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HOMEOMORPHISMS OF POWERS OF METRIC SPACES

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Abstract: We construct a connected metric space X homeomorphic to X^3 but not homeomorphic to X^2. We prove that there exists no countable metric space homeomorphic to X^3 but not to X^2.

Key words: Connected metric spaces, powers of metric spaces.

Classification: Primary 54B10, 54G15
Secondary 54D05, 54E35

In 1973, a metric space X homeomorphic to $X^3 = X \times X \times X$ but not to $X^2 = X \times X$ was constructed, see [3]. The constructed space X was a coproduct (= disjoint union as closed-and-open subsets) of infinitely many metric continua, hence far from being either countable or connected. In the present paper, we construct a connected metric space X homeomorphic to X^3 but not to X^2 and prove that there exists no countable metric space with this property (although there exists a countable strongly paracompact space X homeomorphic to X^3 but not to X^2, see [5]). Some possible strengthenings and generalizations are sketched in 15. at the end of the paper.

1. Lemma. Let X_o, X_1 be non empty countable topologi-

cal spaces. Let X_0 contain two disjoint closed-and-open subsets homeomorphic to X_1 and X_1 contain two disjoint closed-and-open subsets homeomorphic to X_0. Then X_0 is homeomorphic to X_1.

Proof. a) Clearly, both X_0 and X_1 are infinite. Put $\{k,j\} = \{0,1\}$. Let $\{x_{k,1}, x_{k,2}, x_{k,3}, \dots\}$ be a sequence of all elements of X_k, $x_{k,n} \neq x_{k,m}$ for $n \neq m$. One can find easily disjoint closed-and-open subsets $A_{k,1}$, $B_{k,1}$ of the space X_k such that

$A_{k,1}$ is homeomorphic to X_k and $x_{k,1} \notin A_{k,1}$,

$B_{k,1}$ is homeomorphic to X_j.

Since $A_{k,1}$ is homeomorphic to X_k, there exist disjoint closed-and-open subsets $A_{k,2}$, $B_{k,2}$ of the space $A_{k,1}$ such that

$A_{k,2}$ is homeomorphic to X_k and $x_{k,2} \notin A_{k,2}$,

$B_{k,2}$ is homeomorphic to X_j.

By induction, we construct disjoint closed-and-open subsets $A_{k,n}$, $B_{k,n}$ of the space $A_{k,n-1}$ and such that $A_{k,n} \simeq X_k$, $B_{k,n} \simeq X_j$ and $x_{k,n} \notin A_{k,n}$. Consequently

$$\bigcap_{n=1}^{\infty} A_{k,n} = \emptyset.$$

b) Let h_n be a homeomorphism of $A_{0,n}$ onto $B_{1,n+1}$ and g_n a homeomorphism of $A_{1,n}$ onto $B_{0,n+2}$. Moreover, denote by h_0 a homeomorphism of X_0 onto $B_{1,1}$ and by g_0 a homeomorphism of X_1 onto $B_{0,2}$. We define

$$V_0 = X_0 \setminus A_{0,1},$$

$$W_0 = X_1 \setminus (A_{1,1} \cup h_0(V_0)),$$

and, by induction

$$V_n = A_{o,n} \setminus (A_{o,n+1} \cup g_{n-1}(W_{n-1})),$$

$$W_n = A_{1,n} \setminus (A_{1,n+1} \cup h_n(V_n)).$$

We define $\lambda : X_o \longrightarrow X_1$ by

$$\lambda(x) = h_n(x) \text{ for } x \in V_n,$$

$$\lambda(x) = g_n^{-1}(x) \text{ for } x \in g_n^{-1}(W_n).$$

Then λ is a homeomorphism of X_o onto X_1.

2. Theorem. Let n be a natural number, $n > 2$. Let X be a countable metric space homeomorphic to $X^n = X \times \dots \times X$ (n-times). Then X is homeomorphic to X^2.

Proof. If X is finite, then necessarily card $X \in \{0,1\}$, hence $X \simeq X^2$. Let us suppose that X is infinite. If X contains no isolated point, then X is homeomorphic to the ordered space of all rational numbers, hence $X \simeq X^2$ again. If X contains isolated points, then either it contains precisely one isolated point or it contains infinitely many isolated points. In the former case, $X = \{x_o\} \cup R$, where x_o is the isolated point and R is homeomorphic with the space of rational numbers. Then, clearly, $X \simeq X^2$ again. Finally, let us suppose that X contains infinitely many isolated points. Then X contains two disjoint closed-and-open subsets homeomorphic to $X \times X$, namely $h^{-1}(X \times X \times \{a_1\} \times \dots \times \{a_1\})$ and $h^{-1}(X \times X \times \{a_2\} \times \times \dots \times \{a_2\})$, where h is a homeomorphism of X onto X^n and a_1, a_2 are two distinct isolated points of X. Clearly, X^2 contains two disjoint closed-and-open subsets of X. Consequently, we can use the lemma with $X_o = X$, $X_1 = X^2$.

3. The aim of the rest of the paper is to present a construction of a connected metric space X homeomorphic to

X^3 but not to X^2. The construction is done in the category $\mathbb{M}$ of all metric spaces of the diameter ≤ 1 and all their contractions, i.e. mappings $f:(X,d) \longrightarrow (X',d')$ such that $d'(f(x),f(y)) \leq d(x,y)$. Hence, let us present some important properties of the category $\mathbb{M}$, first. Isomorphisms of $\mathbb{M}$ are precisely isometric bijections. The category $\mathbb{M}$ has all products; the product of a collection $\{(X_\alpha,d_\alpha) \mid \alpha \in A\}$ is the space (X,d), $X = \prod_\alpha X_\alpha$, $d = \sup_{\alpha \in A} d_\alpha$, with the usual projections $\pi_\alpha:(X,d) \longrightarrow (X_\alpha,d_\alpha)$. We denote it by $\prod_\alpha (X_\alpha,d_\alpha)$. The category $\mathbb{M}$ has also all coproducts; the coproduct of a collection $\{(X_\alpha,d_\alpha) \mid \alpha \in A\}$ is the space (X,d), $X = \bigcup_\alpha X_\alpha \times \{\alpha\}$, $d((x,\alpha),(y,\alpha)) = d_\alpha(x,y)$, $d((x,\alpha),(y,\alpha')) = 1$ for $\alpha \neq \alpha'$, with the coproduct injections $\iota_\alpha:(X_\alpha,d_\alpha) \longrightarrow (X,d)$ sending $x \in X_\alpha$ to (x,α). We denote it by $\coprod_\alpha (X_\alpha,d_\alpha)$.

We also can make identifications of points in objects of $\mathbb{M}$. If (X,d) is an object of $\mathbb{M}$ and $R \subset X \times X$ is given, then there exists a morphism $q:(X,d) \longrightarrow (\bar{X},\bar{d})$ such that $q(x) = q(y)$ whenever $(x,y) \in R$ and every morphism f of (X,d) into an arbitrary object with $f(x) = f(y)$ for all $(x,y) \in R$ factorizes uniquely through q. The space $(\bar{X},\bar{d})$ is obtained as follows. First, denote by $q_0:X \longrightarrow X/_E$ the factor-mapping, where E is the smallest equivalence containing R, and for $x,y \in X/_E$ put $d_0(x,y) = \inf \sum_{n=1}^{k} d(a_n,b_n)$, where the infimum is taken over all tupples $a_1,b_1,a_2,b_2,\dots,a_k,b_k$ such that $q_0(a_1) = x$, $q_0(b_k) = y$ and $q(b_n) = q(a_{n+1})$ for $n=1,\dots,k-1$. Then d_0 is a pseudometric on $X/_E$; define a surjective mapping $p:X/_E \longrightarrow \bar{X}$ by $p(x) = p(y)$ iff $d_0(x,y) = 0$ and, for any $\bar{x},\bar{y} \in \bar{X}$, put $\bar{d}(\bar{x},\bar{y}) = d_0(p^{-1}(x),p^{-1}(y))$. Then $q = q_0 \cdot p:(X,d) \rightarrow$

$\longrightarrow (\bar{X},\bar{d})$ has all the required properties. We say that $(\bar{X},\bar{d})$ is obtained from (X,d) by the identifications of x with y for all $(x,y) \in R$.

4. Denote by $\mathcal{M}$ the class of all isometric injections. An <u>$\mathcal{M}$-chain</u> is every presheaf in $|\mathbb{M}|$ over a well-ordered scheme $(\{(X_\alpha,d_\alpha)\}_\alpha , \{f_\alpha^\beta\}_{\alpha \le \beta})$ such that every f_α^β is in $\mathcal{M}$. Every $\mathcal{M}$-chain has a colimit in $|\mathbb{M}|$ created as follows. Denote by $(X,\{f_\alpha\}_\alpha)$ a colimit of the presheaf of the underlying sets and define a metric d on X such that for every $x,y \in X$ find an α with $x,y \in f_\alpha(X_\alpha)$ and put $d(x,y) = d_\alpha(f_\alpha^{-1}(x), f_\alpha^{-1}(y))$. Then, clearly, $((X,d),\{f_\alpha\})$ is a colimit of the given $\mathcal{M}$-chain.

If there is no danger of confusion, a space (X,d) will be denoted only by X.

<u>Lemma</u>. In $|\mathbb{M}|$, colimits of $\mathcal{M}$-chains commute with finite products. More precisely, if $\mathcal{P}_i = (\{X_{i,\alpha}\}_\alpha , \{f_{i,\alpha}^\beta\}_{\alpha \in \beta})$ are $\mathcal{M}$-chains over the same scheme, $i=1,\dots,n$, colim $\mathcal{P}_i = (X_i,\{f_{i,\alpha}\}_\alpha)$ and $\mathcal{P} = (\{\prod_i X_{i,\alpha}\}_\alpha, \{\prod_i f_{i,\alpha}^\beta\}_{\alpha \le \beta})$, then

$$\operatorname{colim} \mathcal{P} = (\prod_i X_i, \{\prod_i f_{i,\alpha}\}_\alpha).$$

<u>Proof</u> is straightforward.

5. We shall use the lemma in the following situation. We have a metric space Z and an isometric injection $h: Z \longrightarrow Z^3 = Z \times Z \times Z$ ($\times$ denotes the product in $|\mathbb{M}|$). We define a presheaf $\mathcal{P}$ over the set N of all non-negative integers as follows.

$X_0 = Z$, $X_1 = Z^3$, $h_0^1 = h$,

and, by induction

$$X_{n+1} = X_n^3,\ h_n^{n+1} = (h_{n-1}^n)^3.$$

Clearly, we obtain an $\mathcal{M}$-chain $\mathcal{P} = (\{X_n\}_n, \{h_n^m\}_{n\le m})$. Put $(X,\{h_n\}) = \operatorname{colim} \mathcal{P}$. Then, by the above lemma,

$$X \text{ is isometric to } X^3.$$

Clearly, if Z is connected, then X is also connected.

In what follows, we construct a connected space Z and an isometric injection $h: Z \longrightarrow Z^3$ such that, for the colimit space X, we shall be able to prove the non-homeomorphism of X to X^2.

<u>Observation.</u> If V is an open subset of Z such that $h(V) = V \times V \times V$ and for every $x \in V$ there exists $d(x) > 0$ such that $\operatorname{dist}(x, Z \setminus V) \ge d(x)$ and $d(x) = \min(d(x_1), d(x_2), d(x_3))$, where $(x_1, x_2, x_3) = h(x)$, then $h_0(V)$ is an open subset of X.

6. We recall that N denotes the set of all non-negative integers. Denote by N^N the set of all mappings of N into itself and by $\mathbf{0}$ the constant zero. We consider the addition on N^N given by

$$(f+g)(n) = f(n)+g(n),$$

where + on the right side is the usual addition of numbers. For $F, G \subset N^N$, we put

$$F + G = \{f+g \mid f \in F, g \in G\}.$$

By [4], there exists a set $T \subset N^N \setminus \{\mathbf{0}\}$ such that

$$T = T + T + T,\ T \cap (T+T) = \emptyset.$$

Put $S = T \times N$. For every $s = (f,n)$ put $\bar{s} = f$. Since $T = T + T + T$, one can find a bijection

$$\lambda : S \longrightarrow S \times S \times S$$

such that, for every $s \in S$,

$$\bar{s} = \bar{s}_1 + \bar{s}_2 + \bar{s}_3,$$

where $(s_1, s_2, s_3) = \lambda(s)$.

For every $f \in N^N \setminus \{0\}$, put

$$L(f) = \{(n,j) \mid 0 < j \leq f(n)\}.$$

Since $f \neq 0$, the set $L(f)$ is non empty. For every $s \in S$, define a bijection

$$\wp_s : L(\bar{s}_1) \amalg L(\bar{s}_2) \amalg L(\bar{s}_3) \longrightarrow L(\bar{s}),$$

(where $(s_1, s_2, s_3) = \lambda(s)$) such that

$$\wp_s(n,j) = (n,j) \text{ for } (n,j) \in L(\bar{s}_1),$$

$$\wp_s(n,j) = (n, \bar{s}_1(n) + j) \text{ for } (n,j) \in L(\bar{s}_2),$$

$$\wp_s(n,j) = (n, \bar{s}_1(n) + \bar{s}_2(n) + j) \text{ for } (n,j) \in L(\bar{s}_3).$$

7. Let $\mathcal{C}$ be a Cook continuum, i.e. a connected compact metric space such that for every subcontinuum $D \subset \mathcal{C}$ and every continuous mapping $f : D \longrightarrow \mathcal{C}$ either f is constant or $f(x) = x$ for all $x \in D$. (A continuum with this property was constructed by H. Cook in [1].) Let $\{A_n \mid n \in N\} \cup \{B_k \mid k \in N\}$ be a pairwise disjoint collection of its non-degenerate subcontinua. We may suppose diam $A_n = \frac{1}{2}$ for all $n \in N$, diam $B_k = 2^{-(k+2)}$. Choose $a_n \in A_n$ and $b_{k,1}$, $b_{k,2}$ in B_k in the distance $2^{-(k+2)}$. Denote by V_n the space which we obtain from the coproduct $A_n \amalg \coprod_{k \in N} B_k$ by the identification of the image of the coproduct injection of a_n with that of $b_{0,1}$ and the image of $b_{k,2}$ with that of $b_{k+1,1}$. To simplify the notation, we will suppose $A_n \subset V_n$, $B_k \subset V_n$ and $a_n = b_{0,1}$, $b_{k,2} = b_{k+1,1}$ for all $k, n \in N$. Hence diam $V_n \leq 1$, so V_n is in $\mathbb{M}$. Denote by V_n^* the comple tion of V_n. Clearly, it is obtained by the adding of a single point to $\bigcup_{k=0}^{\infty} B_k \subset V_n$, denote it by σ.

8. For every $f \in N^N \setminus \{0\}$ and every $\ell = (n,j) \in L(f)$ put $\bar{\ell} = n$. Given $f \in N^N \setminus \{0\}$, we investigate the product (in $\mathbb{M}$!) $\prod_{\ell \in L(f)} V_{\bar{\ell}}$, which is only another description of the space $\prod_{n \in N} (V_n)^{f(n)}$, more suitable for the manipulation with coordinates. Denote by $V(f)$ its subspace consisting of all those points x such that

(α) only finitely many coordinates of x are outside of $\bigcup_{k \in N} B_k$ (i.e. in $A_n \setminus \{a_n\}$),

(β) the others form a finite subset of $\bigcup_{k \in N} B_k$.

Moreover, denote by $\sigma(f)$ the point with all coordinates equal to σ. Put $V^*(f) = V(f) \cup \{\sigma(f)\}$ (considered as a subspace of $\prod_{\ell \in L(f)} V^*_{\bar{\ell}}$).

Observation. The space $V^*(f)$ is connected.

9. Let S, λ, φ_s be as in 6. For every $s \in S$, with $\lambda(s) = (s_1, s_2, s_3)$, define

$$\psi_s : V^*(\bar{s}) \longrightarrow V^*(\bar{s}_1) \times V^*(\bar{s}_2) \times V^*(\bar{s}_3)$$

such that $\psi(\sigma(\bar{s})) = (\sigma(\bar{s}_1), \sigma(\bar{s}_2), \sigma(\bar{s}_3))$ and, if $x \in V(\bar{s})$, $\psi_s(x) = (x_1, x_2, x_3)$ with $\pi_\ell(x_i) = \pi_{\varphi_s(\ell)}(x)$ for all $i=1,2,3$, $\ell \in L(\bar{s}_i)$ (where π_ℓ denotes the ℓ-th projection).

Observation. ψ_s is an isometric injection which maps $V(\bar{s})$ onto $V(\bar{s}_1) \times V(\bar{s}_2) \times V(\bar{s}_3)$.

10. Put $V = \coprod_{s \in S} V(\bar{s})$ (i.e. the underlying set of V is $\bigcup_{s \in S} (V(\bar{s}) \times \{s\})$). For $(x,s) \in V$ put $\psi(x,s) = ((x_1,s_1), (x_2,s_2), (x_3,s_3))$, where $(s_1,s_2,s_3) = \lambda(s)$, $(x_1,x_2,x_3) = \psi_s(x)$. Then ψ is an isometric bijection of V onto V^3.

Proposition. V is not homeomorphic to V^2.

Proof. One can verify easily that V^2 is isometric to

$\coprod_{f \in T+T, m \in N} V(f) \times \{n\}$. Since $T \cap (T + T) = \emptyset$ and every $V(f)$ is connected, it is sufficient to prove the following assertion.

If $V(f)$ is homeomorphic to $V(g)$, then $f = g$.

This follows from the fact that, for every $f \in N^N \setminus \{0\}$ and every $n \in N$, the value $f(n)$ is equal to $\log(c_n+1)$, where c_n is the cardinality of a maximal system $\mathcal{H}$ of homeomorphisms of A_n into $V(f)$ with the following properties.

(i) If $h \in \mathcal{H}$, $y \in h(A_n)$, then, for any $m \neq n$, any subcontinuum D of V_m such that $a_m \notin D$ and any continuous mapping $g: D \longrightarrow V(f)$ such that $y \in g(D)$, g is constant;

(ii) if $h, h' \in \mathcal{H}$, then $h(a_n) = h'(a_n)$.

For, by the properties of the Cook continuum $\mathcal{C}$, $h \circ \pi_\ell$ is either constant or $\bar{\ell} = n$ and $h \circ \pi_\ell$ is the inclusion $A_n \longrightarrow V_n$. If $\bar{\ell} = m \neq n$, then the value of $h \circ \pi_\ell$ is equal to a_m, by (i). If $\bar{\ell} = n$ and $h \circ \pi_\ell$ is constant, then the value of $h \circ \pi_\ell$ is equal to a_n, by (ii) and the maximality of $\mathcal{H}$. Hence, the homeomorphisms from $\mathcal{H}$ are in one-to-one correspondence with non-empty subsets of the set $\{1, \dots, f(n)\}$.

11. Denote by $\mathcal{J}$ the set of all non-zero integers. Let us suppose that $\{C_k | k \in \mathcal{J}\}$ is a system of non-degenerate subcontinua of the Cook continuum $\mathcal{C}$ such that the system $\{A_n | n \in N\} \cup \{B_k | k \in N\} \cup \{C_k | k \in \mathcal{J}\}$ is pairwise disjoint. We may suppose diam $C_k = 2^{-(|k|+1)}$. Choose $c_{k,1}$, $c_{k,2}$ in C_k in the distance $2^{-(|k|+1)}$. Denote by C the space which we obtain from $\coprod_{k \in \mathcal{J}} C_k$ by the identification of (the image of the coproduct injection of) $c_{-1,2}$ with $c_{1,1}$ and $c_{k,2}$ with $c_{k+1,1}$ for all $k \in \mathcal{J} \setminus \{-1\}$. Clearly, diam $C = 1$ (a simple counting of the diameter of C is the reason why zero is omitted in $\mathcal{J}$,

i.e. we glue C_{-1} immediately with C_1). To simplify the notation, we suppose again that $C_k \subset C$ for all k and $c_{-1,2} = c_{1,1}$, $c_{k,2} = c_{k+1,1}$. Denote by C^* a completion of C. It is obtained by the adding of two points to C, let us denote them by c_+ and c_- (where c_+ is the limit of the sequence $\{c_{k,1}\}$ with $k \to +\infty$ and c_- with $k \to -\infty$). Denote by W the subspace of the space $C^{\aleph_0}$ consisting of all points x such that the set of all coordinates of x form a finite subset of C; denote by σ_+ (or σ_-) the point of $(C^*)^{\aleph_0}$ with all coordinates equal to c_+ (or c_-, respectively). Put $W^* = W \cup \{\sigma_+, \sigma_-\}$. Then W^* is connected. Now, let

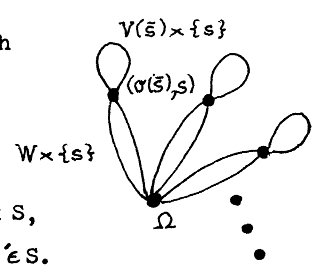
$$\sigma: \aleph_0 \sqcup \aleph_0 \sqcup \aleph_0 \longrightarrow \aleph_0$$

be a bijection. We define an isometric injection

$$\psi_W: W^* \longrightarrow W^* \times W^* \times W^*$$

by $\psi_W(\sigma_+) = (\sigma_+, \sigma_+, \sigma_+)$, $\psi_W(\sigma_-) = (\sigma_-, \sigma_-, \sigma_-)$ and, for $x \in W$, $\psi_W(x) = (x_1, x_2, x_3)$, where $\pi_n(x_i) = \pi_{\sigma(n)}(x)$ for $i = 1,2,3$, $n \in \aleph_0$. Clearly, ψ_W maps W onto $W \times W \times W$.

12. Denote by Z the space which we obtain from $V \sqcup \coprod_{s \in S} W_s^*$, where $W_s^* = W^*$ for all $s \in S$, by the identifications



$(\sigma(\bar{s}), s)$ with (σ_+, s) for all $s \in S$,

(σ_-, s) with (σ_-, s') for all $s, s' \in S$.

We may suppose $V \subset Z$, $\bigcup_{s \in S} (W^* \times \{s\}) \subset Z$ and $(\sigma(\bar{s}), s) = (\sigma_+, s)$, $(\sigma_-, s) = (\sigma_-, s')$. Denote the last point by Ω.

Now, we define an isometric injection h of Z into Z^3. We put

$h(x) = \psi(x)$ for $x \in V$,

$h(w,s) = ((w_1,s_1),(w_2,s_2),(w_3,s_3))$ for $w \in W^*$, where

$(s_1,s_2,s_3) = \lambda(s)$, $(w_1,w_2,w_3) = \psi_W(w)$,

(particularly, $h(\Omega) = (\Omega, \Omega, \Omega)$). One can verify that V is an open subset of Z.

13. We have constructed a connected space Z and an isometric injection $h: Z \to Z^3$. From these data, we construct an $\mathcal{M}$-chain $\mathcal{P}$ as in 5. Denote $(X, \{h_n\}) = \operatorname{colim} \mathcal{P}$. Then, by 5., X is isometric to X^3 and $h_o(V)$ is an open subspace of X.

<u>Proposition.</u> The set $h_o(V)$ is precisely the set of all $x \in X$ which fulfil the following property (p).

(p) There exists a neighbourhood $\mathcal{O}$ of x such that, for every subcontinuum D of any continuum of the system $\{C_k \mid k \in \mathcal{J}\}$ and every continuous mapping $g: D \to \mathcal{O}$, g is constant.

<u>Proof</u>. If $x \in h_o(V)$, it is sufficient to put $\mathcal{O} = h_o(V)$. Let us suppose that $x \in X \setminus h_o(V)$. Find the smallest n such that $x \in h_n(X_n)$ and put $y = h_n^{-1}(x)$. Then $y \in X_n = Z^{3^n}$. Denote by $(y_1, \dots, y_{3^n})$ its coordinates in Z^{3^n}. Since $y \notin h_o^n(V)$, at least one of the coordinates is not in V, say y_1. Then every neighbourhood of y in X_n contains a set $\mathcal{U} \times \{y_2\} \times \dots \times \{y_{3^n}\}$, where $\mathcal{U}$ is a neighbourhood of y_1 in Z. Since y_1 is in $Z \setminus V$, every its neighbourhood contains a homeomorphic image of some non-degenerate subcontinuum of some C_k.

14. <u>Proposition</u>. X is not homeomorphic to X^2.

Proof. The set of all $x \in X$ which fulfil (p) is homeomorphic to V. The set of all $x \in X^2$ which fulfil (p) is homeomorphic to $V \times V$. But V is not homeomorphic to V^2, by 10.

15. Concluding remarks. One can see that we have constructed a connected metric space X isometric to X^3 but not homeomorphic to X^2. By a minor modification of the construction, one can obtain, for every natural number $n \geq 3$, a connected metric space X isometric to X^n but not homeomorphic to X^k, $k=2,\dots,n-1$. Moreover, any metric space of the diameter ≤ 1 can be embedded by an isometric injection onto a closed subspace of X with this property. To obtain this, it is sufficient to embed it in a connected metric space Y of the diameter ≤ 1, to choose $y \in Y$ and to replace the space C in the above construction by the space $C \times Y$ and the points c_+, c_- by the points (c_+,y), (c_-,y).

16. Open problems. Let us denote, for shortness, by T the class of all topological spaces X homeomorphic to X^3 but not to X^2. By the presented construction, T contains a connected metrizable space. On the other hand, answers to the following questions are still unknown (though, by [2], there exist two non-homeomorphic metric continua with homeomorphic squares).

a) Does T contain a compact Hausdorff (or even metrizable) connected space ? (It contains a compact metrizable space, by [6].)

b) Does T contain at least a separable connected metrizable space ?

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