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REPRESENTATIONS OF COUNTABLE COMMUTATIVE SEMIGROUPS BY PRODUCTS OF WEAKLY HOMOGENEOUS SPACES

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Abstract: A weakly homogeneous topological space X which is homeomorphic to X^3 but not to X^2 is constructed.

Key words: Semigroup, representation, product, weakly homogeneous topological space.

Classification: Primary 54H10
Secondary 20M30

Let us recall that a topological space X is said to be weakly homogeneous iff for every $x,y \in X$ there are open neighbourhoods U, V, $x \in U$, $y \in V$, and a homeomorphism h of U onto V such that $h(x)=y$.

The aim of this paper is to prove the following:

Theorem. For any countable commutative semigroup (S,+) there exists a collection $\{r(s); s \in S\}$ of weakly homogeneous metrizable spaces such that for every $s,s' \in S$ the following conditions hold:

(i) $r(s+s')$ is homeomorphic to $r(s) \times r(s')$,

(ii) $r(s)$ is homeomorphic to $r(s')$ iff $s = s'$.

Remark. As a special case of Theorem (S being the additive group of integers modulo 2) we obtain a weakly homogeneous topological space X homeomorphic to X^3 but not to X^2.

Representations of semigroups by products have been investigated for various algebraic, relational or topological structures. A survey of this subject is given in [4].

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1. Conventions and notations. We shall use the symbol $\cong$ for the homeomorphism of spaces. Products will be denoted by $\prod$ (or $\times$ for finite collections), coproducts by $\coprod$. A product of an empty collection of numbers (topological spaces, resp.) is equal to 1 (a one-point space, resp.). A coproduct of an empty collection of spaces is an empty space. N denotes the set of all non-negative integers.

2. There is a natural additive operation + on the power-set exp N^N defined by

$A+A' = \{g \in N^N; (\exists f \in A, f' \in A')(\forall n \in N)(g(n)=f(n)+f'(n))\}$.

Clearly, (exp N^N,+) is a commutative semigroup.

According to [3], any countable commutative semigroup is isomorphic to a subsemigroup of (exp N^N,+). Thus, it suffices to consider representations of subsemigroups of (exp N^N,+) by products of weakly homogeneous spaces.

In other words, our aim is to construct for any subset A of N^N a weakly homogeneous space X(A) such that for every

$A, A' \in \exp N^N$ the following two conditions hold:

(i) $X(A + A') \cong X(A) \times X(A')$,

(ii) $X(A) \cong X(A')$ iff $A = A'$.

As the distributivity of finite products and arbitrary coproducts of weakly homogeneous spaces is fulfilled, it suffices - due to Trnková's result (see [3]) - to construct for any mapping $f \in N^N$ a weakly homogeneous space $X(f)$ such that for every $f, g \in N^N$ and $A, A' \in \exp N^N$ the following conditions hold:

(1) $X(f+g) \cong X(f) \times X(g)$,

(2) $\coprod_{2^N} \coprod_{h \in A} X(h)$ is weakly homogeneous,

(3) $\coprod_{2^N} \coprod_{h \in A} X(h) \cong \coprod_{2^N} \coprod_{k \in A'} X(k)$ iff $A = A'$.

(Having constructed $X(f)$'s satisfying the conditions (1)-(3), we can put $X(A) = \coprod_{2^N} \coprod_{f \in A} X(f)$. Hence, for any $A, A' \in \exp N^N$, $X(A + A')$ is isomorphic to $X(A) \times X(A')$ and (i) is fulfilled. The condition (ii) is just another formulation of the condition (3).)

Trnková's general method for constructing $X(f)$'s satisfying (1)-(3) is the following: find a collection $\{X_n; n \in N\}$ of objects of a given category $\underline{K}$ such that for every $f \in N^N$ and $A, A' \in \exp N^N$ the following three conditions hold:

(a) $\prod_{n \in N} X_n^{f(n)} \in \operatorname{obj} \underline{K}$,

(b) $\coprod_{2^N} \coprod_{h \in A} \prod_{n \in N} X_n^{h(n)} \in \operatorname{obj} \underline{K}$,

(c) $\coprod_{2^N} \coprod_{h \in A} \prod_{n \in N} X_n^{h(n)} \cong \coprod_{2^N} \coprod_{k \in A'} \prod_{n \in N} X_n^{k(n)}$ iff $A = A'$.

In our case ($\underline{K}$ being a category of weakly homogeneous spaces and continuous mappings), a topological product of countably many weakly homogeneous spaces need not be weakly homogeneous. Hence, we must a bit modify the Trnková's general method. We shall construct X(f) using special subobjects of $\prod X_n^{f(n)}$ preserving the property (1).

3. <u>Construction</u>. Let I be the open real interval $]0,1[$ with the usual metric topology. Let $\{k_n; n\in M\subset N\}$ be a collection of non-negative integers, A_n a subspace of I^{k_n}. Denote by $\prod^*_{n\in M} A_n$ a topological space with the underlying set

$$\{(a_{n,i})_{n\in M, 1\le i\le k_n};\ 0<\inf_{n\in M, 1\le i\le k_n} a_{n,i}\ ,\ \sup_{n\in M, 1\le i\le k_n} a_{n,i} < 1\}$$

and the topology induced by the metric

$$\rho((a_{n,i}),(a'_{n,i})) = \sup_{n\in M, 1\le i\le k_n} | a_{n,i} - a'_{n,i} | .$$

(Evidently, for finite collections $\prod^*$ coincides with the usual product $\prod$.) If $A_n = A$ for all $n\in M$ then we shall use the notation $\prod^*_{n\in M} A = (A^M)^*$.

Denote by p_n the n-th prime number and for every $(m,n)\in N^2$ put

$$B_{m,n} = I^2 \setminus \bigcup_{k=0}^{p_n-1} \left[\frac{3k+1}{3p_n}, \frac{3k+2}{3p_n}\right] \times \left[\frac{1}{m+n+3}, \frac{1}{m+n+2}\right]$$

(where $[\cdot,\cdot]$ denotes a closed interval).

Let $B = \{b_m\}_{m\in N}$, C be two disjoint countable dense subspaces of I. Put $Y = ((I^2\times C)^N)^*$. Clearly, $Y^2 \cong Y$. For every $n\in N$ put

$$B_n = \bigcup_{m\in N} (B_{m,n}\times\{b_m\}),\ X_n = B_n \cup I^2\times C,$$

for every $f \in N^N$ put

$$X(f) = \prod^*_{n \in N} X_n^{f(n)} \times Y$$

and for every subset A of N^N put

$$X(A) = \coprod_{2^N} \coprod_{f \in A} X(f).$$

Clearly, $X(f+g) \cong X(f) \times X(g)$ and (1) holds. Every point $x \in X(A)$ has a neighbourhood homeomorphic to Y. Therefore, $X(A)$ is weakly homogeneous and (2) holds, too.

The rest of the paper is devoted to verify the most difficult condition: $X(A) \cong X(A')$ iff $A = A'$. Roughly speaking, our tool will be homotopy equivalence ($\simeq$) and non-equivalence ($\not\simeq$) of components of $X(A)$ and $X(A')$.

4. Lemma. $(I^N)^*$ is a homotopically trivial space.

Proof. $(I^N)^* \simeq \{(\frac{1}{2})_{n \in N}\}$.

5. Proposition. For every component K of $X(f)$ there exists a function $k \in N^N$, $k \leq f$ (i.e. $k(n) \leq f(n)$ for every $n \in N$), and a function $M: \bigcup_{n \in N} (\{n\} \times \{1, \ldots, k(n)\}) \longrightarrow M$ such that

$$K \cong \prod^*_{n \in N} \prod_{i=1}^{k(n)} B_{m(n,i),n} \times T$$

where T is a homotopically trivial space.

Proof. Choose a point $x \in X(f)$, $x = ((x_{n,i})_{n \in N, 1 \leq i \leq f(n)}, (y_n, c_n)_{n \in N})$ where $x_{n,i} \in X_n$, $y_n \in I^2$, $c_n \in C$. For every $n \in N$ define $k(n)$ as a number of coordinates i such that $x_{n,i} \in B_n$. One can assume that the coordinates with this property are just $1, \ldots, k(n)$. Then for every $n \in N$ and $1 \leq i \leq k(n)$ there exist $m(n,i) \in N$ and $a_{n,i} \in B_{m(n,i),n}$ such that $x_{n,i} =$ $= (a_{n,i}, b_{m(n,i)})$; for any $n \in N$ and $k(n) < i \leq f(n)$ there exist $a_{n,i} \in I^2$ and $c_{n,i} \in C$ such that $x_{n,i} = (a_{n,i}, c_{n,i})$.

Hence, the component K containing the chosen point x is homeomorphic to

$$\prod^*_{n\in N}\Big(\prod_{i=1}^{k(n)} (B_{m(n,i),n} \times \{b_{m(n,i)}\}) \times \prod_{i=k(n)+1}^{f(n)} (I^2 \times \{c_{n,i}\})\Big) \times$$

$$\times \prod^*_{n\in N} (I^2 \times \{c_n\}).$$

Thus, $K \cong \prod^*_{n\in N} \prod_{i=1}^{k(n)} B_{m(n,i),n} \times T$ where $T \cong (I^N)^*$. By Lemma 4, T is homotopically trivial.

6. For every $n \in N$ put

$$Z_n = [0,1] \times \{0,1\} \cup \bigcup_{k=0}^{p_n} \{\tfrac{k}{p_n}\} \times [0,1] \subset [0,1]^2.$$

Denote by J the additive group of all integers, $H_q(X)$ the q-th singular homology group of the space X. Tensor products of Abelian groups will be denoted by $\otimes$, their direct sums by $\oplus$.

One can prove easily the following two lemmas:

7. __Lemma__. $B_{m,n} \cong Z_n$ for any $(m,n) \in N^2$.

8. __Lemma__. $H_1(Z_n) = \underbrace{J \oplus \ldots \oplus J}_{p_n}$, $H_q(Z_n) = 0$ if $q > 1$.

9. __Lemma__. Let M be a finite subset of N, $k: M \longrightarrow N$ be a mapping. Then

$$H_q\Big(\prod_{n\in M} Z_n^{k(n)}\Big) = 0 \text{ if } q > \sum_{n\in M} k(n)$$

and $H_q(\prod_{n\in M} Z_n^{k(n)}) = \underbrace{J \oplus \dots \oplus J}_{\prod_{n\in M} p_n^{k(n)}}$ if $q = \sum_{n\in M} k(n)$.

Proof. For $\sum_{n\in M} k(n) = 0$ the assertion holds trivially. For $\sum_{n\in M} k(n) = 1$ it holds due to Lemma 8.

In the general case, $H_q(\prod_{n\in M} Z_n^{k(n)}) = 0$ for $q > \sum_{n\in M} k(n)$ holds trivially. Now, one can prove by induction using the Künneth formula (see e.g. [2]) that all the homology groups of $\prod_{n\in M} Z_n^{k(n)}$ are free and that

$$H_{\sum_{n\in M} k(n)}(\prod_{n\in M} Z_n^{k(n)}) = \bigotimes_{n\in M} (\underbrace{H_1(Z_n) \otimes \dots \otimes H_1(Z_n)}_{k(n)}) =$$

$$= \bigotimes_{n\in M}(\underbrace{(\underbrace{J\oplus \dots \oplus J}_{p_n}) \otimes \dots \otimes (\underbrace{J\oplus \dots \oplus J}_{p_n})}_{k(n)}) =$$

$$= \underbrace{J \oplus \dots \oplus J}_{\prod_{n\in M} p_n^{k(n)}}, \text{ q.e.d.}$$

10. Lemma. For any finite $M \subset N$ and mappings $k: M \longrightarrow N$, $m: \bigcup_{n\in M} (\{n\} \times \{1,\dots,k(n)\}) \longrightarrow N$ there is

$$H_q(\prod_{n\in M} \prod_{i=1}^{k(n)} B_{m(n,i),n}) = 0 \text{ if } q > \sum_{n\in M} k(n),$$

$$H_q(\prod_{n\in M} \prod_{i=1}^{k(n)} B_{m(n,i),n}) = \underbrace{J \oplus \dots \oplus J}_{\prod_{n\in M} p_n^{k(n)}} \text{ if } q = \sum_{n\in M} k(n).$$

Proof follows from Lemmas 7 and 9.

11. Lemma. Let $k:N \longrightarrow N$, $m: \bigcup_{n\in N} (\{n\}\times\{1,\dots,k(n)\}) \longrightarrow$ $\longrightarrow N$ be mappings. Then $H_q(\prod^*_{n\in M} \prod_{i=1}^{k(n)} B_{m(n,i),n}) \neq 0$ whenever M is a finite subset of N, $q = \sum_{n\in M} k(n)$.

Proof. Let $M \subset N$ be finite, $q = \sum_{n\in M} k(n)$. Then

$\prod_{n\in M} \prod_{i=1}^{k(n)} B_{m(n,i),n}$ is a retract of $\prod^*_{n\in N} \prod_{i=1}^{k(n)} B_{m(n,i),n}$.

By Lemma 10, $H_q(\prod_{n\in M} \prod_{i=1}^{k(n)} B_{m(n,i),n}) \neq 0$; therefore,

$H_q(\prod^*_{n\in N} \prod_{i=1}^{k(n)} B_{m(n,i),n}) \neq 0$, too.

12. For every $x \in X(A)$ put
$F(x) = \{g \in N^N; (x\in U,\ U \text{ open in } X(A), n\in N) \Longrightarrow (\exists \text{ component } K \text{ of } X(A),\ K \simeq Z_n^{g(n)},\ K \cap U \neq \emptyset)\}$.

Let $f \in A$ be given; then using $F(x)$ one can characterize the given f, as it follows from the following:

13. Proposition. If $x \in X(f)$ then $F(x) = \{g \in N^N;\ g \leq f\}$. Hence, $f = \sup F(x)$ and $A = \{\sup F(x);\ x \in X(A)\}$.

Proof. A. Let $x \in X(f)$, $g \in N^N$, $g(n_0) > f(n_0)$. Choose an open $U \subset X(f)$ such that $x \in U$. Let K be a component of $X(A)$ such that $K \cap U \neq \emptyset$. Then, by Proposition 5,
$K \cong \prod^*_{n\in N} \prod_{i=1}^{k(n)} B_{m(n,i),n} \times T$ where T is a homotopically trivial space, $k \leq f$.
Consider two cases:
(1) $\sum_{n\in N} k(n)$ is finite. Then there is a finite $M \subset N$ such

that $k(n) = 0$ for every $n \in N \setminus M$. Lemmas 9 and 10 imply that $K \not\cong Z_{n_o}^{g(n_o)}$.

(ii) $\sum_{n\in N} k(n)$ is infinite. Then there exists a finite subset $M \subset N$ such that $\sum_{n\in M} k(n) > g(n_o)$. Lemmas 9 and 11 imply that $K \not\cong Z_{n_o}^{g(n_o)}$.

Therefore, $F(x) \subset \{g \in N^N;\ g \leq f\}$.

B. Let $g \leq f$, $x \in X(f)$, $n \in N$ be given. Denote

$$x = ((x_{ij}, z_{ij})_{i\in N, 1\leq j\leq f(i)}, y)$$

where $x_{ij} \in I^2$, $z_{ij} \in B \cup C$, $y \in Y$. Let U be an open subspace of $X(A)$ such that $x \in U$. We shall find a component K of $X(A)$ such that $K \simeq Z_n^{g(n)}$.

One can choose for $1 \leq j \leq g(n)$ a $v_{nj} = b_{w_j} \in B$ and for every $(i,j) \in (\bigcup_{m\in N} \{m\} \times \{1,\dots,f(m)\}) \setminus \{n\} \times \{1,\dots,g(n)\}$ a $v_{ij} \in C$ such that $\bar{x} = ((x_{ij}, v_{ij})_{i\in N, 1\leq j\leq f(i)}, y) \in X(f) \cap U$. Let K be the component of $X(A)$ containing $\bar{x}$. Then

$$K \cong \prod_{j=1}^{g(n)} (B_{w_j,n} \times \{v_{nj}\}) \times \prod_{j=g(n)+1}^{f(n)} (I^2 \times \{v_{nj}\}) \times$$

$$\times \prod^*_{i\in N\setminus\{n\}} \prod_{j=1}^{f(i)} (I^2 \times \{v_{ij}\}) \times ((I^2)^N)^* \cong \prod_{j=1}^{g(n)} B_{w_j,n} \times (I^N)^*.$$

By Lemmas 4 and 7, $K \simeq Z_n^{g(n)}$.

Therefore, $\{g \in N^N,\ g \leq f\} \subset F(x)$, q.e.d.

14. <u>Observation</u>. Let A, A' be subsets of N^N. Then $X(A) \cong X(A')$ iff $A = A'$.

15. Observation 14 finishes the proof of Theorem. Actually, we have obtained a bit stronger result:

For any countable commutative semigroup (S,+) there exists a collection $\{r(s); s \in S\}$ of weakly homogeneous metric spaces such that for every $s, s' \in S$ the following conditions hold:

(i) $r(s + s')$ is <u>isometric</u> to $r(s) \times r(s')$

(ii) if $s \neq s'$ then $r(s)$ is <u>not homeomorphic</u> to $r(s')$.

16. <u>Concluding remark</u>. J. Adámek and V. Koubek introduced in [1] a concept of the sum-productive representation of an ordered semigroup. Using the above construction one sees immediately (the condition 2.2 (iii) from [1] is clearly satisfied) that any countable ordered commutative semigroup has a sum-productive representation in the category of weakly homogeneous topological spaces and continuous mappings.

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