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SELECTIONS USING ORDERINGS (NON-SEPARABLE CASE)

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Abstract: Two selection theorems with their proofs using the lexicographic ordering of sequences of positive integers are extended for correspondences of complete (non-separable) metric spaces.

Key words: Point-analytic space, point-Luzin space, Suslin set, Baire set, σ-dd-preserving correspondence, σ-db-preserving correspondence.

Classification: 54C65, 54H05

The main result is Theorem below which generalizes the selection Theorem of von Neumann [N] and partially the "uniformization type" Theorem of Mazurkiewicz [M] to the non-separable case. The proofs follow the pattern of the proofs of von Neumann (Lemma 1(a) corresponds to [N, Lemma 16]) and of K. Kuratowski (Lemma 1(b) corresponds to [K, Th. 3, p. 491]) respectively. Lemma 1(b) is proved also in [Ho].

The proofs are using the lexicographic order on $\varkappa^{\omega}$. In what follows, $\varkappa$ is an infinite cardinal conceived as the set of all ordinals of cardinal $< \varkappa$, and endowed with its well order and the discrete uniformity. The product space $\varkappa^{\omega}$ is a metrizable complete uniform space endowed with the lexicographic order $\prec$ defined as follows:

$$\{\alpha_n\} \prec \{\beta_n\} \text{ iff } \{\alpha_n\} \neq \{\beta_n\} \text{ and } \alpha_k < \beta_k$$

for the smallest k such that $\alpha_k \neq \beta_k$.

In what follows, we shall need the following two elementary facts about the order: each non-void closed set in $\varkappa^\omega$ has the smallest element, and the set

$$\{\langle d_1, d_2\rangle \mid d_1 \preceq d_2\}$$

is closed in the product space $\varkappa^\omega \times \varkappa^\omega$.

On the other hand, we need to know the concepts of analytic, point-analytic and point-Luzin spaces, and several basic properties from [$FH_{1,2}$]. Also, the term Baire set is used for a more general notion (corresponding to extended Borel of Hansell in metric spaces, and to hyper-Baire used by the first author in his earlier work). For convenience of the reader we quickly recall what is needed.

By a space we always mean a uniform space, and the topologically fine uniformity (called fine by J. Isbell) consists of all continuous pseudometrics. "Discrete" is understood in the uniform sense. A family $\{X_a \mid a \in A\}$ is called to be σ-discretely decomposable (abb. σ-dd) if there exists a family $\{X_{an} \mid a \in A, n \in \omega\}$ such that each family $\{X_{an} \mid a \in A\}$ is discrete, and $X_a = \bigcup \{X_{an} \mid n \in \omega\}$ for each a. A family $\{X_a\}$ is said to be σ-db (σ-discretely base-like refinable) if there exists a σ-discrete collection $\mathcal{B}$ such that each X_a is the union of a subfamily of $\mathcal{B}$. Clearly σ-dd implies σ-db. For Lemma 2 we need to know that in a metric space $\{X_a\}$ is σ-dd or σ-db iff it is such in the fine uniformity, and locally σ-dd implies σ-dd (see [Ha, Lemma 2 and Corollary 1]).

A correspondence F=gr $F: X \longrightarrow Y$ (gr $F \subset X \times Y$) is said to be σ-dd-preserving or σ-db-preserving provided that if $\{X_a\}$ is σ-dd or σ-db in X, then so is $\{F[X_a]\}$ in Y. To check the properties, it is enough to check the images of discrete families $\{X_a\}$.

A space X is called point-analytic if there exists a continuous σ-dd-preserving mapping of a complete metric space P onto X; if f may be chosen 1-1, then X is called point-Luzin. One obtains the definition of analytic if f is allowed to be an upper-semicontinuous compact-valued correspondence.

We need to know that if $X \subset Y$ and X is point-analytic, then X is Suslin in Y (derived by the Suslin operation from the closed sets of Y) - [FH_1, Corollary 4.3(a)], and if X is Suslin in Y and Y is point-analytic, then so is X [FH_2, Corollary 3.4].

We also need to know that if $f: X \longrightarrow Y$ is a surjective continuous σ-db-preserving mapping, then Y is point-analytic whenever X is, and if f is moreover injective, then Y is point-Luzin whenever X is [FH_2, Th. 3.6(a)].

We denote by Ba(X) (see [FH_2, § 1.1]) the smallest σ-algebra containing the zero sets of uniformly continuous functions, that is closed under the operation of taking arbitrary discrete unions. The elements of Ba(X) are called Baire sets. If we replace "zero sets of uniformly continuous functions" by the collection $\mathcal{S}(X)$ of all Suslin sets in X, we obtain the definition of $\overline{\mathcal{S}}(X)$.

Finally, a correspondence $F: X \longrightarrow Y$ is said to be $(\mathfrak{M} \leftarrow \mathfrak{N})$ measurable if $F^{-1}[N] \in \mathfrak{M}$ for each N in $\mathfrak{N}$.

It should be remarked that Lemma 1(a) can be proved by the method of [KRN] and [KP].

1. The purpose of this section is to prove the following result.

Lemma 1. Let P be a closed subspace of $\varkappa^{\omega}$, and let h be a continuous mapping from P onto a uniform space X. Consider the selection $s:X \to P$ for h^{-1} such that $s[x]$ is the smallest element of $h^{-1}[x]$ for each $x \in X$. Then:

(a) If h is σ-db-preserving, then s is $(\overline{\mathcal{S}}(X) \leftarrow Ba(P))$-measurable (and of course, s^{-1} is σ-dd-preserving as the inverse to any selection of h^{-1} is).

(b) If h is σ-dd-preserving the set $s[X]$ is co-Suslin in P (i.e. $P \setminus s[X]$ is Suslin).

Proof of (a): Let t be any selection for h^{-1}. If $\{D_a\}$ is a disjoint family in $t[X]$, then $\{t^{-1}[D_a]\}$ is disjoint and $h[D_a] = t^{-1}[D_a]$ for each a. Thence, if $\{D_a\}$ is discrete, then $\{h[D_a]\}$ is σ-db, and being disjoint, it is σ-dd. Thus t^{-1} is σ-dd-preserving, particularly, s^{-1} is σ-dd-preserving. It follows now that to show the measurability it is enough to find a σ-discrete open base $\mathcal{B}$ for the topology of P such that s is $(\overline{\mathcal{S}}(X) \leftarrow \mathcal{B})$-measurable. We take the usual basis consisting of sets of the form

$$B(a) = \{b \in P \mid b|n+1 = a\}$$

where $a=(a_0,\dots,a_n) \in \varkappa^{n+1}$, $n \in \omega$, and prove that $s^{-1}[B(a)]$ is the difference of two Suslin sets of X. To this end, for each $d \in \varkappa^{\omega}$ let

$$I(d) = \{c \mid c \in \varkappa^{\omega}, c \prec d\}.$$

Clearly $I(d)$ is an open set, and it is easy to see that for each finite sequence a ranging in $\varkappa$ there exist c and d in $\varkappa^{\omega}$ such that

$$B(a) = I(d) \setminus I(c).$$

(Put $c = \{a_0, \dots, a_{n-1}, a_n, 0, 0, \dots\}$ and $d = \{a_0, \dots, a_{n-1}, a_n + 1, 0, 0, \dots\}$.)

Now the proof is concluded by showing that $s^{-1}[I(d)]$ is analytic, hence Suslin, for each d. Observe

$s^{-1}[I(d)] = \{x \mid s[x] \prec d\} = \{x \mid \exists\, c \in h^{-1}[x] \text{ with } c \prec d\} = h[I(d)]$.

Now $h[I(d)]$ is analytic, because $I(d)$ is analytic (it is complete metrizable), and h is a continuous σ-db-preserving mapping [FH_2, Th. 3.6(a)].

Remark. Without changing the proof, the assumption "h is σ-db-preserving" in Lemma 1(a) may be weakened to "h is σ-dr-preserving" whenever we know that the image of an analytic space under a continuous σ-dr-preserving mapping is analytic, and this is actually true. One can do that by a slight modification of the proof for σ-db in [FH_1, Th. 4].

For the proof of Lemma 1(b) we need the following

Lemma 2. Let h be a σ-dd-preserving continuous mapping from a metric space P onto a uniform space X. Let

$M = \{\langle d_1, d_2\rangle \in P \times P \mid h[d_1] = h[d_2]\}$.

Then the projections

$\pi_1 = \{\langle x,y\rangle \longrightarrow x\} : P \times P \longrightarrow P$ and

$\pi_2 = \{\langle x,y\rangle \longrightarrow y\} : P \times P \longrightarrow P$

restricted to M are σ-dd-preserving.

Proof of Lemma 2. Because of symmetry it suffices to prove the assertion for π_2.

Let $\{D_a \mid a \in A\}$ be a discrete family in M. There exist σ-discrete open covers $\mathcal{U}$ and $\mathcal{V}$ of P such that if $U \in \mathcal{U}$ and $V \in \mathcal{V}$, then $(U\times V)\cap D_a\neq\emptyset$ for at most one $a\in A$. Since $\mathcal{V}$ is σ-discrete and P is σ-dd-simple, by [FH_1, Prop. 1.2] it is enough to show that $\{V\cap\pi_2[D_a]\}$ is σ-dd. Clearly $V\cap\pi_2[D_a]=$ $=\pi_2[(P\times V)\cap D_a]$, and the family $\{\pi_1[(P\times V)\cap D_a]\}$ is discrete because each $U\in\mathcal{U}$ meets at most one of its members. Thence we may and shall assume that $\{D_a\}$ is a discrete family in M such that $\{\pi_1[D_a]\}$ is discrete in P. Since h is σ-dd-preserving, the family $\{h[\pi_1[D_a]]\}$ is σ-dd in X. Since the mappings $h\circ\pi_1$ and $h\circ\pi_2$ coincide on M, we have that $\{h[\pi_2[D_a]]\}$ is σ-dd in X, and since h is continuous, necessarily

$$(*)\qquad \{h^{-1}[h[\pi_2[D_a]]]\}$$

is σ-dd in the fine uniformity of P, and since P is metric, the family is σ-dd in P [Ha, Lemma 2]. Finally, $\{\pi_2[D_a]\}$ is σ-dd because it is dominated by the σ-dd family (*).

Proof of Lemma 1(b). It is easy to check

$$s[X]=P\setminus\pi_1[\{\langle d_1,d_2\rangle\in P\times P\mid h[d_1]=h[d_2],\ d_1\neq d_2\}]$$

where π_1 is the projection on the first factor. The set in the brackets is equal to

$$\{\langle d_1,d_2\rangle\in P\times P\mid h[d_1]=h[d_2]\}\cap\{\langle d_1,d_2\rangle\in P\times P\mid d_1\neq d_2\}.$$

Since the first set is closed and the second one is open, the intersection is analytic [FH_2, Th. 3.3], hence the image under π_1 is analytic because π_1 restricted to $\{\langle d_1,d_2\rangle\mid h[d_1]=h[d_2]\}$ is σ-dd-preserving by Lemma 2. Thus s[x] is the complement of a Suslin set in P.

2. Corollaries. The main result reads as follows:

Theorem. Let $F:X\to Y$ be a correspondence of uniform spaces X and Y. Then:

(a) If the graph grF of F is point-analytic and if the projection π_1:grF $\longrightarrow$ X is σ-db-preserving, then F admits a $(\overline{\mathcal{G}}(X) \longleftarrow Ba(Y))$-measurable selection f.

(b) If grF is point-Luzin and if π_1:grF $\longrightarrow$ X is σ-dd-preserving, then there exists a $(\overline{\mathcal{G}}(X) \longleftarrow Ba(Y))$-measurable selection f for F such that grF$\setminus$grf is analytic.

Proof. (a) Since grF is point-analytic, by definition there exists a σ-dd-preserving continuous mapping g of a closed subspace P of some $\varkappa^{\omega}$ onto grF.

Put h = $\pi_1 \circ g$ and apply Lemma 1(a) (we may suppose that DF = X, i.e. π_1[grF] = X) to obtain a $(\overline{\mathcal{G}}(X) \longleftarrow Ba(Y))$-measurable selection s for h. Put f = $\pi_2 \circ g \circ s$. All three maps are $(\overline{\mathcal{G}} \longleftarrow Ba)$-measurable, and so is then f.

(b) In this case we may assume that g is a bijection. Lemma 1(b) applies to h, and since g is bijective

$$g[P \setminus s[X]] = grF \setminus grf,$$

and hence the set is analytic as the image of an analytic space by a continuous σ-dd-preserving mapping.

We conclude with several consequences of Theorem; in each of the cases the hypothesis would imply that of Theorem. Of course, we need to apply further results.

Corollary 1. There exists a $(\overline{\mathcal{G}}(X) \longleftarrow Ba(Y))$-measurable selection f for a closed-valued-correspondence F:X $\longrightarrow$ Y provided that the following three conditions are satisfied:

(α) F is Suslin measurable (i.e. $(\mathcal{G}(X) \longleftarrow \mathcal{G}(Y))$-measurable))

(β) F^{-1} is σ-dd-preserving

(γ) X is point-analytic and Y is a subspace of a point-Luzin space.

If, in addition, F is Baire measurable and X is point-Luzin, then F can be chosen such that, in addition, $grF \setminus grf$ is analytic.

Proof. The projection $grF \to X$ is σ-dd-preserving by [FH_1 Lemma 2.5(a)] and [FH_1, Prop. 3.1(b)]. The graph of F is Suslin [FH_2, Prop. 4.2], $X \times \overline{Y}$ is point-analytic [FH_2, Prop. 3.2(b)], and thus grF is point-analytic [FH_2, Cor. 3.4].

The validity of the assumptions of Theorem (b) can be derived similarly from the extended assumptions.

Corollary 2. If $F^{-1}: Y \to X$ is a mapping in Corollary 1, then there exists a $(\overline{\mathcal{G}}(X) \leftarrow Ba(Y))$-measurable selection f for F provided that (α) and (γ) are satisfied, and F^{-1} is σ-db-preserving.

Proof. The projection $grF \to X$ is σ-db-preserving by [FH_1, Lemma 2.5(b)] and [FH_1, Prop. 3.1(b)]. The graph of F is point-analytic by the same arguments as in the proof of Corollary 1.

Remark. If the assumption (β) in Corollary 1 was supplied by

(β') F is σ-dd-preserving,

then the same assertions are valid for $F^{-1}: Y \to X$ instead of $F: X \to Y$.

Similar change could be done in Corollary 2 (when $F: X \to$ $\to Y$ is a mapping).

R e f e r e n c e s

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