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INTEGRAL REPRESENTATION OF n-VARIABLE POSITIVE REAL FUNCTIONS

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Abstract: Multivariable functions analytic in a half-plane may have some interesting properties which cannot be directly derived from the known results on functions analytic in polydiscs, but follow from their integral representation. Such representation for functions with positive real part in the right half-plane is given here.

Key words: Several complex variables, functions with positive real part, integral representations.

Classification: 32A25
30D50, 32A30

In recent years, the class $\mathcal{R}^{(n)}$ of positive real functions of several complex variables gained considerable attention due to their significance in electrical network theory. Their integral representation can be the starting point of detailed mathematical description of these functions and, in due course, a tool for solving difficult problems in approximation, analysis and synthesis of multivariable electrical methods.

In what follows, we shall denote $z = (z_1, z_2, \dots, z_n)$ the points of complex Euclidean n-space C^n, $\underline{1} = (1,1,\dots,1)$, and further the following notation of sets will be used (with the superscript n occasionally omitted):

$\Gamma^{(n)} = \{z \in C^n,\ \mathrm{Re}\ z_i > 0,\ i=1,2,\dots,n\}$,

$V^{(n)} = \{z \in C^n,\ \mathrm{Re}\ z_i = 0,\ i=1,2,\dots,n\}$,

$\bar{\Gamma}^{(n)} = \{z \in C^n,\ \mathrm{Re}\ z_i \geqq 0,\ i=1,2,\dots,n\} \cup \{\infty\}$,

where $\{\infty\}$ denotes the set compactifying C^n. $V^{(n)}$ is isomorphic to a real n-dimensional Euclidean space and forms the Bergman-Šilov boundary of $\Gamma^{(n)}$. Considering intervals I in this space and denoting $t_i = \mathrm{Im}\ z_i$ we may set

$$\mathrm{mes}(I) = \frac{1}{\pi^n} \int_I \frac{dt_1\, dt_2 \dots dt_n}{(1+t_1^2)\,(1+t_2^2)\dots(1+t_n^2)},$$

which can be extended to a measure μ on V. Evidently, $\mathrm{mes}(V) = 1$. The set of functions $f: V \longrightarrow C^1$, for which $\int_V |f|^2 d\mu < +\infty$, will be denoted by $L^2(V)$; this set is a Hilbert space with the scalar product $\langle f,g\rangle = \int_V f\,\bar{g}\,d\mu$. The set of functions holomorphic on $\Gamma^{(n)}$ will be $A(\Gamma)$, the set of functions continuous on $\bar{\Gamma}^{(n)}$ except perhaps at infinity will be $C(\Gamma)$. Finally, $G = \{f \in A(\Gamma) \cap C(\Gamma);\ f|_V \in L^2(V)\}$.

A function $f: \Gamma^{(n)} \longrightarrow C$ is called positive ($f \in \mathcal{P}^{(n)}$) if

(1)
i) $f \in A(\Gamma^{(n)})$,
ii) $f(\Gamma^{(n)}) \subset \Gamma^{(1)}$;

it is called positive real ($f \in \mathcal{R}^{(n)}$), if, in addition,

iii) $f(\bar{z}) = \overline{f(z)}$ for all $z \in \Gamma^{(n)}$.

Positive real functions are analytic in an unbounded domain, their behaviour cannot be discussed directly in terms of H^p or L^p spaces by simply transforming the corresponding results for functions on polydiscs. Nevertheless, these functions follow a relatively simple pattern and, as for their boundary behaviour, extensions from subsets of $\Gamma^{(n)}$ and related problems similar results to those for functions on polydiscs can be proved. The basis of such investigation is their integral representation, which will be summarized below.

Lemma 1. The set S of functions

$$S = \left\{ \left(\frac{1 - z_1}{1 + z_1}\right)^{m_1} \left(\frac{1 - z_2}{1 + z_2}\right)^{m_2} \cdots \left(\frac{1 - z_n}{1 + z_n}\right)^{m_n}, \right.$$

$$\left. - \infty < m_1, m_2, \ldots, m_n < + \infty \ , \ m_i \text{ integers} \right\}$$

is complete and orthonormal in $L^2(V)$.

▶ The ortonormality is obvious e.g. form

$$\frac{1}{\pi} \int_{-\infty}^{+\infty} \left(\frac{1 - jt_k}{1 + jt_k}\right)^m \frac{dt_k}{1 + t_k^2} = \delta_{0,m}; \ (j^2 = -1);$$

the completeness follows from earlier and more general results (see e.g. [5], chap. XVII).

S^+ and S^-, respectively, will further denote the subsets of the countable set S such that $m_i \geqq 0$ for all $i=1,2,\ldots,n$ and $m_i \leqq 0$ for all $i=1,2,\ldots,n$, respectively. The set S^+ can be ordered so that $S^+ = \{ \varphi_k(z), k=0,1,2,\ldots \}$. Evidently, $S^+ \cap S^- = \{ \varphi_0(z) \} = \{1\}$. S^+ is a complete orthonormal system in G.

Lemma 2. For all $u \in \Gamma^{(n)}$, $z \in \bar{\Gamma}^{(n)}$ the series

$$\sum_{k=0}^{\infty} \varphi_k(u) \overline{\varphi_k(z)} = H(u,z) \tag{2}$$

converges absolutely and uniformly on any compact subset of $\Gamma^{(n)} \times \bar{\Gamma}^{(n)}$;

i) $H(u,z) = \frac{1}{2^n} \prod_{k=1}^{n} \left(1 + \frac{1 + u_k \bar{z}_k}{u_k + \bar{z}_k}\right)$;

ii) $H(u,z) = \overline{H(z,u)}$ for all $(u,z) \in \Gamma^{(n)} \times \bar{\Gamma}^{(n)}$ and $H(z,z) > 0$ for all $z \in \Gamma^{(n)}$;

iii) $H(u,\underline{1}) = H(\underline{1},z) = 1$;

iv) denoting H_z the function assuming the value $H(u,z)$ at a point $u \in \Gamma^{(n)}$, then for any $z \in \bar{\Gamma}^{(n)}$ there is $H_z \in A(\Gamma)$;

v) for any $z \in \Gamma^{(n)}$ there is $H_z \in A(\Gamma) \cap C(\Gamma)$.

For $u \in \Gamma^{(n)}$, $z \in \bar{\Gamma}^{(n)}$ there is

$$\left|\frac{1-u}{1+u}\,\frac{1-\bar{z}}{1+\bar{z}}\right| < 1$$

and therefore the series (2) is majorized by $\sum_{m=1}^{\infty} mq^m$ for some $q<1$, which is convergent. Further, i) holds true for n=1. The statement follows by induction when considering that there exists an ordering $\varphi_k^{(n)}$ such that

$$\sum_{k=0}^{\infty} \varphi_k^{(n)}(u)\,\varphi_k^{(n)}(\bar{z}) = \Big(\sum_{k=0}^{\infty} \varphi_k^{(n-1)}(u)\,\varphi_k^{(n-1)}(\bar{z})\Big) \Big(\sum_{k=0}^{\infty} \Big(\frac{1-u_n}{1+u_n}\,\frac{1-\bar{z}_n}{1+\bar{z}_n}\Big)^k\Big)$$

where the superscripts n denote the dimension of the corresponding complex space. The rest of Lemma 2 is obvious; it is quoted here for convenience only.

<u>Lemma 3</u>. If $f \in G$ then

$$f(z) = \langle f, H_z \rangle \text{ for all } z \in \Gamma^{(n)}$$

and

$$f(\underline{1}) = \langle f, \underline{1} \rangle = \int_V f\, d\mu$$

► We have $f(z) = \sum_{k=0}^{\infty} \alpha_k \varphi_k(z)$ and therefore

$$\langle f, H_z \rangle = \Big\langle \sum_{k=0}^{\infty} \alpha_k \varphi_k(u), \sum_{\ell=0}^{\infty} \varphi_\ell(u)\,\overline{\varphi_\ell(z)} \Big\rangle =$$
$$= \sum_{k=0}^{\infty} \alpha_k \langle \varphi_k(u), \varphi_k(u)\,\overline{\varphi_k(z)} \rangle = \sum_{k=0}^{\infty} \alpha_k \varphi_k(Z).$$

(In the sequel, the notation like $\langle f(u), g(u)\, h(z) \rangle$ means $\overline{h(z)} \int_V f\, \bar{g}\, d\mu$, i.e. u denotes the "integration variable", while z is a "parameter" of integration.) ◄

Let P and $\tilde{P}$ respectively, denote the projection in $L^2(V)$ onto the subspace spanned by the set S^+ and S^-. Clearly, $PL^2(V) \cap \tilde{P}L^2(V) = \{f : f \equiv \text{const.}\}$. If $f \in G$, then $Pf = f$ and, moreover, $\tilde{P}f = \tilde{P}Pf = f(\underline{1}) = \langle f, \underline{1} \rangle$. We have $H_z \in PL^2(V)$ for all $z \in \bar{\Gamma}^{(n)}$. The projection in $L^2(V)$ onto the subspace spanned by the set $S^+ \cup S^-$ will further be denoted by Q.

Consider now a function $f \in PL^2(V)$ and an ordering of $S^+ =$ $= \{\varphi_k(z), k=1,2,\dots, \varphi_0(z) \equiv 1\}$. Then the set S^- can be so ordered, that $S^- = \{\psi_k(z)\}$ and $\psi_k(z)\, \varphi_k(z) = 1$, $k=0,1,\dots$. Due to completeness of S^+ in $PL^2(V)$ and that of S^- in $\tilde{P}L^2(V)$ the relation

$$\langle f^*, \psi_k \rangle = \overline{\langle f, \varphi_k \rangle};\ k=0,1,\dots$$

defines for any $f \in PL^2(V)$ a certain function $f^* \in \tilde{P}L^2(V)$:

$$f^* = \sum_{k=0}^{\infty} \overline{\langle f, \varphi_k \rangle}\, \psi_k. \tag{3}$$

If, in addition, $f \in G$, then f and f^* have complex conjugate restrictions to the set V, i.e.

$$f|_V = \bar{f}^*|_V \tag{3a}$$

and therefore $f + f^* \in QL^2(V)$ and $(f + f^*)|_V = 2 \operatorname{Re} f|_V$, where Re means the real part.

<u>Lemma 4</u>. Suppose $f \in G$, then

i) for all $z \in \bar{\Gamma}^{(n)}$ there is

$$\langle f^*, H_z \rangle = \langle \bar{f}, H_z \rangle = \overline{f(\underline{1})};$$

ii) for all $z \in \Gamma^{(n)}$ there is

$$f(z) = -\overline{f(\underline{1})} + \langle 2 \operatorname{Re} f, H_z \rangle .$$

The proof may be omitted.

As usual, if a set B of μ-measurable functions is given and $\int_V f\, d\mu = 0$ for all $f \in B$, we shall say that the measure μ is orthogonal to B. Now the first part of the main Theorem can be proved.

<u>Theorem 1</u>. Suppose $f \in A(\Gamma^{(n)})$ and $\operatorname{Re} f(\Gamma^{(n)}) > 0$, i.e.

$f \in \mathcal{P}^{(n)}$. Then there exists a positive measure μ orthogonal to the subspace $S \setminus (S^+ \cup S^-)$ such that for all $z \in \Gamma^{(n)}$

$$f(z) = -\overline{f(\underline{1})} + 2\int_V \overline{H}_z \, d\mu \quad , \quad \int_V d\mu < +\infty \, .$$

► Let r_i be real numbers, $0 < r_i < 1$, $i=1,2,\ldots,n$, and $\rho_i = \dfrac{1 - r_i}{1 + r_i}$, $0 < \rho_i < 1$. Define a function f_r by

$$f_r = f\left(\frac{z_1 + \rho_1}{1 + \rho_1 z_1}, \frac{z_2 + \rho_2}{1 + \rho_2 z_2}, \ldots, \frac{z_n + \rho_n}{1 + \rho_n z_n}\right).$$

Since $f \in A(\Gamma^{(n)})$ and $\operatorname{Re} \dfrac{z_i + \rho_i}{1 + \rho_i z_i} > 0$ for all $\operatorname{Re} z_i > 0$, we have $f_r \in A(\Gamma^{(n)})$. Moreover, $\operatorname{Re} \dfrac{\rho_i + jt_i}{1 + j\rho_i t_i} = \dfrac{1 + t_i^2}{1 + \rho_i^2 t_i^2} \rho_i > 0$ and therefore $f_r \in C(\Gamma^{(n)})$.

Using the maximum principle, we may state that for $z \in V$

$$(4) \qquad |f_r(z)| = \left| f\left(\frac{\rho + jt}{1 + j\rho t}\right)\right| \leqq \max \left| f\left(\frac{\rho' + jt}{1 + j\rho' t}\right)\right|$$

for some $1 > \rho' > \rho$. Here, $\rho' > \rho$ means $\rho'_i > \rho_i$ for all $i = 1,2,\ldots,n$ and the notation is similarly shortened in (4). This inequality implies $f_r \in L^2(V)$ for all $0 < r < \underline{1}$. Summarizing we obtain $f_r \in G$ for all $0 < r < \underline{1}$. According to ii) of Lemma 4, there is

$$(5) \qquad f_r(z) = -\overline{f(\underline{1})} + 2\langle \operatorname{Re} f_r, H_z \rangle$$

or otherwise

$$f_r(z) = -\overline{f(\underline{1})} + 2\int_V \overline{H}_z \, d\mu_r,$$

where $d\mu_r = \operatorname{Re} f_r \dfrac{dt_1, dt_2, \ldots, dt_n}{(1 + t_1^2)(1 + t_2^2) \ldots (1 + t_n^2)}$.

In (5), $\operatorname{Re} f_r > 0$ is the restriction of $f_r + f_r^* \in QL^2(V)$ to the set V and therefore it is orthogonal to all functions $\Phi \in L^2(V)$

which does not belong to QL (V). The total variation of ^$_r$ equals

$$/ d\hat{}a_r * \langle Re\ f\ \hat{}\ ) \ll Re\ f_r(p * Re\ fCl).$$

All the previous conclueions are valid for any vector r<lj the set of positive measures *(A,* is equally bounded and has the orthogonality property as stated. The function H^ is continuous and boundsd on V by v) in Lemma 2. Using now a generalization of Helly's theorem, which is due to H,X. Bray (see *Í4l$_f$* p. 192), the assertion of the theorem follows.

To prove the converse to the previous theorem, we need some simple lemma*.

Lemma 5» Suppose f eG, then

$$\langle Re\ f, H_u\ H_z\rangle = H(u,z)^{f}\ W? + * (\ll)\ {}_{for\ all\ 8>u\ 6}\ p<n)_{\#}$$

• Let ze $P^{(n)}$, then H(u,z) f(u) « *{H$_{\%}$f)(\i)e* Q **and therefore** (see Lemma 3)

(6') $\quad H(u,z)\ fCu) = \langle H_2 f_f H_u\rangle = \langle f_f H_u I\hat{}\rangle.$

Similarly (see Lemma 2 and (3a)),

(6") $\quad H(u,z)\ fl\check{z}) * H(z,u)\ f(z) \gg \langle f,H_8\ H_u\rangle * \langle f_f H_u \backslash / .$

Adding (6') and (6") we obtain the desired result.

Lemma 6. For all z,u c $n\hat{}^{n'}$ there is

$$Q(H_U\ 5J[) \ll H(z,u)ÍH_u + 1\hat{}-1),$$

where Q is the projection operátor introduced above.

Let f be an arbitrary function belonging to G. Then (using Lemma 5 and ii) of Lemma 4)

$$\langle Re\ f, H_u H\hat{}\rangle = |H(u,z)C\langle f \bullet f^*, H_u\rangle \bullet \langle T + f^*, H\hat{}\rangle - til) - *0>3 \bullet$$

Using Lemma 3 and i) of Lemma 4 we obtain

$$\langle \operatorname{Re} f, H_u \overline{H_z} \rangle = \frac{1}{2} H(u,z) \langle f + f^*, H_u + \overline{H_z} - 1 \rangle$$

or, otherwise

$$(7) \quad \langle f + f^*; H_u \overline{H_z} - \overline{H(u,z)}(H_u + \overline{H_z} - 1) \rangle = 0.$$

This is an orthogonality condition and, moreover, $f + f^* \in QL^2(V)$. In order to satisfy (7), either $H_u \overline{H}_z - H(u,z)(H_u + \overline{H}_z - 1)$ belongs to the orthogonal complement of $QL^2(V)$ or $Q(H_u \overline{H}_z) =$ $= Q(H(u,z)(H_u + \overline{H}_z - 1))$. The first condition cannot be met because $H_u + \overline{H}_z - 1 \in QL^2(V)$; and therefore $Q(H_u \overline{H}_z) = H(u,z)\, Q(H_u +$ $+ \overline{H}_z - 1)$ which implies the statement.

<u>Theorem 2</u>. Suppose μ is a positive measure finite on V and orthogonal to the subspace spanned by the set $S \setminus (S^+ \cup S^-)$; then the function f,

$$f(z) = -\overline{f(\underline{1})} + 2\int_V \overline{H}_z \, d\mu ,$$

is analytic in $\Gamma^{(n)}$ and has a positive real part there.

The function $\varphi(z) = \int_V \overline{H}_z \, d\mu$ is holomorphic in $\Gamma^{(n)}$ since the integral converges uniformly and absolutely on any compact subset of $\Gamma^{(n)}$. For $z = \underline{1}$ we obtain

$$\int_V d\mu = \operatorname{Re} f(\underline{1})$$

and therefore

$$\operatorname{Re} f(z) = \int_V (H_z + \overline{H}_z - 1) d\mu .$$

From Lemma 6 it follows that

$$H_u + \overline{H}_z - 1 = \frac{1}{H(z,z)} Q(H_z \overline{H}_z)(u)$$

and therefore

$$\operatorname{Re} f(z) = \frac{1}{H(z,z)} \int_V Q(H_z \overline{H}_z)(u) d\mu = \frac{1}{H(z,z)} \int_V (H_z \overline{H}_z)(u) d\mu ;$$

since $H(z,z) > 0$ for $\operatorname{Re} z > 0$, the proof is completed.

The two above theorems give necessary and sufficient conditions for a function f to be positive. To meet condition iii) in (1), additional assumptions must be made. We may summarize the corresponding result as follows:

<u>Theorem 3</u>. The function $f: \Gamma^{(n)} \longrightarrow C$ is positive if and only if it admits a representation

$$f(z) = -\overline{f(\underline{1})} + 2\int_V \overline{H_z}\, d\mu$$

with a finite positive measure μ on $V^{(n)}$ which is orthogonal to the subspace spanned by the set $S \setminus (S^{+} \cup S^{-})$. The function f is positive real if and only if it is positive and the measure μ is orthogonal to the set Ω of functions $\phi \in L^2(V)$ whose real part equals zero μ-almost everywhere.

▶ Only the second part remains to be proved. A positive function f is positive real if and only if $f(\overline{z}) = \overline{f(z)}$ for all $z \in \Gamma^{(n)}$. According to Theorems 1, 2

$$f(\overline{z}) = \overline{-f(\underline{1})} + 2\int_V (\Sigma\, \overline{\varphi_k(u)}\;\; \overline{\varphi_k(z)})d\mu,$$

$$\overline{f(z)} = -f(\underline{1}) + 2\int_V (\Sigma\, \varphi_k(u)\;\; \overline{\varphi_k(z)})d\mu .$$

The right-hand sides of these two equations are equal if and only if

$$\int_V (\Sigma\, (\varphi_k(u) - \overline{\varphi_k(u)})\;\; \overline{\varphi_k(z)}d\mu = \mathrm{Re}\, f(\underline{1}),$$

which is equivalent to

$$\int_V (\Sigma\, \mathrm{Im}\;\varphi_k(u)\; \mathrm{Re}\;\varphi_k(z))d\mu(u) = 0.$$

The last condition will be satisfied iff

$$\int_V \mathrm{Im}\;\varphi_k(u)\, d\mu(u) = 0 \text{ for all } k=0,1,\dots .$$

The system $\{\varphi_k\}$ is complete and therefore the measure has to be orthogonal to all functions which have the imaginary part of their reduction to $V^{(n)}$ equivalent to zero.

Specifying the above theorems to $n=1$, a form of integral representation slightly different from the usual Herglotz theorem (see e.g. [2]) is obtained. Analyzing the two proofs it can be seen that this is mainly due to different handling of the behaviour of function f at infinity. Among others the above result enables us to generalize the Wolf's theorem on positive functions, interpolation theorems in the class of positive and positive real functions, results on boundary behaviour of positive function and similar topics.

We hope to devote another paper to these problems.

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