, \|\ \|^*)$ *be a paranormed space,* K *be a non-empty, closed and convex subset of* E *and* $F:K \longrightarrow \mathcal{R}(K)$ *be a closed mapping such that* F(K) *is relatively compact and satisfies the Zima condition. Then for every* $\varepsilon > 0$ *there exists a finite dimensional, closed mapping* $F_\varepsilon : K \longrightarrow \mathcal{R}(K)$ *such that* $F_\varepsilon(K)$ *is relatively compact and:*

$$F_\varepsilon(x) \subseteq F(x) + V_\varepsilon, \quad \forall x \in K$$

where: $V_\varepsilon = \{x \mid x \in E, \ \|x\|^* \leq \varepsilon\}$.

Proof: Since the set F(K) is relatively compact, there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subset F(K)$ such that:

(2) $\quad F(K) \subseteq \bigcup_{i=1}^{n} \{x_i + U_{\frac{\varepsilon}{C}}\} \quad (U_{\frac{\varepsilon}{C}} = \{x \mid x \in E, \ \|x\|^* < \frac{\varepsilon}{C}\})$

Let:

$$F_\varepsilon(x) = [F(x) + \overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K)))] \cap \overline{co}\,M$$

for every $x \in K$, where $M = \{x_1, x_2, \dots, x_n\}$.

For every $x \in K$ we have that $F_\varepsilon(x) \neq \emptyset$. Indeed, if $u \in F(x)$ it follows that there exists $x_i \in M$ and $z \in U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K))$ so that $u-z=x_i$ from which we conclude that:

$$u-z \in [F(x)+\overline{co}(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K)))] \cap \overline{co}\,M$$

(this follows from (2)).

Further $\overline{co}\,F_\varepsilon(x) = F_\varepsilon(x)$, for every $x \in K$ since $\overline{co}\,F(x) = F(x)$ for every $x \in K$, and $F_\varepsilon(K) \subseteq \overline{co}\,M$ which implies that the mapping F_ε is finite dimensional. Let us prove that the mapping F_ε is closed. Suppose that $\{x_\alpha\}_{\alpha \in \mathcal{A}} \subseteq K$ is a convergent net such that $\lim_{\alpha \in \mathcal{A}} x_\alpha = x$, $y_\alpha \in F_\varepsilon(x_\alpha)$, for every $\alpha \in \mathcal{A}$ and $\lim_{\alpha \in \mathcal{A}} y_\alpha = y$. We shall prove that $y \in F_\varepsilon(x)$ which means that the mapping F is closed. Since $y_\alpha \in F_\varepsilon(x_\alpha)$, for every $\alpha \in \mathcal{A}$ it follows that there exists $z_\alpha \in F(x_\alpha)$, for every $\alpha \in \mathcal{A}$ and $u_\alpha \in \overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K)))$, for every $\alpha \in \mathcal{A}$ such that:

$$y_\alpha = z_\alpha + u_\alpha \in \overline{co}\,M$$

Since the set $\overline{F(K)}$ is compact there exists a convergent subnet $\{z_{\alpha_\delta}\}$ of the net $\{z_\alpha\}$ and let $\lim_\delta z_{\alpha_\delta} = z$. Since $y_{\alpha_\delta} \longrightarrow y$ it follows that $u_{\alpha_\delta} = y_{\alpha_\delta} - z_{\alpha_\delta} \longrightarrow u=y-z$. Further, the mapping F is closed and since $\lim_\delta x_{\alpha_\delta} = x$, $\lim_\delta z_{\alpha_\delta} = z$ and $z_{\alpha_\delta} \in F(x_{\alpha_\delta})$ it follows that $z \in F(x)$. From the relation:

$$u_{\alpha_\delta} \in \overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K))), \text{ for every } \delta$$

it follows that $u \in \overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K)))$ and so:

$y=u+z \in (F(x)+\overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K)))) \cap \overline{co}\,M = F_\varepsilon(x)$.

Now, we shall prove that for every $x \in K$ we have:

$$F_\varepsilon(x) \subseteq F(x) + V_\varepsilon$$

Since $F_\varepsilon(x) \subseteq F(x) + \overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K))$ it remains to prove that:

(3) $$\overline{co}\,(U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K))) \subset V_\varepsilon$$

Suppose that $u \in co\,(U_{\frac{\varepsilon}{C}} \cap (G(K)-F(K)))$. Then $u = \sum_{i=1}^{m} t_i z_i$, where $\sum_{i=1}^{m} t_i = 1$, $t_i \geq 0$ $(i=1,2,\ldots,n)$ and $z_i \in U_{\frac{\varepsilon}{C}} \cap (F(K)-F(K))$. So we have:

$$\|u\|^* = \|\sum_{i=1}^{m} t_i z_i\|^* \leq C \sum_{i=1}^{n} t_i \frac{\varepsilon}{C} = \varepsilon$$

and so $u \in V_\varepsilon$. Since V_ε is closed it follows that the relatiom (3) is proved. It is obvious that $F_\varepsilon(K)$ is relatively compact since $F_\varepsilon(K) \subseteq \overline{co}\,\{x_1, x_2, \ldots, x_n\}$, and so the proof is complete.

From Theorem 1 it is easy to obtain the following Corollary.

<u>Corollary 1.</u> <u>Let</u> $(E, \|\ \|^*)$ <u>be a paranormed space</u>, K <u>be a non-empty, closed and convex subset of</u> E <u>and</u> $F: K \longrightarrow \mathcal{R}(K)$ <u>be a closed mapping such that</u> F(K) <u>is relatively compact and satisfies the Zima condition. Then there exists</u> $x \in K$ <u>such that</u> $x \in F(x)$.

Proof: From Theorem 1 it follows that there exists, for every $\varepsilon > 0$, a compact finite dimensional mapping $F_\varepsilon: K \longrightarrow \mathcal{R}(K)$ such that:

$$F_\varepsilon(x) \subseteq F(x) + V_\varepsilon\,, \quad \forall x \in K$$

and that $F_\varepsilon(M) \subseteq M$ where M is $\overline{co}\,\{x_1, x_2, \ldots, x_n\}$ (see Theorem 1). If we apply Kakutani's fixed point theorem we conclude that for every $\varepsilon > 0$ there exists x_ε such that $x_\varepsilon \in F_\varepsilon(x_\varepsilon)$ and so:

(4) $$x_\varepsilon \in F(x_\varepsilon) + V_\varepsilon$$

Since the set $\overline{F_{\varepsilon}(K)}$ is compact there exists a sequence ($\varepsilon_n \to 0$) $\{\varepsilon_n\}_{n\in N}$ such that $x_{\varepsilon_n} \longrightarrow x \in K$ and since F is closed it is easy to see that $x \in F(x)$.

Remark: From Theorem 1 it follows that every closed and convex subset of E which satisfies the Zima condition is $\mathfrak{S}$-admissible [4] and so we can apply a result of S. Hahn in order to obtain a fixed point theorem for multivalued mapping [4].

Corollary 2. Let $(E, \|\ \|^*)$ be a paranormed space, W be a closed neighbourhood of $b \in E$, K be a closed, convex subset of E and satisfies the Zima condition. Let $F: W \cap K \longrightarrow \mathcal{R}(K)$ be a compact mapping such that:

$$x \in \partial W \cap K, \quad \beta > 1 \Longrightarrow \beta x + (1 - \beta) b \notin F(x)$$

Then there exists a point $x_0 \in W \cap K$ such that $x_0 \in F(x_0)$.

Now, we shall prove a theorem on almost continuous selection property for multivalued mapping in paranormed space.

First, we shall give a definition [2], introducing the notion of uniformly u-continuous mapping.

Definition 2. Let X be a topological vector space, K be a non-empty subset of X, $F: K \longrightarrow 2^X$ and $\mathcal{U}$ be the fundamental system of symmetric neighbourhoods of zero in X. The mapping F is uniformly u-continuous iff for every $V \in \mathcal{U}$ there exists $U \in \mathcal{U}$ such that the following implication holds:

$$x_1, x_2 \in K, x_1 - x_2 \in U \text{ and } y_1 \in F(x_1) \Longrightarrow \text{there exists } y_2 \in F(x_2),\ y_1 - y_2 \in V$$

Definition 3 ([1]). Let X be a topological vector space, $\mathcal{U}$ be the fundamental system of neighbourhoods of zero in X, K be a non-empty subset of X and $F:K \longrightarrow \mathcal{R}(K)$. The mapping F has the almost continuous selection property iff for every $V \in \mathcal{U}$ there exists a continuous mapping $h_V:K \longrightarrow K$ such that:

$$h_V(x) \in F(x) + V, \text{ for every } x \in K$$

Theorem 2. Let $(E, \| \|^*)$ be a paranormed space, K be a compact and convex subset of E which satisfies the Zima condition. Then every uniformly u-continuous mapping $F:K \longrightarrow \mathcal{R}(K)$ has the almost continuous selection property.

Proof: Let $\varepsilon > 0$. Since the mapping F is uniformly u-continuous on K there exists $\delta > 0$ such that the following implication holds:

$$x_1,x_2 \in K,\ x_1-x_2 \in U_\delta,\ y_1 \in F(x_1) \Longrightarrow \text{there exists } y_2 \in F(x_2),\ y_1-y_2 \in V_\varepsilon$$

From the compactness of the set K it follows that there exists $\{x_1,x_2,\dots,x_n\} \subseteq K$ such that:

$$K \subseteq \bigcup_{i=1}^{n} \{x_i + U_\delta\}$$

and let $\{g_i\}_{i=1}$ be the partition of the unity subordinated to the open cover $\{x_i + U_\delta\}_{i=1}^{n}$. Then $g_i(x)=0$ if $x \notin x_i + U_\delta$ $(i = 1,2,\dots,n)$, $g_i(x) \geq 0$ $(x \in K,\ i=1,2,\dots,n)$ and $\sum_{i=1}^{n} g_i(x)=1$. Let us define the mapping $h_\varepsilon :K \longrightarrow K$ in the following way:

$$h_\varepsilon(x) = \sum_{i=1}^{n} g_i(x)y_i, \text{ for every } x \in K$$

where $y_i \in F(x_i)$ $(i=1,2,\dots,n)$. Let us prove that:

(5) $\qquad h_\varepsilon(x) \subseteq F(x) + V_\varepsilon$, for every $x \in K$

i.e. that for every $x \in K$ there exists $z(x) \in F(x)$ such that $h_\varepsilon(x)-z(x) \in V_\varepsilon$. Let $x \in K$ and $g_i(x) > 0$. Then $x-x_i \in U_\delta$ and so there exists $u_i \in F(x)$ such that $y_i-u_i \in V_\varepsilon$, since F is uniformly u-continuous and $y_i \in F(x_i)$. Let:

$$z(x) = \sum_{i: g_i(x) > 0} g_i(x) u_i$$

Then $z(x) \in F(x)$ since $F(x) \in \mathcal{R}(K)$. Further:

$$\| h_\varepsilon (x) - z(x) \|^* = \| \sum_{i: g_i(x) > 0} g_i(x)(y_i - u_i) \|^* \leq$$

$$\leq \sum_{i: g_i(x) > 0} C\, g_i(x) \cdot \frac{\varepsilon}{C} = \varepsilon$$

and so the proof of (5) is complete.

In the next Corollary we shall use the notion of ε-fixed point of the multivalued mapping G in the following sense.

Definition 4. Let $(E, \| \|^*)$ be a paranormed space, K be a non-empty subset of E and $F: K \to 2^E$. Then $x \in K$ is an ε-fixed point of the mapping F iff:

$$x_\varepsilon \in F(x_\varepsilon) + V_\varepsilon$$

Corollary 3. Let $(E, \| \|^*)$ be a paranormed space, K be a compact, convex subset which satisfies the Zima condition. Then for every $\varepsilon > 0$ and every uniformly u-continuous multivalued mapping $F: K \to \mathcal{R}(K)$ there exists ε-fixed point of the mapping F.

Proof: Since from Theorem 2 it follows that there exists $h_\varepsilon : K \to K$, with (4), such that h_ε is continuous and $h_\varepsilon : K \to \overline{co}\,\{x_1, x_2, \dots, x_n\}$ from Brouwer fixed point theorem it follows that there exists $x_\varepsilon \in K$ such that $x_\varepsilon = h_\varepsilon(x_\varepsilon)$ and so:

$$x_\varepsilon = h_\varepsilon(x_\varepsilon) \in F(x_\varepsilon) + V_\varepsilon$$

which means that x_ε is an ε-fixed point of the mapping F.

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Faculty of Science
Dept.of Mathematics
21000 Novi Sad
Jovana Subotića 7
Jugoslavija

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