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Abstract: Using Palais-Terng theorem on natural bundles, we determine all smooth principal fibre bundles with the property that the group of all automorphisms can be expressed as a semi-direct product of a prescribed type.

Key words: Principal fibre bundle, natural bundle, jet, gauge transformation.

Classification: 58A20

This research was inspired by a discussion with Prof. A. Trautman and by his paper on gauge transformations [3].

Consider a principal fibre bundle $\pi : P \longrightarrow M$ with structure group G. Let Aut P be the group of all automorphisms of P. We have an exact sequence

(1) $$0 \longrightarrow \mathrm{Aut}_M P \longrightarrow \mathrm{Aut}P \longrightarrow \mathrm{Diff}M,$$

where $\mathrm{Aut}_M P$ means the group of all vertical automorphisms of P, [3]. An interesting problem is: under what conditions AutP can be expressed as a semi-direct product of Aut_M and DiffM? In general, given an exact sequence of groups

(2) $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

B can be expressed as a semi-direct product of A and C iff there exists a splitting of (2). We shall determine all P such that there is a splitting of (1) with the following "local" property. Denoting by LDiffM the pseudogroup of all local diffeomorphisms of M and by LAutP the pseudogroup of all local automorphisms of P, we shall assume that the splitting DiffM $\longrightarrow$ AutP is the restriction of a splitting S: LDiffM $\longrightarrow$ LAutP defined on the whole pseudogroup LDiffM.

Such a splitting S is a lifting functor on P in the sense of A. Nijenhuis [1], which endows P with the structure of a natural bundle. According to a recent theorem by R. S. Palais and C.L. Terng (and a related result by D.B.A. Epstein and W. Thurston) [2], any natural bundle has finite order. Given an r-th order natural bundle $E \longrightarrow M$ with lifting functor F and an element $c \in M$, any element X of the group $L^r_c M$ of all invertible isotropic r-jets on M at c determines a diffeomorphism FX: $E_c \longrightarrow E_c$. The assignment $X \longmapsto FX$ is a smooth left action of $L^r_c M$ on E_c [2]. Conversely, given a smooth left action φ of the group $L^r_n = L^r_0 \mathbb{R}^n$ on a manifold Q, n = dimM, we can construct the associated fibre bundle $Q_M = (M, Q, L^r_n, H^r M)$, where $H^r M$ means the r-th order frame bundle of M. The bundle Q_M is natural with respect to the following lifting functor F. Any local diffeomorphism f: $U \longrightarrow V$ on M is prolonged into a principal bundle isomorphism $H^r f$: $H^r U \longrightarrow H^r V$ and we define Ff: $p^{-1}(U) \longrightarrow p^{-1}(V)$ by $Ff(u,q) = (H^r f(u), q)$, where p denotes the bundle projection of Q_M.

In our case, S is a functor into the category of principal fibre bundles, so that SX: $P_c \longrightarrow P_c$ satisfies SX(ug) = = (SX(u))g. If we fix an element $u \in P_c$, we obtain a map S_u: : $L^r_c M \longrightarrow G$ defined by

$$SX(u) = uS_uX.$$

As $S(YX)(u) = SY(uS_uX) = u(S_uY)(S_uX)$, S_u is a group homomorphism. Conversely, let G be a Lie group and $\sigma: L^r_n \longrightarrow G$ an analytic homomorphism. Then $(X,g) \longmapsto \sigma(X)g$ is a left action of L^r_n on G and we can construct the associated fibre bundle $P = (M,G,L^r_n,H^rM)$. Any element of P being an equivalence class of the equivalence relation $(u,g) \sim (uX, \sigma(X^{-1})g)$, $u \in P$, $g \in G$, $X \in L^r_n$, we have a well-defined right action $P \times G \longrightarrow P$, $((u,g),h) \longmapsto (u,gh)$. One verifies directly that P(M,G) is a principal fibre bundle and the induced lifting functor S is a splitting S: LDiffM $\longrightarrow$ LAutP. Thus, we have deduced

Theorem. If P is a principal fibre bundle such that there exists a splitting S: LDiffM $\longrightarrow$ LAutP, then there is an integer r and an analytic homomorphism $\sigma: L^r_n \longrightarrow G$ such that P coincides with the corresponding bundle (M,G,L^r_n,H^rM) and S is constructed as above.

R e f e r e n c e s

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