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Barry J. Gardner

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EXTENSION-CLOSURE AND ATTAINABILITY FOR VARIETIES OF ALGEBRAS WITH INVOLUTION

B. J. GARDNER

Abstract: Some information is obtained about varieties of algebras with involution which are closed under extensions. A result of Mal'tsev asserts that varieties with attainable identities must be closed under extensions. It is shown that the converse is false for algebras with involution. Consequently not every extension-closed variety is a semi-simple radical class.

Key words: Algebra with involution, variety, radical class.

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Salavova [9],[10] has studied Kurosh-Amitsur radical theory for the universal class of all associative rings with involution. In this note we shall investigate semi-simple radical classes (i.e. radical classes which are the semi-simple classes corresponding to other radical properties) in this context.

We begin by summarizing some results from [1]. (While these results were obtained for varieties of algebras, they hold also in varieties of algebras with involution. In the category of algebras with involution over a ring, the normal subobjects are those ideals which are closed under the involution; they will be called *-ideals and indicated by the symbol $\triangleleft^*$. The involution itself will always be called $*$.

Let $\mathcal{V}$ be a variety of algebras with involution. (Such objects have an extra unary operation - the involution $*$ - as well as the ring operations, and polynomial identities will in general involve this operation too.) $\mathcal{V}$ is a radical class if and only if it's closed under extensions (i.e. $A \in \mathcal{V}$ if $I \triangleleft^* A$ and $\mathcal{V}$ contains I and A/I) ([1], Theorem 1.4). For each A, let $A(\mathcal{V}) = \cap\{I \mid I \triangleleft^* A$ and $A/I \in \mathcal{V}\}$. Then $\mathcal{V}$ is said to have <u>attainable identities</u> if $A(\mathcal{V})(\mathcal{V}) = A(\mathcal{V})$ for every A. $\mathcal{V}$ is a semi-simple radical class if and only if it has attainable identities ([1], Theorem 1.5). Varieties with attainable identities are closed under extensions not only in the case of rings, but whenever the two variety properties make sense [4]. The converse is often true (cf. [1], Corollary 1.12). In [2] we gave an example of a universal variety of (non-associative) rings in which the identity $x^2 = x$ defines an extension-closed variety without attainable identities. As we see below, such an example also exists for (associative) algebras with involution.

We shall work with algebras the ring $Z^{(2)} = \{m/2^n \mid m,n \in Z\}$. The possibility of dividing by 2 is essential to our arguments in a couple of places. It is not clear what happens in the case of rings (i.e. Z-algebras).

Our first result presents some "large" extension-closed varieties.

<u>Theorem 1</u>. Let $\mathcal{V}_s$, $\mathcal{V}_k$ be the varieties defined by the identities $x^* = x$, $x^* = -x$ respectively.

(i) $\mathcal{V}_s$ and $\mathcal{V}_k$ are closed under extensions.

(ii) If $\mathcal{U} \neq \{0\}$ is an extension-closed proper subvariety of $\mathcal{V}_s$, then $\mathcal{U}$ contains no nilpotent algebras.

(iii) If $\mathcal{W} \neq \{0\}$ is an extension-closed subvariety of $\mathcal{V}_k$, then $\mathcal{W} = \mathcal{V}_k$.

Proof. (i) If $I \lhd^* A$ and if $I, A/I \in \mathcal{V}_s$, let a be any element of A. Then $a - a^* \in I$, so

$$a - a^* = (a-a^*)^* = a^* - a,$$

whence $2(a-a^*) = 0$, i.e. $a = a^*$. Thus A is in $\mathcal{V}_s$. A similar argument disposes of $\mathcal{V}_k$.

(ii) Suppose $\{0\} \subsetneqq \mathcal{U} \subseteq \mathcal{V}_s$ and A is a non-zero nilpotent member of $\mathcal{U}$. Now effectively $\mathcal{V}_s$ is just the class of all commutative algebras - we can forget about involutions - so arguing as in the proof of Corollary 1.9 of [1], we see that $\mathcal{U} = \mathcal{V}_s$.

(iii) In $\mathcal{V}_k$, we have

$$-ab = (ab)^* = b^*a^* = (-b)(-a) = ab$$

for all elements a, b of any algebra. We are thus effectively dealing with all anticommutative algebras or, equivalently (since we can divide by 2), with all algebras satisfying $x^2 = 0$. Such algebras are nilpotent of index 3 (see, e.g. [11], § 2). If $\{0\} \subseteq \mathcal{W} \subseteq \mathcal{V}_k$ and $0 \neq A \in \mathcal{W}$, choose $a \in A$ such that $a \neq 0 = a^2$. Then $(a^2)^* = 0^* = 0 = -a^2$, so the <u>algebra</u> $\langle a \rangle$ generated by a is in $\mathcal{W}$. An argument like the one involved in (ii) establishes that $\mathcal{W}$ contains all algebras R in $\mathcal{V}_k$ for which $R^2 = 0$. But if $T \in \mathcal{V}_k$, then $T^2 \lhd^* T$, $(T^2)^2 = 0$, $(T/T^2)^2 = 0$ and $T^2, T/T^2 \in \mathcal{V}_k$. Since $\mathcal{W}$ is extension-closed, we have $T \in \mathcal{W}$.

Theorem 1 implies that if an extension-closed variety contains a non-zero nilpotent algebra from $\mathcal{V}_s$ or $\mathcal{V}_k$, it must contain all of $\mathcal{V}_s$ or $\mathcal{V}_k$ respectively. We therefore consider extension-closed varieties containing no such nilpotent algebras.

Theorem 2. Let $\mathcal{V}$ be an extension-closed variety containing no non-zero nilpotent algebras from $\mathcal{V}_s$ or $\mathcal{V}_k$. Then $\mathcal{V}$ is generated by fields and 2×2 matrix rings over fields.

Proof. If $a \in A \in \mathcal{V}$ and $a^* = a$, then the subalgebra $\langle a \rangle$ generated by a is closed under $*$ and therefore $\langle a \rangle \in \mathcal{V}$. Hence $\langle a \rangle / \langle a \rangle^2 \in \mathcal{V} \cap \mathcal{V}_s$, so $\langle a \rangle = \langle a \rangle^2 = \langle a^2 \rangle$. Thus we have

$$(+) \ldots \qquad a = r_2a^2 + r_3a^3 + \ldots + r_na^n$$

for some $r_2, \ldots, r_n \in Z^{(2)}$. If $b \in \langle a \rangle$ then $b^* = b$ so b satisfies a relation like (+). Hence, by Theorem 13.2, p. 321 of Osborn [8], which holds with $Z^{(2)}$ replacing Z, $\langle a \rangle$ is periodic. Thus A satisfies the condition

$$a \in A,\ a^* = a \Longrightarrow n > 1 \text{ such that } a^n = a.$$

By results of Montgomery [6],[7] (see [7], Theorem 3; see also Herstein [3]) $J(A)^3 = 0$ (where J is the Jacobson radical of A) and $A/J(A)$ is a subdirect product of fields and 2×2 matrix rings over fields. Let c be in $J(A)$. Then $c + c^* \in J(A)$ and $(c+c^*)^* = c + c^*$, so $\langle c+c^* \rangle \in \mathcal{V} \cap \mathcal{V}_s$. Since $\langle c+c^* \rangle$ is nilpotent, we have $c + c^* = 0$, i.e. $c^* = -c$. Since $J(A)$ is closed under $*$, this means that $J(A) \in \mathcal{V} \cap \mathcal{V}_k = \{0\}$. Hence A is semiprime, and therefore, by Theorem 6 of [5], a subdirect product, qua algebra with involution of prime algebras with involution. The latter are in $\mathcal{V}$, whence the result follows.

We turn now to the question of attainability.

Theorem 3. Let $\mathcal{V}$ be an extension-closed variety which contains no non-zero nilpotent algebras from $\mathcal{V}_s$ or $\mathcal{V}_k$. Then $\mathcal{V}$ has attainable identities.

Proof. Salavova ([9], Lemma 2.12) notes that rings with involution satisfy Andrunakievich's Lemma, i.e. if $I \lhd^* B \lhd^* A$ and if B is the $*$-ideal of A generated by I, then $B^3 \subseteq I$. Consider any algebra R with involution. We have

$$R(\mathcal{V})(\mathcal{V}) \lhd^* R(\mathcal{V}) \lhd^* R$$

and by the argument used in the proof of Theorem 1.10 of [1], $R(\mathcal{V})$ is the $*$-ideal of R generated by $R(\mathcal{V})(\mathcal{V})$. Thus C = $= R(\mathcal{V})/R(\mathcal{V})(\mathcal{V}) \in \mathcal{V}$ and $C^3 = 0$. But by Theorem 2 (proof) C (if non-zero) is a subdirect product of $*$-prime algebras. Hence C = 0 and $\mathcal{V}$ has attainable identities.

We have not been able to determine whether or not $\mathcal{V}_s$ has attainable identities, but we do have the following theorem .

Theorem 4. $\mathcal{V}_k$ does not have attainable identities.

Proof. We'll show that $\mathcal{V}_k$ is not a semi-simple class by adapting an example which was used in [9] to obtain an example of a non-hereditary semi-simple class.

Let $R = Z^{(2)}[y]/(y^4)$, where the involution is that induced by the involution

$$\alpha(y) \mapsto \alpha(-y)$$

in $Z^{(2)}[y]$. Let $u = y + (y^4)$, $v = y^2 + (y^4)$, $w = y^3 + (y^4)$. Then $u^* = -u$, $v^* = v$ and $w^* = -w$ and multiplication is controlled by the table

	u	v	w
u	v	w	0
v	w	0	0
w	0	0	0

It is a routine matter to show that the $*$-ideals of R are:

$$0, M = \{sw \mid s \in Z^{(2)}\},\ L = \{rv + sw \mid r, s \in Z^{(2)}\},\ R.$$

Now $L' = \{rv \mid r \in Z^{(2)}\} \triangleleft^* L$ and $L/L' \in \mathcal{V}_k$. Also $R/L \in \mathcal{V}_k$ and M itself is in $\mathcal{V}_k$. Thus every non-zero $*$-ideal of R has a non-zero homomorphic image in $\mathcal{V}_k$. But $R \notin \mathcal{V}_k$, since $v^* = v \neq -v$. Hence $\mathcal{V}_k$ is not a semi-simple class, so it does not have attainable identities.

Although the attainability question remains open for varieties containing $\mathcal{V}_s$, for those which do not, we can say something.

<u>Theorem 5</u>. Let $\mathcal{V}$ be an extension-closed variety such that $\mathcal{V} \cap \mathcal{V}_s$ has no non-zero nilpotent members. Then $\mathcal{V}$ has attainable identities if and only if $\mathcal{V} \cap \mathcal{V}_k = \{0\}$.

Proof. If $\mathcal{V} \cap \mathcal{V}_k \neq \{0\}$, then by Theorem 1, $\mathcal{V}_k \subseteq \mathcal{V}$. The algebra R of the preceding proof has the property that each of its non-zero $*$-ideals has a non-zero homomorphic image in $\mathcal{V}_k$ and hence in $\mathcal{V}$. If $\mathcal{V}$ has attainable identities, then $R \in \mathcal{V}$. But $v^* = v$ and $v^2 = 0$, so $0 \neq \langle v \rangle \in \mathcal{V} \cap \mathcal{V}_s$ - a contradiction, since $\langle v \rangle$ is nilpotent.

Our final result merely paraphrases Theorems 4 and 5.

<u>Theorem 6</u>. Let $\mathcal{V}$ be an extension-closed variety such that $\mathcal{V} \cap \mathcal{V}_s$ has no non-zero nilpotent members. Then $\mathcal{V}$ is a semi-simple radical class if and only if $\mathcal{V} \cap \mathcal{V}_k = \{0\}$.

In particular, there are extension-closed varieties which are not semi-simple radical classes.

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Mathematics Department
University of Tasmania
GPO Box 252C, Hobart, Tas. 7001
Australia

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