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ON THE EMBEDDING THEOREM

LE VAN HOT

Abstract: Radström [6], Godet-Thobie and Pham The Lai [3], Urbański [10] have proved that the space of all convex closed non-empty subsets of a locally convex space can be embedded into a locally convex space $\hat{X}$. In this paper, we consider the properties of the space $\hat{X}$, which will be used in our subsequent papers dealing with the differentiability of multivalued mappings.

Key words: Embedding theorem, multivalued mapping, locally convex spaces.

Classification: Primary 58C06
Secondary 57R35

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1. Introduction. Through this work, all linear spaces are assumed to be real.

We shall consider the space $\mathcal{C}_0(X)$ of all bounded convex closed non-empty subsets of a locally convex space X, and the embedding of the space $\mathcal{C}_0(X)$ into a locally convex space $\hat{X}$. In section 2, we recall some concepts of the space exp X of all closed nonempty subsets of a uniform space X and the space $\mathcal{C}(X)$ (resp. $\mathcal{C}_0(X)$) of all bounded (resp. bounded convex) closed non-empty subsets of a locally convex space X. Section 3, deals with some elementary properties of the spaces $\mathcal{C}_0(X)$ and $\hat{X}$. Our main results are contained in section 4.

2. Preliminaries. Let X be a uniform space and let its uniformity $\mathcal{U}$ have a base $\mathcal{B}$ of symmetric entourages. We denote the family of all closed non-empty subsets of X by exp X. We introduce a uniformity structure into exp X as follows: for each $U \in \mathcal{B}$, we set $\exp U = \{(A,B) \in \exp X \times \exp X \mid A \subseteq \overline{U(B)}$ and $B \subseteq \overline{U(A)}\}$, where $U(B) = \{x \in X \mid$, there exists an $y \in B$ such that $(x,y) \in U\}$. Then the family $\exp \mathcal{B} = \{\exp U \mid U \in \mathcal{B}\}$ forms a base of a uniformity of exp X, which is denoted by $\exp \mathcal{U}$.

If the uniformity $\mathcal{U}$ of X is induced by a bounded metric d then the uniformity $\exp \mathcal{U}$ is induced by the metric $\hat{d}$ defined by:

$$\hat{d}(A,B) = \max\{\sup_{x \in A} \inf_{y \in B} d(x,y), \sup_{y \in B} \inf_{x \in A} d(x,y)\}.$$

Let M be a closed nonempty subset of X and $\mathcal{U}_M$ be a restriction of $\mathcal{U}$ on M, then it is easy to see that $\exp \mathcal{U}_M = (\exp \mathcal{U})_{\exp M}$. We shall use the following

Theorem 1. [9] Let X be a metrizable uniform compact space, then the metrizable uniform space exp X is compact.

Let X be a locally convex space (l.c.s.), its topology τ is induced by a family of seminorms $\mathcal{P} = (p)$. We always suppose that the family $\mathcal{P}$ has the following property: for each $p,q \in \mathcal{P}$ there exists an $r \in \mathcal{P}$ such that $r \geq p$ and $r \geq q$.

We denote the family of all bounded (bounded closed, bounded convex closed resp.) non-empty subsets of a locally convex space X by $\mathcal{B}(X)$ ($\mathcal{C}(X)$ $\mathcal{C}_0(X)$) resp.). Let $\mathcal{N}$ be a base of convex circled neighborhoods of zero in X. We define a uniformity $\mathcal{U}$ on $\mathcal{B}(X)$, with a base $\mathcal{B} = \{U_N \mid N \in \mathcal{N}\}$, where

U_N is defined by

$$U_N = \{(A,B) \mid A \subseteq \overline{B + N} \text{ and } B \subseteq \overline{A + N}\},$$

where $\overline{A}$ denotes the closure of the set A in X.
The uniformity $\mathcal{U}$ is induced by a family of pseudometrics $\{d_p \mid p \in \mathcal{P}\}$ defined by

$$\begin{aligned} d_p(A,B) &= \inf\{\lambda > 0 \mid A \subseteq \overline{B + \lambda S_p} \text{ and } B \subseteq \overline{A + \lambda S_p}\} \\ &= \max\{\sup_{x \in A} \inf_{y \in B} p(x - y),\ \sup_{y \in B} \inf_{x \in A} p(x - y)\}, \end{aligned}$$

where $S_p = \{x \in X \mid p(x) \leq 1\}$.
The restriction $\mathcal{U}_c$ of $\mathcal{U}$ on $\mathcal{C}(X)$ is a Hausdorff's uniformity; i.e. $\cap\{U \mid U \in \mathcal{U}_c\} = \Delta = \{(A,A) \mid A \in \mathcal{C}(X)\}$.

It is clear that $(U_N) \cap \mathcal{C}(X) \times \mathcal{C}(X) = (\exp V_N)_c$, where $V_N = \{(x,y) \mid x - y \in N\}$ and $(\exp V_N)_c$ is the restriction of $\exp V_N$ on $\mathcal{C}(X)$.

If X is normable with the norm $\| \ \|$, then the uniformity $\mathcal{U}$ restricted on $\mathcal{C}(X)$ is induced by the metric d defined by

$$\begin{aligned} d(A,B) &= \inf\{\lambda > 0 \mid A \subseteq \overline{B + \lambda S_1} \text{ and } B \subseteq \overline{A + \lambda S_1}\} \\ &= \max\{\sup_{x \in A} \inf_{y \in B} \|x - y\|,\ \sup_{y \in B} \inf_{x \in A} \|x - y\|\}, \end{aligned}$$

where

$$S_1 = \{x \in X \mid \|x\| \leq 1\}.$$

Let A, B be subsets of X, $\lambda \in R$; we define

$$A + B = \{x + y \mid x \in A, y \in B\},$$

$$\lambda A = \{\lambda x \mid x \in A\},$$

$$A +^* B = \overline{A + B}.$$

Then we have the following theorem (see [6],[3] and [10]).

Theorem 2 ([6],[3],[10]). Let X be a locally convex space with the topology τ induced by a family of seminorms $\mathcal{P}$. Let $\mathcal{N}$ be a base of convex circled 0-neighborhoods in X. We put $\hat{X} = \mathcal{C}_o(X) \times \mathcal{C}_o(X)/_{\sim}$, where $\sim$ is an equivalence defined by

$$(A,B)\sim(C,D) \text{ iff } A +^* D = B +^* C.$$

Let $[A,B]$ denote an equivalence class containing the element (A,B). We define:

$[A,B]+[C,D] = [A +^* C, B +^* D]$ for $[A,B],[C,D] \in \hat{X}$,

$\lambda[A,B] = [\lambda A, \lambda B]$ for $\lambda \geq 0$ $[A,B] \in \hat{X}$,

$\lambda[A,B] = [|\lambda| B, |\lambda| A]$ for $\lambda < 0$ $[A,B] \in \hat{X}$.

Then:

1) $\hat{X}$ is a linear (real) space.

2) The family $\hat{\mathcal{P}}$ of seminorms $\{\hat{p} | p \in \mathcal{P}\}$ given by $\hat{p}([A,B]) = dp(A,B)$ defines a locally convex topology $\hat{\tau}$, having the following base of 0-neighborhoods:

$$\hat{\mathcal{B}} = \{\hat{U}_N | N \in \mathcal{N}\} \text{ , where } \hat{U}_N = \{[A,B] \mid (A,B) \in U_N\}.$$

If X is normable with norm $\|\cdot\|$, then $\hat{X}$ is normable under the norm $\|[A,B]\| = d(A,B)$.

3) The map $\varkappa : \mathcal{C}_o(X) \longrightarrow \hat{X}$ defined by $\varkappa(A) = [A,\{0\}]$ is an isometry in the following sense $d_p(A,B) = \hat{p}(\varkappa(A) - \varkappa(B))$ and $\varkappa(A +^* B) = \varkappa(A) + \varkappa(B)$ and $\varkappa(\lambda A) = \lambda \varkappa(A)$ for all $A,B \in \mathcal{C}_o(X)$ and $\lambda \geq 0$.

Example 1. Let $X = R_1$; $e = [\{1\},\{0\}]$; $E = [[0,1],\{0\}] \in \hat{R}_1$. If $A \in \mathcal{C}_o(R_1)$, then A is a bounded closed interval of R_1; i.e. $A = [a_1, a_1 + a]$ where $a_1 \in R_1$; $a \geq 0$. For each $\alpha \in \hat{R}_1$ there exist $a_1, b_1 \in R_1$, $a \geq 0$ $b \geq 0$ such that $\alpha = [[a_1, a_1 + a],[b_1, b_1 + b]]$ $\alpha = (a_1 - b_1)e + (a - b)E$.

Of course e and E are linearly independent. It follows that $\dim \hat{R}_1 = 2$ and R_1 is complete.

If we define: $(ae + bE)\cdot(ce + dE) = (ac)e + (ad + cb + db)E$, then it is easy to verify that $\hat{R}_1$ is commutative B-algebra with the unit e and the maps $\varphi, \psi : \hat{R}_1 \longrightarrow R_1$ defined by

$$\varphi([A,B]) = \max A - \max B, \quad \psi([A,B]) = \min A - \min B$$

are homomorphisms of algebra $\hat{R}_1$ onto algebra R_1. If $[A,B] = ae + bE$, then $\varphi([A,B]) = a + b$, $\psi([A,B]) = a$. If $a \neq 0$ and $a + b \neq 0$, then $(ae + bE)$ has inverse and

$$(ae + bE)^{-1} = \frac{1}{a} e + \frac{b}{a(a + b)} E.$$

Example 2. The following example is due to Aumann and Kakutani [2], who shows that the space $\hat{R}_2$ is not complete. Let $\{\alpha_i\}$ be a decreasing sequence of positive real numbers such that $\alpha_1 < \frac{\pi}{2}$; $\sum_{i=1}^{\infty} \sin \alpha_i < +\infty$. Given an angle α denote by E_α the closed straight line segment, whose extremities have coordinates $(0,0)$, $(\cos\alpha, \sin\alpha)$. Let $X_p = \sum_{i=1}^{p} E_{\alpha_i}$, $Y_p = pE_0$; $Z_p = [X_p, Y_p]$. Then $\{Z_p\}$ is a Cauchy sequence in $\hat{R}_2$, but $\{Z_p\}$ does not converge in $\hat{R}_2$.

3. Some basic properties. In [3], Godet-Thobie and Pham The Lai, have proved that if X is an F-space, then the uniform space $\mathcal{C}_0(X)$ is complete. It is easy to verify that if X is a space of type LF, i.e. is a strict inductive limit of sequence of F-spaces ($X = \varinjlim_n X_n$, where X_n is a subspace of X_{n+1}, and X_n is an F-space for all n), then the uniform space $\mathcal{C}_0(X)$ is sequentially complete. In fact, let $\{A_n\}$ be a Cauchy se-

quence in $\mathscr{C}_0(X)$, then it is clear that the set $\bigcup_1^{\infty} A_n$ is bounded in X. (In fact let U be an O-neighborhood in X, then there exists n_0 such that for all $n \geq n_0$ we have $A_{n_0} \subseteq \overline{A_n + U} \subseteq$ $\subseteq A_n + 2U$ and $A_n \subseteq \overline{A_{n_0} + U} \subseteq A_{n_0} + 2U$. On the other hand, $\bigcup_1^{n_0} A_i$ is a bounded subset of X, hence there exists $k > 0$ such that for all $\lambda : \lambda > k$, $\bigcup_1^{n_0} A_i \subseteq \lambda U$. Then

$$\bigcup_1^{\infty} A_i = \bigcup_1^{n_0} A_i \cup \bigcup_{n_0+1}^{\infty} A_i \subseteq (\lambda U) \cup (A_{n_0} + 2U) \subseteq (\lambda U) \cup (\lambda + 2)U \subseteq \subseteq (\lambda + 2)U.)$$

By theorem II.6.5 [8], there exists an integer n_1 such that $\bigcup_1^{\infty} A_i \subseteq X_{n_1}$. That is, $A_n \in \mathscr{C}_0(X_{n_1})$ for all n. Of course $\{A_n\}$ is a Cauchy sequence in $\mathscr{C}_0(X_{n_1})$. Since we know that $\mathscr{C}_0(X_{n_1})$ is complete [3], there exists $A \in \mathscr{C}_0(X_{n_1})$ such that $\lim A_n =$ $= A$ in $\mathscr{C}_0(X_{n_1})$. It follows $\lim A_n = A$ in $\mathscr{C}(X)$ and this proves that $\mathscr{C}_0(X)$ is sequentially complete.

<u>Proposition 1.</u> Let X be a semi-reflexive locally convex space ([8]), then the uniform space $\mathscr{C}_0(X)$ is sequentially complete.

Proof. Let $\{A_n\}$ be a Cauchy sequence in $\mathscr{C}_0(X)$. We set $B_n = \overline{\text{conv}} \left(\bigcup_m^{\infty} A_i \right)$ (where $\overline{\text{conv}}\, A$ denotes the closed convex hull of set A). We claim that $\{B_n\}$ is a Cauchy sequence and if $B = \lim B_n$ then $B = \lim A_n$. In fact, let U be a convex circled O-neighborhood in X. There exists an integer N such that for all $n,m \geq N$ we have:

$$A_n \subseteq \overline{A_m + U} \subseteq \overline{B_m + U} \text{ and } A_m \subseteq \overline{A_n + U} \subseteq \overline{B_n + U}.$$

Then $B_n = \overline{\text{conv}} \bigcup_m^{\infty} A_i \subseteq \overline{B_m + U}$ and $B_m = \overline{\text{conv}} \bigcup_m^{\infty} A_i \subseteq \overline{B_n + U}$.

This shows that $\{B_n\}$ is a Cauchy sequence. Let $B = \lim B_n$ and let U be a convex circled O-neighborhood in X. Then there exists N such that for all $m,n \geq N$ we have $A_m \subseteq \overline{A_n + \frac{1}{2} U}$, $A_n \subseteq \overline{A_m + \frac{1}{2} U}$, $B_n \subseteq \overline{B + \frac{1}{2} U}$ and $B \subseteq \overline{B_n + \frac{1}{2} U}$. Then $A_n \subseteq B_n \subseteq \overline{B + \frac{1}{2} U}$ and $B \subseteq \overline{B_n + \frac{1}{2} U} \subseteq \overline{(\overline{A_n + \frac{1}{2} U}) + \frac{1}{2} U} \subseteq \overline{A_n + U}$, which gives $\lim A_n = B$. Now our proof will be completed, if we prove the existence of $\lim B_n$. Of course $B_n \supseteq B_{n+1} \supseteq \dots$. Since X is semireflexive, B_n is weakly compact for all n. Then $B = \bigcap_1^{\infty} B_n \neq \emptyset$, $B \in \mathcal{C}_0(X)$. If $B \neq \lim B_n$, then there exists a convex circled closed 0-neighborhood U such that for each n there exists $x_n \in B_n$ such that $x_n \notin (B + U)$ (of course $B \subseteq B_n$ for all n). Let n_0 be a positive integer such that for all $n \geq n_0$ we have $B_{n_0} \subseteq \overline{B_n + \frac{1}{2} U} \subseteq B_n + U$. Put $K_n = (x_{n_0} + U) \cap B_n \neq \emptyset$, $K_n \in \mathcal{C}_0(X)$; $K_n \supseteq K_{n+1}$. Then $\bigcap_1^{\infty} K_n \neq \emptyset$ because K_n is weakly compact for all n. Let $x \in \bigcap_1^{\infty} K_n \in \bigcap_1^{\infty} B_n = B$. It is $x \in K_{n_0} \subseteq x_{n_0} + U$, whence $x_{n_0} \in x + U \subseteq B + U$, a contradiction with the assumption that $x_{n_0} \notin B + U$. The proof is complete.

Corollary 1. If X is an LF-space or semi-reflexive space, then $\varkappa(\mathcal{C}_0(X))$ is sequentially closed in $\hat{X}$.

It is easy to see that if M is bounded convex subset of X, then the set $\{[A,B] \mid A \subseteq \overline{B + M}$ and $B \subseteq \overline{A + M}\}$ is a bounded set of $\hat{X}$.

Proposition 2. Suppose that (X, τ) is a regular inductive limit of a sequence of metrizable locally convex spaces (X_n, τ_n) (for instance when (X_n, τ_n) is a closed subspace of (X_{n+1}, τ_{n+1}) for all n), M, N are closed convex subsets of X. Put $\mathfrak{M} = \{[A,B] \mid A \subseteq M, B \subseteq N\}$. Then

1) If M, N are compact, then $\mathfrak{M}$ is compact,

2) If M, N are separable and weakly compact (i.e. $w(X,X')$-compact, where X' denotes the dual space of X), then $\mathfrak{M}$ is $\hat{w}(X,X')$-compact.

Proof: It is easy to see that if p is a continuous seminorm on X, $A,B \in \mathcal{C}(X)$, then $d_p(\operatorname{conv} A, \operatorname{conv} B) \leq d_p(A,B)$, where conv A denotes the convex hull of A. For each closed convex subset M of X, put $\mathcal{C}(M) = \{A \in \mathcal{C}(X) \mid A \subseteq M\}$; $\mathcal{C}_0(M) = \{A \in \mathcal{C}_0(X) \mid A \subseteq M\}$. Then it is easy to verify that $\mathcal{C}_0(M)$ is a closed subset of $\mathcal{C}(M)$. Since $\mathfrak{M} = \varkappa(\mathcal{C}_0(M)) - \varkappa(\mathcal{C}_0(N))$, it follows that the proof of our Proposition will be complete if we prove that $\mathcal{C}(M)$ and $\mathcal{C}(N)$ are exp $\mathcal{U}_\tau$-compact (respectively exp $\mathcal{U}_w$-compact), where $\mathcal{U}_\tau$ (respectively $\mathcal{U}_w$) is the translation invariant uniformity with respect to the topology τ (the topology $w(X,X')$ respectively) on X. By Theorem 1, it is sufficient to prove that $\mathcal{U}_\tau$ (respectively $\mathcal{U}_w$) restricted on M and N is metrizable. But M, N are τ-compact ($w(X,X')$-compact respectively), so it is sufficient to prove that the topology τ (topology $w(X,X')$ respectively) restricted on M, N is metrizable, because for the Hausdorff compact space M(N) there exists a unique uniform structure, which induces its topology.

1) If M, N are τ-compact, then $M \cup N$ is τ-bounded. There exists an integer n_0 such that $M \cup N \subseteq X_{n_0}$, as (X,τ) is a regular inductive limit of (X_n, τ_n). It follows that the topology τ restricted on $M \cup N$ is metrizable, because τ_{n_0} is metrizable.

2) If M, N are $w(X,X')$-compact, then $M \cup N$ is $w(X,X')$-bounded. Therefore $M \cup N$ is τ-bounded. There exists n_0 such that $M \cup N \subseteq X_{n_0}$. To prove that the topology $w(X,X')$ restric-

ted on M or on N is metrizable, it is sufficient to prove that there exists a countable family of real weakly continuous functions, defined on M(N), which distinguish the points of M (or N, respectively).

Let $\{x_n\}$ be a dense subset of M. Let p_j, $j = 1,2,\ldots$ be a sequence of continuous seminorms on X such that $p_1 \leq p_2 \leq \ldots$, and $\{p_j \, X_{n_o}\}$ induces the topology τ_{n_o}. For each n,m,j (n,m, j = 1,2,...) there exists $x'_{n,m,j} \in X'$ such that $x'_{n,m,j}(x_n - x_m) = p_j(x_n - x_m)$ and $|x'_{n,m,j}(x)| \leq p_j(x)$ for all $x \in X$. We claim that $\{x'_{n,m,j} | n,m,j = 1,2,\ldots\}$ distinguishes the points of M. Let $x,y \in M$, $x'_{n,m,j}(x) = x'_{n,m,j}(y)$ for all n,m,j. There exist subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $\lim x_{n_k} = x$, $\lim x_{m_k} = y$.

We have that:

$$
\begin{aligned}
p_j(x_{n_k} - x_{m_k}) &= x'_{n_k,m_k,j}(x_{n_k} - x_{m_k}) - x'_{n_k,m_k,j}(x - y) \\
&\leq p_j(x_{n_k} - x_{m_k} - x - y) \\
&\leq p_j(x_{n_k} - x) + p_j(x_{m_k} - y), \\
p_j(x - y) &= \lim p_j(x_{n_k} - x_{m_k}) \\
&\leq \lim p_j(x_{n_k} - x) + \lim p_j(x_{m_k} - y) = 0.
\end{aligned}
$$

Therefore $p_j(x - y) = 0$ for all j. Since $\{p_j \, X_{n_o}\}$ induces the topology τ_{n_o} on X_{n_o} we have that $x = y$. This means that $w(X,X')$ restricted on M is metrizable. Similarly $w(X,X')$ restricted on N is metrizable and this completes the proof.

4. Main results

Proposition 3. Let X, Y be locally convex spaces, $T \in L(X,Y)$, where $L(X,Y)$ denotes the space of all linear continuous mappings of X into Y. We define a map $T_c: \mathscr{C}_0(X) \longrightarrow \mathscr{C}_0(Y)$:

$T_c(A) = \overline{T(A)}$ for all $A \in \mathscr{C}_0(X)$.

Then:

1) $T_c(A +^* B) = T_c(A) +^* T_c(B)$ for all $A,B \in \mathscr{C}_0(X)$;

2) $T_c(\lambda A) = \lambda T_c(A)$ for $\lambda \geq 0$ and $A \in \mathscr{C}_0(X)$;

3) If Z is anl.c.s., $P \in L(Y,Z)$, then

$(P \circ T)_c = P_c \circ T_c$;

4) If X, Y are normed spaces then

$d(T_c(A),T_c(B)) \leq \|T\| d(A,B)$ for $A,B \in \mathscr{C}_0(X)$.

Proof. 1) $T_c(A +^* B) = \overline{T(\overline{A + B})} \supseteq \overline{T(A + B)} = \overline{T(A) + T(B)} =$
$= T_c(A) +^* T_c(B)$. On the other hand we have:

$T(A +^* B) = T(\overline{A + B}) \subseteq \overline{T(A + B)} = T_c(A) +^* T_c(B)$

Hence $T_c(A +^* B) = T_c(A) +^* T_c(B)$.

The proofs of 2),3) and 4) are obvious. Q.E.D.

Let A, B, C, $D \in \mathscr{C}_0(X)$ and $[A,B] = [C,D]$, then $A +^* D =$
$= B +^* C$ and $T_c(A) +^* T_c(D) = T_c(B) +^* T_c(C)$. This shows that $[T_c(A),T_c(B)] = [T_c(C),T_c(D)]$. So, we can define a map $\hat{T}:\hat{X} \longrightarrow \hat{Y}$ by:

$$\hat{T}([A,B]) = [T_c(A),T_c(B)].$$

<u>Proposition 4.</u> The following conclusions are valid:

1) $\hat{T} \in L(\hat{X},\hat{Y})$;

2) If $P \in L(Y,X)$, where Z is a l.c.space, then $(\widehat{P \circ T}) =$
$= \hat{P} \circ \hat{T}$;

3) If X, Y are normed linear spaces, then $\|\hat{T}\| = \|T\|$.

Proof. 1) It is easy to verify that $\hat{T}$ is a linear map of $\hat{X}$ into $\hat{Y}$ and if V is an 0-neighborhood in Y, N is an 0-neighborhood in X such that $T(N) \subseteq V$, then $\hat{T}(\hat{U}_N) \subseteq \hat{U}_V$. This implies $\hat{T} \in L(\hat{X},\hat{Y})$.

2) The property $\widehat{P\circ T} = \hat{P}\circ\hat{T}$ follows immediately from the equality $(P\circ T)_c = P_c \circ T_c$.

3) From $d(T_c(A),T_c(B)) \leq \|T\|\, d(A,B)$ we have $\|\hat{T}\| \leq \|T\|$. On the other hand

$$\|\hat{T}\| \geq \sup_{\|x\|\leq 1} \|\hat{T}([\{x\},\{0\}])\| = \sup_{\|x\|\leq 1} \|T(x)\| = \|T\|.$$

Hence $\|\hat{T}\| = \|T\|$. Q.E.D.

It is obvious that $\hat{I}_X = I_{\hat{X}}$ (where I_X denotes the identity mapping of X). It follows that if T is an isomorphism of X onto Y, then $\hat{T}$ is also an isomorphism of $\hat{X}$ onto $\hat{Y}$.

<u>Remark 1.</u> Let $F:X \longrightarrow Y$ be an affine continuous map, $F(0) = a$, then the map T defined by $T(x) = F(x) - a$, belongs to $L(X,Y)$. If we define $\hat{F}:\hat{X} \longrightarrow \hat{Y}$ by $\hat{F}([A,B]) = [\overline{F(A)},\overline{F(B)}]$, then $\hat{F} = \hat{T}$.

<u>Remark 2.</u> If $T \in L(X,Y)$ and T is 1-1 and onto (i.e. an algebraic isomorphism), then $\hat{T}$ need not be either 1 - 1 or onto.

<u>Example 3.</u> Let $X = C([0,1])$, and Y be a subspace of X such that $Y = \{x:[0,1] \longrightarrow R \mid x$ is continuously differentiable on $[0,1]$ and $x(0) = 0\}$. We define:

$$(Tx)(t) = \int_0^t x(\tau)d\tau \quad \text{for all } x \in X;\ t \in [0,1].$$

Then, of course, $T \in L(X,Y)$; $\|T\| \leq 1$ and T is a map 1 - 1 and onto.

1) Let $S_1 = \{x \mid x \in X, \|x\| \leq 1\}$,

$D_1 = \{x \mid x \in X; \|x\| \leq 1$ and $x(0) = 0\}$.

Then $S_1, D_1 \in \mathcal{C}_0(X)$ and $[S_1,D_1] \neq 0$. It is easy to verify that for each $\varepsilon > 0$ and each $x \in S_1$, there exists $\bar{x} \in D_1$, such that $\bar{x}(t) = x(t)$ for all $\frac{\varepsilon}{2} \leq t \leq 1$. Then $\|Tx - T\bar{x}\| \leq \varepsilon$. This shows $T_c(D_1) \supseteq T(S_1)$. It follows $\hat{T}([S_1,D_1]) = [T_c(S_1),T_c(D_1)] =$

$= 0$, which shows that $\hat{T}$ is not 1 - 1.

2) Let $Q = \{y \in Y \mid \|y\| \leq 1\}$; then $[Q, \{0\}] \in \hat{Y}$. Suppose that $\hat{T}$ is onto; then there exist $A, B \in \mathcal{C}_0(X)$ such that $\hat{T}([A,B]) = [T_c(A), T_c(B)] = [Q, \{0\}]$ or $T_c(A) = T_c(B) +^{*} Q \supseteq$ $\supseteq T_c(B) + Q$. It follows that

$$Q \subseteq T_c(A) - T_c(B) \subseteq \overline{T(A) - T(B)}.$$

Let M be a positive number such that for all $x \in A \cup B$ $\|x\| \leq M$. Then $|y(t) - y(t')| \leq 2M|t - t'|$ for all $y \in T(A) -$ $- T(B)$ and for all $t, t' \in [0,1]$. The set $\{y \in Y \mid |y(t) - y(t')| \leq$ $\leq 2M|t - t'|$ for all $t, t' \in [0,1]\}$ is closed in Y. Hence for all $y \in Q \subseteq \overline{T(A) - T(B)}$ and for all $t, t' \in [0,1]$ we have

$$|y(t) - y(t')| \leq 2M|t - t'| ,$$

a contradiction with the fact that for all $k > 2$ there is $y \in Q$ and $t_1, t_2 \in [0,1]$ such that $|y(t_1) - y(t_2)| \geq k|t_1 - t_2|$. For instance, put $y(t) = \int_0^t x(\tau)\,d\tau$, where:

$$x(t) = \begin{cases} K & \text{for } 0 \leq t \leq \frac{1}{2K}, \\ \frac{3}{2}K - K^2 t & \text{for } \frac{1}{2K} \leq t \leq \frac{3}{2K}, \\ 0 & \text{for } t \geq \frac{3}{2K}, \end{cases}$$

then $y \in Q$ and $|y(\frac{1}{2K}) - y(0)| = K\frac{1}{2K}$. Hence $\hat{T}$ is not onto.

Let X, Y be locally convex spaces, $T \in L(X,Y)$. We denote the adjoint operator of T by T', the range of T' by $R(T')$, the strong topology in the dual space X' by $\beta(X',X)$.

<u>Proposition 5.</u> Let X, Y be locally convex spaces, $T \in$ $\in L(X,Y)$. If $\overline{R(T')}^{\beta(X',X)} = X'$, then $\hat{T}$ is 1 - 1.

Proof. Let $[A,B] \neq 0$, then $A \nsubseteq B$ or $B \nsubseteq A$. Assume for instance $A \nsubseteq B$. There is $x_0 \in A$ and $x_0 \notin B$. By the Hahn-Banach theorem is $x' \in X'$ such that $\langle x', x_0 \rangle = \beta > \alpha = \sup \{\langle x', x \rangle \mid x \in$

$\in B\}$. Set $\varepsilon = \frac{1}{3}(\beta - \alpha) > 0$, $V = (A \cup B)^o = \{x' \in X' \mid |\langle x', x\rangle| \leq 1$ for all $x \in A \cup B\}$. Then V is an 0-neighborhood in the topology $\beta(X', X)$ in X'. By our assumption we have $(x' + \varepsilon V) \cap R(T') \neq \emptyset$. Let $y' \in Y'$ be such that $T'(y') \in x' + \varepsilon V$, then $|\langle x' - T'(y'), x\rangle| \leq \varepsilon$ for all $x \in A \cup B$. We have

$$\langle y', Tx_o\rangle = \langle T'y', x_o\rangle = \langle x', x_o\rangle + \langle T'y' - x', x_o\rangle \geq \beta - \varepsilon = \frac{2\beta + \alpha}{3} .$$

For all $x \in B$ we have

$$\langle y', Tx\rangle = \langle T', y', x\rangle = \langle x', x\rangle + \langle T'y' - x', x\rangle \leq \alpha + \varepsilon = \frac{\beta + 2\alpha}{3} .$$

Therefore $\langle y', Tx_o\rangle > \sup\{\langle y', Tx\rangle \mid x \in B\}$. Hence $Tx_o \notin \overline{T(B)} = T_c(B)$; which shows that $\hat{T}([A,B]) = [T_c(A), T_c(B)] \neq 0$. This completes our proof.

Theorem 3. Let X be a locally convex space, with the topology τ induced by the family of seminorms $\mathcal{P} = (p)$, M a subspace of X, p_M the restriction of p on M. Let $i: M \longrightarrow X$ be an inclusion map of M into X. Then:

1) $\hat{i}: \hat{M} \longrightarrow \hat{X}$ is isometric in the following sense: $\hat{p}(\hat{i}([A,B])) = \hat{p}_M([A,B])$ for all $[A,B] \in \hat{M}$ and $p \in \mathcal{P}$.

2) If X is a normed linear space, then the isometry $\hat{i}$ is an isomorphism of $\hat{M}$ onto $\hat{X}$ if and only if $\overline{M} = X$.

Proof. 1) Let $[A,B] \in \hat{M}$, then

$$\hat{p}(\hat{i}([A,B]) = \hat{p}([\overline{A}, \overline{B}]) = d_p(\overline{A}, \overline{B}) = d_{p_M}(A,B) = \hat{p}_M([A,B]) .$$

2)a) Let X be a normed linear space and $\overline{M} = X$. We denote by $S_1 = \{x \mid x \in X;\ \|x\| \leq 1\} \in \mathcal{C}_0(X)$ the unit closed ball of X and the unit open ball of X by $S_1^o = \{x \in X \mid \|x\| < 1\}$. Let

$[A,B] \in \hat{X}$, then $[A,B] = [A +^* S_1, B +^* S_1]$. We set $A_1 = (A +^* S_1) \cap$ $\cap M \in \mathcal{C}_0(M)$ and $B_1 = (B +^* S_1) \cap M \in \mathcal{C}_0(M)$. We have $\bar{A}_1 =$ $= \overline{(A +^* S_1) \cap M} \supseteq \overline{(A + S_1^o) \cap M} \supseteq (A + S_1^o) \cap \bar{M} = A + S_1^o$ and hence $\bar{A}_1 \supseteq \overline{A + S_1^o} = \overline{A + S_1} = A +^* S_1$. On the other hand, $A_1 \subseteq A +^* S_1$. Therefore $\bar{A}_1 = A +^* S_1$. Similarly one can obtain $\bar{B}_1 = B +^* S_1$. Then

$$\hat{i}([A_1, B_1]) = [\bar{A}_1, \bar{B}_1] = [A +^* S_1, B +^* S_1] = [A,B].$$

This shows that $\hat{i}$ is an isomorphism of $\hat{M}$ onto $\hat{X}$.

b) Let $\hat{i}$ be an isomorphism of $\hat{M}$ onto $\hat{X}$, we shall prove that $X \subseteq \bar{M}$. Suppose $x \in X$, then $[\{x\},\{0\}] \in \hat{X}$. By our assumption, there is an $[A,B] \in \hat{M}$ such that $\hat{i}([A,B]) = [\bar{A},\bar{B}] = [\{x\},\{0\}]$. This implies $\bar{A} = \bar{B} + \{x\}$, whence $x \in \{x\} \subseteq \bar{A} - \bar{B} \subseteq \overline{A - B} \subseteq \bar{M}$. The proof is complete.

Remark 3. If X is a normable linear space, then X has the following property:

(*) If $\tilde{X}$ is the completion of X and $i: X \longrightarrow \tilde{X}$ is the inclusion of X into $\tilde{X}$, then $\hat{i}: \hat{X} \longrightarrow \hat{\tilde{X}}$ is an isomorphism of $\hat{X}$ onto $\hat{\tilde{X}}$.

If X is not a normable linear space, then X need not have the property (*). Suppose, for instance, that X is a locally convex space, which is quasicomplete (i.e. every bounded closed subset A of X is complete (see [8])) but not complete. We claim that X has not the property (*). Let $\tilde{X}$ be the completion of X; $i: X \longrightarrow \tilde{X}$ be the inclusion of X into $\tilde{X}$. Suppose that $\hat{i}$ is an isomorphism of $\hat{X}$ onto $\hat{\tilde{X}}$. Assume that $x \in \tilde{X}$ but $x \notin X$. There is an $[A,B] \in \hat{X}$ such that $\hat{i}([A,B]) = [\bar{A},\bar{B}] = [\{x\},\{0\}]$, where $\bar{A}$ denotes the closure of A in $\tilde{X}$. Then $x + \bar{B} = \bar{A}$. By the assumption A, B are complete, it follows that A, B are closed in $\widetilde{X}$. Then we have $x + B = A$ and $x \in A - B \subseteq X$. This contradicts $x \notin X$.

<u>Theorem 4.</u> Let X be a strict inductive limit of a sequence of locally convex spaces X_n (i.e. $X = \varinjlim_n X_n$). If X_n has the property (∗) for all n, then X possesses the property (∗).

Proof: Let $\widetilde{X}_n$ be the completion of X_n, $i_n : X_n \longrightarrow \widetilde{X}_n$, $\widetilde{i}_n : \widetilde{X}_n \longrightarrow \widetilde{X}_{n-1}$ the inclusions. We set $Y = \varinjlim_n \widetilde{X}_n$. By theorem II 6.6 [8] the space Y is complete and it is easy to see that X is dense in Y. Then $Y = \widetilde{X}$.

Let $[A,B] \in \hat{Y} = \hat{\widetilde{X}}$, there exists n_0 such that $A \subseteq \widetilde{X}_{n_0}$ and $B \subseteq \widetilde{X}_{n_0}$ (Theorem II 6.5 [8]). That is $[A,B] \in \hat{\widetilde{X}}_{n_0}$. According to our assumption there exists $[A_1,B_1] \in \hat{X}_{n_0}$ such that $\hat{i}_{n_0}([A_1,B_1]) = [A,B]$. Of course $[\bar{A}_1 \cap X, \bar{B}_1 \cap X] \in \hat{X}$ and $\hat{i}([\bar{A}_1 \cap X, \bar{B}_1 \cap X]) = \hat{i}_{n_0}([A_1,B_1]) = [A,B]$. This completes our proof.

<u>Corollary 2.</u> Let X be a locally convex space, M a closed subspace of X such that M has the complement in X. Then:

1) $\hat{i}(\hat{M})$ is a closed subspace of $\hat{X}$, where $i : M \longrightarrow X$ is the inclusion.

2) If $\dim X \geq 2$, then $\hat{X}$ is not complete.

Proof. 1) From the assumption that M has the complement in X we conclude that there exists $P \in L(X,M)$ such that $P \circ i = I_M$, where I_M denotes the identity of M. Then $\hat{P} \circ \hat{i} = I_{\hat{M}}$. Let $[A,B] \in \hat{i}(\hat{M})$, then there exists a net $\{[A_j,B_k]\}_{j \in J}$ of $\hat{M}$ such that $\{\hat{i}([A_j,B_j])\}_{j \in J}$ converges to $[A,B]$. Then $\{[A_j,B_j]\}_{j \in J} = \{\hat{P} \circ \hat{i}([A_j,B_j])\}_{j \in J}$ converges to $\hat{P}([A,B]) = [P_c(A),P_c(B)] \in \hat{M}$, as $\hat{P}$ is a continuous map. Finally $\{\hat{i}([A_j,B_j])\}_{j \in J}$ converges to

$\hat{i}([P_c(A),P_c(B)]) \in \hat{i}(\hat{M})$

in view of the similar argument. Since $\hat{X}$ is the Hausdorff space, $[A,B] = \hat{i}([P_c(A),P_c(B)])$, for $[A,B]$ is also a limit of $\{\hat{i}([A_j,B_j])\}_{j\in J}$. This proves that $[A,B] \in \hat{i}(\hat{M})$ and $\hat{i}(\hat{M})$ is the closed subspace of $\hat{X}$.

2) Let $\dim X \geq 2$ and suppose that $\hat{X}$ is complete. Take a two dimensional subspace X_2 of X, then X_2 has a complement in X (see Corollary II.4.2 [8]). Then $\hat{i}(\hat{X}_2)$ is the closed subspace of the complete space $\hat{X}$. Hence $\hat{i}(\hat{X}_2)$ is complete. Since $\hat{i}$ is isometric, we conclude that $\hat{X}_2$ is complete. We know that X_2 is isomorphic with R_2. Thus $\hat{X}_2$ is isomorphic with $\hat{R}_2$. This means that $\hat{R}_2$ is complete, a contradiction with the Example 2.

<u>Corollary 3.</u> Let X be a metrizable locally convex space and let X have the property (*) (in particular X is a normed space), then $\varkappa(\mathcal{C}_0(X))$ is a closed subset of $\hat{X}$ if and only if X is complete.

Proof. 1) If X is an F-space, then $\varkappa(\mathcal{C}_0(X))$ is complete (see [3]) and hence $\varkappa(\mathcal{C}_0(X))$ is closed in $\hat{X}$.

2) Let $\varkappa(\mathcal{C}_0(X))$ be closed in $\hat{X}$, $\tilde{X}$ be the completion of X. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{C}_0(X) & \xrightarrow{\ i_c\ } & \mathcal{C}_0(\tilde{X}) \\ \varkappa \downarrow & & \downarrow \varkappa \\ \hat{X} & \xrightarrow{\ \hat{i}\ } & \hat{\tilde{X}} \end{array}$$

Since $\hat{i}$ is an isomorphism of $\hat{X}$ onto $\hat{\tilde{X}}$, $\hat{i}(\varkappa(\mathcal{C}_0(X)))$ is a closed subset of $\varkappa(\mathcal{C}_0(\tilde{X}))$. We know that $\varkappa(\mathcal{C}_0(\tilde{X}))$ is complete as $\tilde{X}$ is an F-space. It follows $\hat{i}\circ\varkappa(\mathcal{C}_0(X))$ is complete and hence $\mathcal{C}_0(X)$ is complete, since $\hat{i}\circ\varkappa$ is isometric. Therefore, X is complete, too. This completes the proof.

<u>Corollary 4.</u> Let X be a metrizable locally convex space,

$[A,B] \in \overline{\varkappa(\mathcal{C}_0(X))}$ and suppose that one of two sets A, B is weakly compact. Then $[A,B] \in \varkappa(\mathcal{C}_0(X))$.

Proof. Let $\tilde{X}$ be a completion of X, then

$$\hat{i}([A,B]) \in \hat{i}(\overline{\varkappa(\mathcal{C}_0(X))}) \subset \overline{\hat{i}(\varkappa(\mathcal{C}_0(X)))} \subseteq \varkappa(\mathcal{C}_0(\tilde{X})).$$

There is $C \in \mathcal{C}_0(\tilde{X})$ such that

$$[C,\{0\}] = \hat{i}([A,B]) = [\bar{A},\bar{B}],$$

where A denotes the closure of A in $\tilde{X}$. Then $\bar{A} = \bar{B} +^* C = \overline{B + C}$.

Assume now that 1) A is weakly compact (i.e. $w(X,X')$-compact), then A is $w(\tilde{X},\tilde{X}')$-compact for $w(\tilde{X},\tilde{X}')|_X = w(X,X')$. Hence A is $w(\tilde{X},\tilde{X}')$-closed in $\tilde{X}$ and A is closed in $\tilde{X}$, since A is convex. Then we have:

$$A = \overline{B + C} \supseteq B + C \text{ or } C \subseteq A - B \subseteq X.$$

This shows that $[A,B] = [C,\{0\}] \in \varkappa(\mathcal{C}_0(X))$.

2) If B is weakly compact, then by the same way as in 1) we prove that B + C is closed in $\tilde{X}$ and we obtain $\bar{A} = B + C$, $A = \bar{A} \cap X = (B + C) \cap X = B + C \cap X$. Put $C_1 = C \cap X$, then we have $A = B + C$, i.e. $[A,B] = [C_1,\{0\}] \in \varkappa(\mathcal{C}_0(X))$, which concludes the proof.

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Matematický ústav UK

Sokolovská 83, 18600 Praha 8

Československo

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