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REMARKS ON MULTIPLE PERIODIC SOLUTIONS OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: We prove the existence and multiplicity of periodic solutions for nonlinear ordinary differential equations of the type

$$u''(x) + g(u(x)) = f(x)$$

under the various conditions upon the function g.

Key words: Nonlinear ordinary differential equations, periodic problems.

Classification: 34C25

I. Introduction. Our starting point have been the papers [1],[2]. There are given in [2, Theorem 10] some conditions upon the right hand side f to obtain at least one solution of periodic problem

$$(1) \qquad \begin{cases} u''(x) + g(u(x)) = f(x) \\ u(0) = u(T),\ u'(0) = u'(T), \end{cases}$$

where $T \in (0,\pi)$ and g is τ-periodic function on $\mathbb{R}$ with some $\tau > 0$. In this article we present some multiplicity results for the solvability of (1) using the approach indicated in [1, 26.10] and in [2, Theorem 10], under the assumption that g is a bounded function, generally not periodic, with bounded derivative on $\mathbb{R}$. The presented sufficient conditions for the solvability of (1) make restriction only on the L_1-norm of the right hand side f in distinction from the conditions presented in [2].

If we put $g(x) = \sin x$, in (1), we obtain the mathematical pendulum equation.

2. Preliminaries. Let $T>0$ and let us denote C_T^0 the Banach space of all continuous and T-periodic functions defined on a real line $\mathbb{R}$ with the norm

$$\| u \|_{C_T^0} = \max_{x \in \mathbb{R}} |u(x)|.$$

Let, further, g be a continuous real-valued function such that g' exists almost everywhere in $\mathbb{R}$ and there exist constants $M>0$, $K>0$, $t_0>0$ such that

$$(2) \qquad |g(\xi)| \leq M, \quad |g'(\xi)| \leq K$$

for all $|\xi| \geq t_0$. Assume, in addition, that g is not a constant function.

Definition. For p, q such that

$$(3) \qquad \underline{G} = \inf_{\xi \in \mathbb{R}} g(\xi) < q \leq p < \sup_{\xi \in \mathbb{R}} g(\xi) = \overline{G}$$

we put $M_{p,q} = M_{p,q}^1 \cup M_{p,q}^2$, where

$$M_{p,q}^1 = \{ d \in \mathbb{R} ; \exists c_1, c_2 \in \mathbb{R},\ 0 \leq c_1 < c_2, \xi \in \langle c_1, c_2 \rangle \Rightarrow$$
$$\Rightarrow g(\xi) > p,\ \xi \in \langle -c_2, -c_1 \rangle \Rightarrow g(\xi) < q, d \leq c_2 - c_1 \},$$

$$M_{p,q}^2 = \{ d \in \mathbb{R} ; \exists c_1, c_2 \in \mathbb{R},\ 0 \leq c_1 < c_2,\ \xi \in \langle c_1, c_2 \rangle \Rightarrow$$
$$\Rightarrow g(\xi) < q,\ \xi \in \langle -c_2, -c_1 \rangle \Rightarrow g(\xi) > p,\ d \leq c_2 - c_1 \}.$$

If $\sup M_{p,q} = \infty$ for each p, q, satisfying (3), then g is called the expansive function.

Assume that the sets $g^{-1}(\underline{G})$, $g^{-1}(\overline{G})$ do not contain a nondegenerated interval. Slightly modifying the proof of Theorem 8 from [2] we obtain

Lemma 1. Let $f \in C_T^0$, $x_0 \in \mathbb{R}$ and $K < \pi^2/T^2$. If u_1, u_2 are solutions of (1) such that

$$u_1(x_0) = u_2(x_0).$$

Then u_1 and u_2 coincide on $\mathbb{R}$.

There is given in [2] a sketch of the proof of

Lemma 2. Let $f \in C_T^0$ and $K < \pi^2/T^2$. Then the Dirichlet problem

$$\text{(4)} \qquad \begin{cases} u''(x) + g(c+u(x)) = f(x), x \in (0,T), \\ u(0) = u(T) = 0 \end{cases}$$

has a unique solution $u \in C^2(\langle 0,T \rangle)$ for arbitrary $c \in \mathbb{R}$ (see also [1, Sec. 4.14, 4.19]).

3. Main result

Theorem. Let $f \in C_T^0$ and $K < \pi^2/T^2$. Then the problem (1) has at least one T-periodic solution if

$$\underline{G} < q \leq \frac{1}{T} \int_0^T f(x)dx \leq p < \overline{G},$$

$$T^2 M + T\int_0^T |f(x)|\, dx < \sup M_{p,q}.$$

Proof. Denote by $\tilde{v}_{c,f}$ the solution of (4) and put

$$v_{c,f}(x) = c + \tilde{v}_{c,f}(x - kT)$$

for $x \in \langle kT,(k+1)T \rangle$ (k is an integer). Then $v_{c,f}$ is a T-periodic solution of (1) if and only if

$$\int_0^T g(v_{c,f}(x))dx = \int_0^T f(x)dx.$$

Let us define a function $\Phi_f \colon \mathbb{R} \to \mathbb{R}$,

$$\Phi_f : c \longmapsto \int_0^T g(v_{c,f}(x))dx.$$

The Rolle's theorem implies the existence of such $x_c \in (0,T)$ that $\tilde{v}'_{c,f}(x_c) = 0$. Using this, we obtain

(5) $$|\tilde{v}'_{c,f}(y)| \leq \left|\int_{x_c}^{y} g(c+\tilde{v}_{c,f}(x))dx\right| + \left|\int_{x_c}^{y} f(x)dx\right| \leq$$
$$\leq TM + \int_0^T |f(x)|dx,\ y \in \langle 0,T\rangle,\ c \in \mathbb{R},$$

(6) $$|\tilde{v}_{c,f}(y_1) - \tilde{v}_{c,f}(y_2)| \leq \sup_{z\in\langle 0,T\rangle} |\tilde{v}'_{c,f}(z)|\,|y_1 - y_2| \leq$$
$$\leq T^2M + T\int_0^T |f(x)|dx,\ y_1,y_2 \in \langle 0,T\rangle,\ c \in \mathbb{R}.$$

From (6) and from the assumption $T^2M + T\int_0^T |f(x)|dx < < \sup M_{p,q}$ we obtain $c_1, c_2 \in \mathbb{R}$ such that

(7) $$\Phi_f(c_1) < Tq \text{ and } \Phi_f(c_2) > Tp.$$

Let us suppose that $\lim_{n\to\infty} d_n = d_0$. Then according to (5),(6) the set $\{\tilde{v}_{d_n,f}\}_{n=1}^{\infty}$ satisfies the assumptions of [3, Theorem 1.5.4] and so it is relatively compact in the space of two times continuously differentiable functions on $\langle 0,T\rangle$. This fact together with Lemma 2 imply that there exists exactly one $\tilde{v}_{d_0,f}$ which is the solution of (4) and $\tilde{\Phi}_f(d_0) = \lim_{n\to\infty} \Phi_f(d_n)$. So $\tilde{\Phi}_f$ is a continuous function and from (7) we obtain $c_3 \in (c_1,c_2)$ such that

$$\tilde{\Phi}_f(c_3) = \int_0^T f(x)\,dx.$$

Then $v_{c_3,f}$ is the solution of (1).

<u>Corollary 1.</u> Let $f \in C_T^0$, $K < \pi^2/T^2$. Suppose, moreover, that g is an expansive function, $\sup M^i_{p,q} = \infty$, $i=1,2$ and $g^{-1}(\underline{G})$, $g^{-1}(\overline{G})$ are both empty or both infinite. Then the problem (1) has infinitely many distinct solutions if and

only if

$\underline{G} < \frac{1}{T}\int_0^T f(x)dx < \overline{G}$, in the case $g^{-1}(\underline{G}) = g^{-1}(\overline{G}) = \emptyset$;

$\underline{G} < \frac{1}{T}\int_0^T f(x)dx < \overline{G}$, $f = \underline{G}$, $f = \overline{G}$, in the case $g^{-1}(\underline{G}) \neq \emptyset$, $g^{-1}(\overline{G}) \neq \emptyset$.

Proof. There are $p,q \in \mathbb{R}$ such that

$$\underline{G} < q \leq \frac{1}{T}\int_0^T f(x)dx \leq p < \overline{G},$$

in the case $g^{-1}(\underline{G}) = \emptyset$, $g^{-1}(\overline{G}) = \emptyset$. Because of $\sup M^i_{p,q} = \infty$, $i=1,2$, we obtain $\{c_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $c_n \neq c_m$ for $n \neq m$, $\Phi_f(c_n) = \int_0^T f(x)dx$. If $g^{-1}(\underline{G}) \neq \emptyset$, $g^{-1}(\overline{G}) \neq \emptyset$ then for each $k_1 \in g^{-1}(\underline{G})$, resp. $k_2 \in g^{-1}(\overline{G})$, the function $u = k_1$, resp. $u = k_2$, is the solution of (1) with $f = \underline{G}$, resp. $f = \overline{G}$. The necessity of the condition follows from the fact that each periodic solution u of (1) must satisfy

$$\int_0^T g(u(x))dx = \int_0^T f(x)dx.$$

Corollary 2. Let $f \in C^0_T$, $K < \pi^2/T^2$ and, moreover, let g be a τ-periodic function. Then the problem (1) has at least two distinct solutions u_1, u_2 such that $|u_i(0)| \leq \tau$, $i=1,2$, if

$$-1 < -p \leq \frac{1}{T}\int_0^T f(x)dx \leq p < 1 \quad \text{and}$$

$$T^2 M + T\int_0^T |f(x)|dx < \sup M_{p,q}.$$

Proof. There are fulfilled all the assumptions of Theorem and moreover Φ_f is a τ-periodic function. There are $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2 < c_1 + \tau$ such that $\Phi_f(c_1) = \Phi_f(c_1 + \tau) < -Tp$, $\Phi_f(c_2) > Tp$. So we obtain $c_3 \in (c_1, c_2)$ and $c_4 \in (c_2, c_1 + \tau)$ such that $\Phi_f(c_3) = \Phi_f(c_4) = \int_0^T f(x)dx$.

<u>Remark</u>. From the Corollary 1 it follows that the equation

$$u''(x) + \sin (u^{\frac{2k-1}{2k+1}}(x)) = f(x)$$

possesses an infinite number of T-periodic solutions if and only if

$$-1 < \frac{1}{T}\int_0^T f(x)dx < 1, \ f = \pm 1.$$

From the Corollary 2 it follows that the mathematical pendulum equation

$$u''(x) + \sin u(x) = f(x)$$

has at least two distinct T-periodic solutions u_1, u_2 such that $|u_i(0)| \le 2\pi$, $i=1,2$, if

$$-1 < -p \le \frac{1}{T}\int_0^T f(x)dx \le p < 1 \quad \text{and} \quad T^2 + T\int_0^T |f(x)|dx < \pi - 2\arcsin p.$$

R e f e r e n c e s

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